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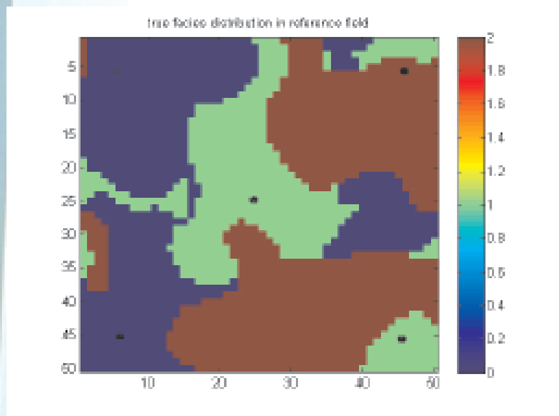

PROCEEDINGS

THE 11th WORKSHOP OF

DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE

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PREFACE

11-th Workshop of Department of Mathematics and Computer Science, Technical University of Civil Engineering was held in Bucharest, Romania, on May 27, 2011.

The aime of the workshop was the exchange of ideas, methods and problems between researchers, professors, practitioners in mathematics and related fields. The program included 36 lectures organized in five sections:

- Mathematical analysis, functional analysis, numerical analysis
- Algebra, geometry
- Differential equations, equations with partial derivates, mechanic
- Probability theory, operational researches, statistics
- Informatics, mathematical applications in engineering sciences, use of calculus software in research and education.

This volume contains 26 papers corresponding to the research talks that cover a wide variety of topics in mathematics.

We thank the authors of the contributed papers for timely submission and participation in the workshop.

The Editors

CUPRINS

DARBOUX – STIELTJES VECTOR INTEGRAL

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Abstract : In this paper we consider a concept of Integrability, noted D–S integrability, which generalizes the well – known notion of Riemann- Stieltjes integrability of a function $f:[a,b]\to\mathbb{R}$ with respect to a function $g:[a,b]\to\mathbb{R}$ having the finite variation.

1. Preliminaries and first results

Let us consider two real valued functions f and g on the compact interval $[a,b]$. We denote by $\mathsf{D}\,[a,b]$ the set of all divisions of the interval $[a,b]$ i.e. the finite systems $\Delta=(a=x_0\le x_1\le x_2\le ...\le x_k=b)$ of numbers of the interval $[a,b]$. As usually, the norm of Δ is, by definition, the positive number $\|\Delta\|:=\sup\{x_i-x_{i-1}|i=1,2,...,k\}$. We shall denote also by $\mathsf{I}\ (\Delta)$ the set of all indermediate system – points ξ of Δ i.e.

$$\xi=(\xi_1,\xi_2,...,\xi_k) \text{ with } x_{i-1}\le\xi_i\le x_i \quad \forall i\in\{1,2,...,k\}.$$

For any $\xi\in\mathsf{I}\ (\Delta)$ as before we denote by $\overline{\xi}$ the division of the interval $[a,b]$, adding to ξ the number $\xi_0=a$, $\xi_{k+1}=b$ i.e.

$$\overline{\xi}=(a=\xi_0\le\xi_1\le...\le\xi_k\le\xi_{k+1}=b)$$

If $\Delta'=(a=x'_0\le x'_1\le x'_2\le...\le x'_p=b)$ and $\{x_i|i\in\{1,2,...k\}\}\subset\{x'_j\,|\,j\in\{1,2,...,p\}\}$ use say that Δ' is finer than Δ and we write $\Delta\le\Delta'$.

Obviously, in this case Δ becomes an intermediate system - points for the division $\overline{\xi}$ i.e. $\xi\in\mathsf{I}\ (\overline{\xi})$ and we have $\|\overline{\xi}\|\le 2\|\Delta\|$, $\|\Delta\|\le 2\|\overline{\xi}\|$.

Let f,g be two real functions defined on $[a,b]$, let $\Delta\in\mathsf{D}\,[a,b]$, $\xi\in\mathsf{I}\ (\Delta)$ as before. As usually we denote $\sigma(f,g;\Delta,\xi):=\sum_{i=1}^{k} f(\xi_i)(g(x_i)-g(x_{i-1}))$ and if f is bounded on $[a,b]$.

$$s(f,g;\Delta):\sum_{i=1}^{k} m_i(g(x_i)-g(x_{i-1})) \text{ with } m_i=\inf\{f(x)|x\in[x_{i-1},x_i]\}$$

$$S(f,g;\Delta):\sum_{i=1}^{k} M_i(g(x_i)-g(x_{i-1})) \text{ with } M_i=\sup\{f(x)|x\in[x_{i-1},x_i]\}.$$

We remember that f in *Riemann- Stieltjes integrable with respect to* g (we write simply $f \in RS(g)$) if *for any sequence* $(\Delta_n)_n$ *of* $\mathcal{D}[a,b]$ *and any* $\xi_n \in I(\Delta_n)$ *the sequence* $(\sigma(f,g;\Delta_n,\xi_n))_n$ *is convergent. The limite of the last sequence does not depend on the sequence* $(\Delta_n)_n$ and $(\xi_n)_n$ and it is denoted by $\int_a^b fdg$. An alternative definition for Riemann-Stieltjes integrability is the following one: *The function* f *is Riemann- Stieltjes integrable with respect to* g *if there exists a real number* I *such that for any* $\varepsilon > 0$ *there exists* $\eta_\varepsilon > 0$ *such that we have* $|I - \sigma(f,g,\Delta;\xi)| \le \varepsilon$, $\forall \Delta \in \mathcal{D}[a,b]$, $\|\Delta\| < \eta_\varepsilon$ and $(\forall)\xi \in I(\Delta)$.

In fact the number I is uniquely determined and we have

$$I = \int_a^b fdg .$$

In $[1]$ we have given the following definition the function f is *Darboux – Stieltjes integrable with respect to* g *if there exists a sequene* $(\Delta_n)_n$ *in* $\mathcal{D}[a,b]$ *with* $\lim_{n\to\infty}\|\Delta_n\| = 0$ *such that for any sequences* $(\Delta'_n)_n$ in $\mathcal{D}[a,b]$ with $\Delta_n \le \Delta'_n$, *for all* $n \in \mathbb{N}$, *the sequence* $(\sigma(f,g,\Delta'_n,\xi'_n))_n$ *converges for all* $\xi'_n \in I(\Delta'_n)$. The limite of the last sequence does not depend on the sequences ($(\Delta'_n)_n$ and $(\xi'_n)_n$ and is denote by $DS\int_a^b fdg$. Some time we write $f \in DSI(g)$ instead of, the function f is Darboux – Stieltjes integrable with respect to g.

We have shown the fallowing results $[2]$.

Theorem 1.1. *If* $f:[a,b] \to \mathbb{R}$ *is bounded and* $g:[a,b] \to \mathbb{R}$ *is increasing the follwing assertions are equivalent.*

1. *For any* $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$ *there exists* $\Delta_\varepsilon \in \mathcal{D}[a,b]$ *such that*

$$S(f,g,\Delta_\varepsilon) - s(f,g,\Delta_\varepsilon) < \varepsilon$$

1'. $\sup\{s(f,g,\Delta) \mid \Delta \in \mathcal{D}[a,b]\} = \inf\{S(f,g,\Delta) \mid \Delta \in \mathcal{D}[a,b]\}$.

2. There exists a sequence $(\Delta_n)_n$ in $\mathcal{D}[a,b]$ *such that the sequence* $(\sigma(f,g,\Delta_n,\xi_n))_n$ *convergens for any* $\xi_n \in I(\Delta_n)$, *for all* $n \in \mathbb{N}$.

2'. There exists a sequence $(\Delta_n)_n$ in $\mathcal{D}[a,b]$ *with* $\lim_{n\to\infty}\|\Delta_n\| = 0$ *such that the sequences* $(s(f,g,\Delta_n))_n$, $(S(f,g,\Delta_n))_n$, $(\sigma(f,g,\Delta_n,\xi_n))_n$ *are convergent for any choise* $\xi_n \in I(\Delta_n)$ and we have $\lim_{n\to\infty} s(f,g,\Delta_n) = \lim_{n\to\infty} S(f,g,\Delta_n) = \lim_{n\to\infty} \sigma(f,g,\Delta_n,\xi_n)$.

3. *The function* f *is Darboux – Stieltjes integrable with respect to* g.

Remark 1. *If the above assertions hold we have, using the above notations,*

$$DS\int_a^b fdg = \lim_{n\to\infty} s(f,g,\Delta_n) = \lim_{n\to\infty} S(f,g,\Delta_n) = \sup_{\Delta \in \mathcal{D}[ab]} s(f,g,\Delta) = \inf_{\Delta \in \mathcal{D}[ab]} S(f,g,\Delta).$$

Remark 2. *If* $f:[a,b] \to \mathbb{R}$ *is bounded and* $g:[a,b] \to \mathbb{R}$ *is increasing then* $f \notin RS(g)$ *if* f *and* g *have a cammon discontinuity point* $x_0 \in [a,b]$.

A similar remark holds if f is bounded and g has finite variation.

One can easyly see that giving two real functions f,g on the interval $[a,b]$ we have $f \in RS(g) \Rightarrow f \in DSI(g)$ and $\int_a^b fdg = DS\int_a^b fdg$.

The following assertion shows that the existence of common discontinuity points of f and g make the difference between Riemann- Stieltjes and Darboux – Stieltjes integrability of the function f with respect to, the function g .

Theorem 1.2. *Suppose that the functions $g:[a,b]\to\mathbb{R}$ and $f:[a,b]\to\mathbb{R}$ are bounded. If the function f is Darboux–Stieltjes integrable with respect to g, and the functions f and g have no common discontinuity print, then f is Riemann- Stieltjes integrable with respect to g .*

Proof. The following well known lemma (see for instance $[5]$) is essential in our approach.

Lemma. Let $h:[a,b]\to\mathbb{R}$ *be on arbitrary function and let K be a compact subset of* $[a,b]$ such that the oscilation $\omega(f,k)$ of the function f *at the print k is dominated by the number* $\varepsilon\in\mathbb{R}$, $\varepsilon>0$ *for all $k\in K$, (i.e. $\omega(f,k)<\varepsilon \quad \forall k\in K$*). Then there exists $\eta\in\mathbb{R}$, $\eta>0$ such that for any interval $[c,d]$ $a\le c<d\le b$ with $d-c<\eta$ and $[c,d]\cap K\ne\phi$ we have

$$\omega(f,[c,d])<\varepsilon$$

i.e.

$$\sup\{f(x)|x\in[c,d]\}<\varepsilon+\inf\{f(x)|x\in[c,d]\}.$$

Let now $I=DS\int_a^b fdg$ and for a given $\varepsilon>0$ let $\Delta_\varepsilon\in \mathrm{D}\,[a,b]$ $\Delta_\varepsilon=(a=x_0<x_1<...<x_p=b)$ be such that

$$|I-\sigma(f,g,\Delta;\xi)|<\varepsilon$$

for any $\Delta\in \mathrm{D}\,[a,b]$, with $\Delta_\varepsilon\subset\Delta$, and any $\xi\in I(\Delta)$.

Since the functions f and g have no common point of discontinuity we may consider $K_f=\{x_i|f \text{ continuons in } x_i\}$; $K_g=\{x_i|g \text{ continuons in } x_i\}$.

Obvionsly we have

$$\{x_0,x_1,...,x_n\}=K_f\cup K_g$$

And using the above lemma we may choose $\delta>0$ such that $2\delta<\|\Delta_\varepsilon\|$ and for any interval $[\alpha,\beta]\subset[a,b]$ such that

$$[\alpha,\beta]\cap K_f\ne\phi\Rightarrow\omega(f,[\alpha,\beta])<\varepsilon,$$
$$[\alpha,\beta]\cap K_g\ne\phi\Rightarrow\omega(g,[\alpha,\beta])<\varepsilon.$$

Let us consider $\Delta\in \mathrm{D}\,[a,b]$, with $\|\Delta\|<\delta$, $\Delta=(a=y_0<y_1<y_2<...<y_k=b)$ and let $\xi\in I(\Delta)$.

Since $y_{j+1}-y_j<\delta$ the interval $[y_j,y_{j+1}]$ contains at most a point of the set $\{x_0,x_1,...,x_n\}=K_f\cup K_g$.

Let $I_f := \left\{ i \in \{0,1,2,...,n\} \middle| x_i \in k_f \right\}$; $I_g := \left\{ i \in \{0,1,2,...,n\} \setminus I_f \right\}$ and let us denote by $[y'_i, y'_{i+1}]$ (resp. $[y"_i, y"_{i+1}]$) those interval $\left[y_j, y_{j+1} \right]$ which contains the point $x_i \in I_f$ (resp. $x_i \in I_g$). With this notations we have

$$\left| S(f,g,\Delta,\xi) - S(f,g,\Delta \cup \Delta_\varepsilon, \xi) \right| \leq \left(\|f\| + \|g\| \right) p \cdot \varepsilon$$

where $\overline{\xi_i} \in [y_i, y_{i+1}]$ is chousen equal y_i if $i \notin I_f \cup I_g$.

Hence

$$\left| S(f,g,\Delta,\xi) - I \right| \leq \left| S(f,g,\Delta,\xi) - S(f,g,\Delta_\varepsilon,\xi) \right| + \left| S\left(f,g,\Delta \cup \Delta_\varepsilon, \tilde{\xi}\right) - I \right| \leq \left(\|f\| + \|g\| + 1 \right) p \cdot \varepsilon$$

for all $\Delta \in \mathcal{D}[a,b]$ and $\xi \in I(\Delta)$ with $\|\Delta\| < \delta$.

2. Vector case

For a function $f : [a,b] \to X$ where X is a Banach space and $g : [a,b] \to \mathbb{R}$ we may introduce a notion of RS and DS integrability and we are able to prove the analogous assertions as Theorem 1.1 and Theorem 1.2 as before.

Bibliografie

[1] Bucur Ileana, *Somemore about Riemann-Stieltjes integral Seminaire des espaces lineaires ordonné topologique*, Nr.16,1997.

[2] Bucur Ileana, *Integrability eriterium for Darboux–Stieltjes integral Seminaire des espaces lineaires ordonné topologique*, Nr. 17, 1998.

[3] Boboc Nicu, *Analiză matemtică*, 1988. Editura Universitățíi București.

[4] R.E.Bradley, *The Riemann- Stieltjes integral*, Missouri Journal of Math. Sci 1994.

[5] Bucur Gheorghe, *Analiză matemtică*, 2007, Editura Universitățíi Bucureşti.

[6] Nicolescu Miron, *Analiză matemtică*, Vol. II, 1953, Editura Academiei Române.

[7] Rudin W., *Principles of Mathematical Analysis,* Mc. Graw-Hill, New-York, 1953.

[8] E.J.McShane, T.A. Rotts, *Real Analysis*, Van Nostrand, Princeton, New Jersey, 1958.

[9] Torchinsky, *A.*, *Real Variables*, Addison-Wesley, Reading, MA, 1988.

CELLULAR CONSTANT SHEAVES

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Abstract : The purpose of this article is to study the basic properties of the cellular constant sheaves. As applications we present the cohomology of a standard 1 and 2-simplex with values in a cellular constant sheaf ; we give explicit formulas.
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Key words: cellular constant sheaf , cohomology.

1. **The basic properties of the cellular constant sheaves**

The cellular constant sheaves appeared in the computation of K_G - groups for some G-spaces (where G is a compact Lie group). This notion was introduced in [2] for a standard simplex (as base), but it was extended in [3] to any CW-complex.

Let X be a standard n-simplex - i.e. generating by exactly n+1 vertices (this section contains definitions from [4] and [1]).

Definition. A sheaf $\mathcal{F}$ (of abelian groups) on X is called *cellular constant* if for any open face Y of X the restriction of $\mathcal{F}$ to Y is constant (i.e. $\mathcal{F}|Y = A_Y \in$ Ab).

Proposition 1. For the cellular constant sheaf $\mathcal{F}$ we have :

a. If Y, Z are two faces of the simplex X such that $Y \cap \overline{Z} \neq \emptyset$ (this condition is equivalent with: Y is a face of Z) then there exists the morphisms $f^Y_Z : A_Y \to A_Z$.

b. If Y, Z , W are faces of the simplex X such that $Z \cap \overline{W} \neq \emptyset$ and $Y \cap \overline{Z} \cap \overline{W} \neq \emptyset$ then f^Y_W is the composition $f^Y_W = f^Z_W \circ f^Y_Z$.

Proof. a. If $a \in A_Y$ and y ∈ $Y \cap \overline{Z}$ we consider a neighborhood V_y of y and a section $s_y \in \mathcal{F}(V_y)$ such that s_y(y) = a . Because the sheaf $\mathcal{F}$ is constant on Y it follows that the restrictions of the sections s_y and s_z to $V_y \cap V_z \cap Y$ are equal and we have this equality on a neighborhood U such that $U \cap Z \neq \emptyset$.

Let be V = $\bigcup_y (V_y \cap Z)$ and because the sheaf $\mathcal{F}$ is constant on Z it follows that the restrictions of the sections s_y and s_z to $V_y \cap V_z \cap Z$ are equal ; then there exist an unique section s ∈ $\mathcal{F}|Z$ (V) such that his restriction to $V_y \cap Z$ is exactly s_y for any y ∈ $Y \cap \overline{Z}$.

Now let us consider the element z∈ V and one denotes s (z) = b $\in A_Z$; then we can define the morphism f^Y_Z by the following formula :

$$f_Z^Y\,(a) = b\,.$$

b. Given y$\in Y\cap\overline{Z}\cap\overline{W}$, let V_y be a neighborhood of y and let be a$\in A_Y$; we choose an element z$\in\bigcup_y(V_y\cap Z)$ as above and similary the element w$\in\bigcup_z(V_z\cap W)$. For the section s $\in$ F (V_y) such that s (y) = a, we have:

$$f_Z^Y\,(a) = s\,(z) \quad \text{and} \quad f_W^Y\,(a) = s\,(w);$$

then it follows that:

$$(f_W^Z \circ f_Z^Y)(a) = f_W^Z\,(s\,(z)) = s\,(w) = f_W^Y\,(a)\,. \qquad \square$$

<u>Proposition 2.</u> Let Y be an open face of the standard simplex X, let (A_Y) be a family of objects in the category Ab and let $g_Z^Y : A_Y \to A_Z$ be a family of morphisms (where $Y\cap\overline{Z}\neq\varnothing$).

Then there exist a cellular constant sheaf F on X : F $_{|}$ Y = A_Y , with values in the category Ab and the morphisms associated as in the proposition 1 are exactly g_Z^Y. This sheaf F is unic in the set of isomorphism classes of sheaves.

<u>Proof.</u> Given an open set U of the standard simplex X we define :

$$\mathrm{F}\,(U) = \prod_Y A_Y$$

where $Y\cap U\neq\varnothing$ and Y runs through the open faces of X ; for another open set V of the standard simplex X such that U$\subset$V, one can define the structural morphisms

$$\rho_U^V : \mathrm{F}\,(V) \to \mathrm{F}\,(U)$$

by the following formula :

$$\rho_U^V\,((a_Y)_{Y\cap V\neq\varnothing}) = (\sum_Z g_Z^Y(a_Y))_{Z\cap U\neq\varnothing}$$

where $a_Y\in A_Y$.

It is easy to check up that F is a presheaf (of abelian groups) and that it is even a sheaf on the standard simplex X; from the above definition it follows that F $_{|}$ Y = A_Y for any open face Y of X and it is obvious that the morphisms associated as in the proposition 1 are exactly g_Z^Y under the condition $Y\cap\overline{Z}\neq\varnothing$. $\square$

2. **Cohomology with values in a cellular constant sheaf**

Let X be the standard 1- simplex generated by the points A_0 and A_1 and we consider that the stalk of the cellular constant sheaf F at the point A_0 is the group G_0, at the point A_1 is G_1 and the stalk of F on any point of the interior of X is the group G_{01}; we also assume that $G_0, G_1 \subset G_{01}$.

Given the close set F = $\{A_0, A_1\}$ one considers the long exact sequence (in cohomology) associated to F (see [4] and [1]):

$$0 \to H^0(X - F;\ \mathsf{F}) \to H^0(X;\ \mathsf{F}) \to H^0(F;\ \mathsf{F}) \xrightarrow{\delta} H^1(X - F;\ \mathsf{F}) \to \ldots \qquad (1)$$

But X-F is just the open segment $(A_0,\ A_1)$ and using the following formula:

$$H_c^n(\mathbb{R}^m; G) = \begin{cases} 0 & if \quad n \neq m \\ G & if \quad n = m \end{cases} \qquad (2)$$

where by H_c one denotes the cohomology with compact supports (see [4]) we obtain:

$$H^1(\text{ X-F }; \mathsf{F}) = G_{01}\ ;$$
$$H^n(\text{ X-F }; \mathsf{F}) = 0 \quad \text{for any } n \neq 1.$$

Using now the exact Mayer-Vietoris sequence (in cohomology) associated to the closed sets $F_0 = \{ A_0 \}$ and $F_1 = \{ A_1 \}$ (see [4]) :

$$0 \to H^0(F;\ \mathsf{F}) \to H^0(F_0;\ \mathsf{F}) \oplus H^0(F_1; \mathsf{F}) \to H^0(F_0 \cap F_1;\ \mathsf{F}) \xrightarrow{\delta}$$
$$H^1(F; \mathsf{F}) \to \ldots$$

one obtains : $H^0(F;\ \mathsf{F}) = G_0 \oplus G_1$ and $H^n(F;\ \mathsf{F}) = 0$ for any n > 0 ; using all the above results it follows that $H^n(X;\ \mathsf{F}) = 0$ for any n > 1 and moreover the exact sequence in cohomology (1) turns in:

$$0 \to H^0(X;\ \mathsf{F}) \xrightarrow{\alpha} G_0 \oplus G_1 \xrightarrow{\delta} G_{01} \xrightarrow{\beta} H^1(X;\ \mathsf{F}) \to 0$$

where the differential is defined by the formula: δ(a,b) = a-b .

Then we obtain finally the cohomology of the standard 1-simplex X with values in the above cellular constant sheaf :

$$H^0(X; \mathsf{F}) \cong \operatorname{Im}\alpha = Ker\,\delta = G_0 \cap G_1;$$
$$H^1(X; \mathsf{F}) \cong \operatorname{Im}\beta \cong G_{01} / Ker\,\beta = G_{01} / \operatorname{Im}\delta = G_{01}/(G_0 + G_1)\,.$$

Remark. One can obtain the same results by considering a close and locally finite covering of X and computing the cohomology associated to this covering with values in the sheaf F (for details see [2]).

Now let X be the standard 2 - simplex generated by the points A_0, A_1 and A_2 (i.e. X is a triangle) and one considers that the stalks of the cellular constant sheaf F at the above vertices are the groups G_0, G_1 and respectively G_2, that on the sides of the triangle the stalks are G_{01}, G_{02} and respectively G_{12} and that on any point of the interior of X the stalk of F is the group G_{012} ; one assumes that there exists the "natural" inclusion relations between the above groups:

$$G_0,\ G_1 \subset G_{01},\ G_0,\ G_2 \subset G_{02}, \ldots, G_{01},\ G_{02},\ G_{12} \subset G_{012}$$

Given the close set F = { A_0, A_1, A_2} it's obvious that BX - F is close in X-F, where BX denotes the boundary of X; moreover the set (X-F) - (BX-F) is exactly the interior of X, noted by $\overset{\circ}{X}$. Then, by the formula (2), we have:

$$H^2(\overset{\circ}{X}\ ; \mathsf{F}) = G_{012} \text{ and } H^1(\text{BX-F} ; \mathsf{F}) = G_{01} \oplus G_{02} \oplus G_{12} .$$

Using two times the long exact sequence (1), firstly for the sets X and X-F, after that for the sets X-F and $\overset{\circ}{X}$, one obtains finally the cohomology of the standard 2-simplex X with values in the above cellular constant sheaf :

$$H^0(\text{X} ; \mathsf{F}) = G_0 \cap G_1 \cap G_2 ;$$

$$H^1(\text{X} ; \mathsf{F}) = \frac{\{(g_{01}, g_{12}) \in G_{01} \oplus G_{12} \,|\, g_{01} + g_{12} \in G_{02}\}}{\{(a+b, b+c) \,|\, a \in G_0, b \in G_1, c \in G_2\}};$$

$$H^2(\text{X} ; \mathsf{F}) = \frac{G_{012}}{G_{01} + G_{12} + G_{02}};$$

$$H^n(\text{X} ; \mathsf{F}) = 0 \quad \text{for any } n > 2.$$

(for the definitions of the differentials and for more details see [2]).

References

1. Bredon, G.: *Sheaf theory*, Mc. Graw-Hill Book Company, 1967.
2. Costinescu, C.N.: Sirul spectral Atiyah-Hirzebruch în K_G-teorie şi aplicații, *St. cerc. mat.*, 27, **4** (1975), 425-442.
3. Dogaru, O. : Fascicule celular simple, *St. cerc. mat.*, 27, **5** (1975), 535 - 545.
4. Godement, R. : *Topologie algébrique et théorie des faisceaux*, Ed. Hermann, Paris, 1958.
5. Spanier E. : *Algebraic topology*, Mc. Graw-Hill, New York, 1966.

SIMULTANEOUS EXTENSION PROBLEMS

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Abstract: Firstly we refer to the common extension problem for two (positive) linear operators: given two vector subspaces G_1 and G_2 in an (ordered) vector space E, a Dedekind complete ordered vector space F and two (positive) linear operators $T_1 : G_1 \to F, T_2 : G_2 \to F$, when does a (positive) linear common extension of T_1, T_2 exist?
Secondly we refer to a more general problem having as a consequence the existence of a common extension for two (positive) linear operators. The framework for this problem is the following: if X is a vector space, $S : X \to F$ is a sublinear operator, A_1, A_2 are two arbitrary sets, $g_1 : A_1 \to X$, $g_2 : A_2 \to X$, $f_1 : A_1 \to F$, $f_2 : A_2 \to F$ are four arbitrary maps such that $f_1 \le S \circ g_1$ on A_1 and $f_2 \le S \circ g_2$ on A_2, give a general condition to extend simultaneously there inequalities replacing S by a linear operator L dominated by S.
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Keywords: common extension of positive linear operators, sublinear operators, Hahn-Banach theorem.

1. Preliminaries

In this paper the terminology and the notation are like in [1], [2] and [11]; X_0 and X will be real vector spaces, E_0 and E will be ordered vector spaces and, generally, F will be a Dedekind complete ordered vector space (that is, every nonempty ordered bounded set in F has a supremum or, equivalently, an infimum).
The first simultaneous extension problem studied in this paper will be the following common extension problem: if G_1, G_2 are two vector spaces (or arbitrary sets) in E_0, $E = span(G_1 \cup G_2)$ and $T_1 : G_1 \to F$, $T_2 : G_2 \to F$ are two linear operators (or arbitrary maps) give (necessary and) sufficient conditions for the existence of a (positive) linear operator $L : E \to F$ such that L extends T_1 and T_2, that is $L(v_1) = T(v_1)$ and $L(v_2) = T(v_2)$, for all $v_1 \in G_1$ and $v_2 \in G_2$. It is immediately that a necessary condition for this is that the operators T_1 and T_2 are *consistent* (see [9]) that is $T_1 = T_2$, on $G_1 \cap G_2$.

The importance of this problem appears, for example, in [9], [13], [14], [15] and [16]. If $F = \mathbb{R}$, this problem was solved in [8] and [12].

After the usage (see for example [13]) we will call this problem 2-*common extension problem*. Also we will consider the generalization of the previous problem for n linear operators instead of two linear operators, and we will name it n-*common extension problem*.

The second simultaneous extension problem studied in this paper will be the following: if A_1, A_2 are two arbitrary sets, $g_1 : A_1 \to X$, $g_2 : A_2 \to X$, $f_1 : A_1 \to F$, $f_2 : A_2 \to F$ are four arbitrary maps and $S : X \to F$ is a sublinear operator such that $f_1 \le S \circ g_1$ and $f_2 \le S \circ g_2$, give (necessary and) sufficient conditions for the existence of a linear operator $L : E \to F$ dominated by S such that the previous inequalities will be extended simultaneously putting L instead of S.

2. Primary Results

Theorem 1. (2-common linear extension) *Let* X_0 *and* Y *be two vector spaces,* G_1 *and* G_2 *two vector subspaces of* X_0 , $X = span(G_1 \cup G_2)$ *and* $T_j : G_j \to Y$, $j \in \{1,2\}$ *, two linear operators. Then, the following are equivalent:*

i) There exists $L : X \to Y$*, a common linear extension of* T_1, T_2.

ii) If $v_1 + v_2 = 0$*, with* $v_j \in G_j$, $j \in \{1,2\}$ *then* $T_1(v_1) + T_2(v_2) = 0$.

iii) $T_1 = T_2$ *on* $G_1 \cap G_2$.

For a finite family $(T_j)_{j\in\{1,...,n\}}$ of linear operators, Theorem 1 becomes:

Theorem 2. (n-common linear extension) *Let* X_0 *and* Y *be two vector spaces,* $(G_j)_{j\in\{1,...,n\}}$ *a family of vector subspaces of* X_0 *and* $T_j : G_j \to Y$, $j \in \{1,...,n\}$ *a family of linear operators. Then, the following are equivalent:*

i) There exists $L : span(G_1 \cup ... \cup G_n) \to Y$ *a common linear extension of* $T_1,...,T_n$.

ii) If $v_1 + v_2 + ... + v_n = 0$*, then* $T_1(v_1) + T_2(v_2) + ... + T_n(v_n) = 0$*, where* $v_j \in G_j$*, for each* $j \in \{1,...,n\}$.

iii) For each two sets N_1, N_2 *so that* $N_1 \cap N_2 = \emptyset$ *and* $N_1 \cup N_2 = \{1,...,n\}$,
$\sum_{k\in N_1} T_k(v_k) = \sum_{j\in N_2} T_j(v_j)$ *if* $\sum_{k\in N_1} v_k = \sum_{j\in N_2} v_j$*, where* $v_k \in G_k$ *for any* $k \in N_1$*, and* $v_j \in G_j$ *for any* $j \in N_2$.

The following result is a version of the Theorem 1 in the ordered vector spaces setting, all the linear operators which appear being positive.

Theorem 3. *Let* E_0 *be an ordered vector space and let* F *be a Dedekind complete ordered vector space. Let also* G_1, G_2 *be two vector subspaces of* E_0 *and let* $T_1 : G_1 \to F$, $T_2 : G_2 \to F$ *be two positive linear operators. Let us consider the following statements, where we denote* $E = span(G_1 \cup G_2)$:

i) There exists $L : E \to F$ *, a positive common linear extension of* T_1 *and* T_2;

ii) If $v_1 + v_2 \leq 0$*, where* $v_j \in G_j$, $j \in \{1,2\}$*, then* $T_1(v_1) + T_2(v_2) \leq 0$;

iii) If $v_1 + v_2 \geq 0$*, where* $v_j \in G_j$, $j \in \{1,2\}$*, then* $T_1(v_1) + T_2(v_2) \geq 0$;

iv) If $v_1 + v_2 = 0$*, where* $v_j \in G_j$, $j \in \{1,2\}$*, then* $T_1(v_1) + T_2(v_2) = 0$;

v) $T_1 = T_2$ *on* $G_1 \cap G_2$.

Then, we have: $i) \Leftrightarrow ii) \Leftrightarrow iii) \Rightarrow iv) \Leftrightarrow v)$

The *proof* of Theorem 3 is immediately. Also, the corresponding result which generalizes this theorem for a family $(T_j)_{j\in\{1,...,n\}}$ of positive linear operators can easily formulated.

3. Common extensions in the line of Mazur-Orlicz and Hahn-Banach theorems

In the following result having as a consequence the Mazur-Orlicz theorem, we meet another meaning for the *common extension problem*. We will consider two nonempty sets A_1, A_2, four maps $g_1 : A_1 \to X$, $g_2 : A_2 \to X$, $f_1 : A_1 \to F$, $f_2 : A_2 \to F$ and a sublinear operator $S : X \to F$ such that all these maps satisfy an inequality which imply that $f_1 \le S \circ g_1$ and $f_2 \le S \circ g_2$. Then we can extend simultaneously these inequalities, obtaining the existence of a linear operator $L : E \to F$ dominated by S and such that $f_1 \le L \circ g_1$ and $f_2 \le L \circ g_2$. Actually, this result will be applied to obtain a *common extension* (for two positive linear operators) in the mainly meaning considered in this paper and in the line of the Hahn-Banach theorem.

Theorem 4. *Let X be a vector space, F a Dedekind complete ordered vector space, A_1 and A_2 two nonempty arbitrary sets, $S : X \to F$ a sublinear operator, and $g_j : A_j \to X$ and $f_j : A_j \to F$, $j \in \{1,2\}$, four maps. Then, the following are equivalent:*

i) There exists $L : X \to F$ a linear operator such that

a) $L \le S$ on X, and

b) $f_1 \le L \circ g_1$ on A_1 and $f_2 \le L \circ g_2$ on A_2.

ii) The inequality

$$\sum_{i=1}^{n} \lambda_i f_1(a_{1i}) + \sum_{j=1}^{m} \mu_j f_2(a_{2j}) \le S\left(\sum_{i=1}^{n} \lambda_i g_1(a_{1i}) + \sum_{j=1}^{m} \mu_j g_2(a_{2j}) \right)$$

holds for all n, $m \in \mathrm{N}^*$, $\{a_{11},...,a_{1n}\} \subset A_1$, $\lambda_1 \ge 0,...,\lambda_n \ge 0$ and $\{a_{21},...,a_{2m}\} \subset A_2$, $\mu_1 \ge 0,...,\mu_m \ge 0$.

Remark 1. We can easily extend Theorem 4 for any p sets $A_1,...,A_p$ and $2p$ maps $g_i : A_i \to X$, $f_i : A_i \to F$, $i \in \{1,..., p\}$ instead of A_1, A_2 and g_1, g_2, f_1, f_2. Obviously, for $p = 2$ and A_2, f_2 and g_2 suitable we obtain the vectorial form of the Mazur-Orlicz theorem (see [10])

The following result is the version of Theorem 4 for ordered vector spaces.

Theorem 5. *Let E be an ordered vector space, F a Dedekind complete ordered vector space, and K_1, K_2 two nonempty convex sets, and $S : E \to F$ a monotone sublinear operator. For each $i \in \{1,2\}$, let $P_i : K_i \to E$ be a convex operator and $Q_i : K_i \to F$ a concave operator. Then, the following conditions are equivalent:*

i) There exists a positive linear operator $L : E \to F$ such that

a) $L \le S$ on E, and

b) $Q_1 \le L \circ P_1$ on K_1 and $Q_2 \le L \circ P_2$ on K_2.

ii)The inequality

$$\lambda Q_1(a_1) + \mu Q_2(a_2) \le S\left(\lambda P_1(a_1) + \mu P_2(a_2)\right)$$

holds for all $a_1 \in K_1$, $a_2 \in K_2$, $\lambda \ge 0$ and $\mu \ge 0$.

Obviously, the previous result extend the Mazur-Orlicz theorem for ordered vector spaces (see [7], theorem 2.4.).

Now we remember two vectorial forms of the Hahn-Banach extension theorem, for cases in which the domain space is an arbitrary *vector space*, and an *ordered vector space*, respectively.

Theorem 6. *Let X be a vector space, F a Dedekind complete ordered vector space, and $S: X \to F$ a sublinear operator. Let G be a vector subspace of X and $T: G \to F$ a linear operator. The following conditions are equivalent:*
i) There exists a linear operator $L: X \to F$ with the properties

a) $L \le S$ on X, and b) $L = T$ on G.

ii) $T \le S$ on G.

Theorem 7. *Let E be an ordered vector space, F a Dedekind complete ordered vector space and $S: E \to F$ a monotone sublinear operator. Let G be a vector subspace of E and $T: G \to F$ a positive linear operator. Then, the following are equivalent:*
i) There exists a positive linear operator $L: E \to F$ such that

a) $L \le S$ on E, and b) $L = T$ on G.

ii) $T \le S$ on G.

Remark that the Mazur-Orlicz theorem is a generalization of Theorem 6 (the vectorial form of the Hahn-Banach extension theorem).
The following common extension result will be formulated in the line of the Hahn-Banach extension theorem with a *vector space* as the domain space (see Theorem 6).

Theorem 8. *Let X be a vector space, F a Dedekind complete ordered vector space, and $S: X \to F$ a sublinear operator. Let G_1 and G_2 be two vector subspaces of X and $T_1: G_1 \to F$, $T_2: G_2 \to F$ two linear operators. The following conditions are equivalent:*
i) There exists a linear operator $L: X \to F$ with the properties:

a) $L \le S$ on X, and

b) $L = T_1$ on G_1, $L = T_2$ on G_2.

ii) $T_1(v_1) + T_2(v_2) \le S(v_1 + v_2)$ for all $v_1 \in G_1$ *and* $v_2 \in G_2$.

The following common extension result will be formulated in the line of the Hahn-Banach extension theorem with an *ordered vector space* as the domain space (see Theorem 7).

Theorem 9. *Let E be an ordered vector space, F a Dedekind complete ordered vector space and $S: E \to F$ a monotone sublinear operator. Let G_1 and G_2 be two vector subspaces of X and $T_1: G_1 \to F$, $T_2: G_2 \to F$ two positive linear operators. Then, the following are equivalent:*
i) There exists a positive linear operator $L: E \to F$ such that

a) $L \le S$ on E, and

b) $L = T_1$ on G_1, $L = T_2$ on G_2.

ii) $T_1(v_1) + T_2(v_2) \le S(v_1 + v_2)$, for all $v_1 \in G_1$ and $v_2 \in G_2$.

The following result is a consequence of Theorem 9.

Corollary 1. *Let E, F, G_1, G_2 and T_1, T_2 be like in the previous theorem. Then, the following are equivalent:*
i) There exists $L: E \to F$ a positive linear operator such that $L = T_1$ on G_1 and $L = T_2$ on

G_2.
ii) There exists $S: E \to F$ *a monotone sublinear operator such that*

$$T_1(v_1) + T_2(v_2) \le S(v_1 + v_2)$$

for all $v_1 \in G_1$ *and* $v_2 \in G_2$.

In the following result, which is a consequence of Corollary 1, the condition that the sublinear operator S is monotone is dropped.

Theorem 10. *Let* E *be an ordered vector space,* F *a Dedekind complete ordered vector space and* G_1, G_2 *two vector subspaces of* E. *Let also* $T_1: G_1 \to F$ *and* $T_2: G_2 \to F$ *be two positive linear operators. Then, the following are equivalent:*
i) There exists a positive linear operator $L: E \to F$ *such that* $L = T_1$ *on* G_1 *and* $L = T_2$ *on* G_2.
ii) There exists $S: E \to F$ *a sublinear operator such that*

$$v_1 + v_2 \le v \Rightarrow T_1(v_1) + T_2(v_2) \le S(v)$$

where $v_1 \in G_1$, $v_2 \in G_2$ *and* $v \in E$.

Remark 2. Many results of this paper, including Theorem 9, can be easily generalized in the line of the Maharam theorem (1972).

Theorem 11. (*Maharam theorem*) *Let* E *be a vector lattice with an order unit* $e \in E_+$ *and* $(G_\delta)_{\delta\in\Delta}$ *a family of subspaces of* E *such that* $e \in span\left(\bigcup_{\delta\in\Delta} G_\delta\right)$. *Let also* F *be a Dedekind complete ordered vector space and let* $\{T_\delta : G_\delta \to F \mid \delta \in \Delta\}$ *be a family of positive linear operators. Then, the following conditions are equivalent:*
i) There exists $T: E \to F$ *a positive linear extension of the family* $(T_\delta)_{\delta\in\Delta}$ *(i.e.* $T(x) = T_\delta(x)$ *for all* $\delta \in \Delta$ *and* $x \in G_\delta$*)*
ii) The inequality $0 \le T_\delta(v_\delta)$ *holds for every family* $(v_\delta)_{\delta\in\Delta} \in \Phi((G_\delta))$, *satisfying* $0 \le \sum_{\delta\in\Delta} v_\delta$, *where* $\Phi\left((G_\delta)_{\delta\in\Delta}\right)$ *denotes the collection of all families* $\{v_\delta \in G_\delta \mid \delta \in \Delta\}$ *such that* $v_\delta \ne 0$ *for at most finitely many* $\delta \in \Delta$.

This theorem was originally proved by D. Maharam in [9] (see also [13], Theorem 6.3).
The following result (see [7], Theorem 5.4) is an easy generalization of Theorem 11, because if the ordered vector space E has an order unit $e > 0$ and $G \subseteq E$ is a vector subspace so that $e \in G$, then G is a majorizing subspace of E.

Theorem 12. *Let* E *be an ordered vector space and let* $(G_\delta)_{\delta\in\Delta}$ *be a family of subspaces of* E, *such that there exists at least one which is majorizing, say* G_{δ_0}. *Let* F *be a Dedekind complete ordered vector space and let* $\{T_\delta : G_\delta \to F \mid \delta \in \Delta\}$ *be a family of positive linear operators. Then the following conditions are equivalent:*
i) The family $\{T_\delta : G_\delta \to F \mid \delta \in \Delta\}$ *has a positive common linear extension* $T: E \to F$.

ii) The implication $\sum_{\delta\in\Delta} v_\delta \geq 0 \Rightarrow \sum_{\delta\in\Delta} T_\delta(v_\delta) \geq 0$ *holds for every family* $(v_\delta)_{\delta\in\Delta} \in \Phi\left((G_\delta)_{\delta\in\Delta}\right)$.

Remark 3. If we generalize Corollary 1 in the line of the Maharam theorem, we obtain Theorem 12, and hence Theorem 11 too, as consequences. To prove this it suffices to prove that Corollary 1 implies the version of Theorem 12 for $\Delta = \{1,2\}$. For this aim it is necessary to prove that *ii')*$\Rightarrow$*ii*) if at least one of the subspaces G_1, G_2, say G_1, is majorizing, where *ii*) and *ii')* are the following statements:

ii) There exists S *a monotone sublinear operator such that*

$$T_1(v_1) + T_2(v_2) \leq S(v_1 + v_2)$$

for all $v_1 \in G_1$ *and* $v_2 \in G_2$.

ii') If $v_1 + v_2 \leq 0$, *then* $T_1(v_1) + T_2(v_2) \leq 0$ *for all* $v_1 \in G_1$ *and* $v_2 \in G_2$.

Suppose that *ii')* is valid. Let us define $T : span(G_1 \cup G_2) \to F$ by the equality

$$T(v_1 + v_2) = T_1(v_1) + T_2(v_2)$$

for all $v_1 \in G_1$ and $v_2 \in G_2$.

The operator T has the following properties:

1) T is well-defined, according to *ii')*;
2) T is linear;
3) T is positive.

Because we supposed that G_1 is a majorizing subspace, it follows that $G = span(G_1 \cup G_2)$ is majorizing, too. Define $S : E \to F$, $S(x) = \overline{T}(x)$, for all $x \in E$, (that is $S(x) = \inf\{T(z) \mid z \in G,\ z \geq x\}$). It is known that S is a monotone sublinear operator and $T \leq S$ on E. We have:

$T_1(v_1) + T_2(v_2) = T(v_1 + v_2) \leq S(v_1 + v_2)$, for all $v_1 \in G_1$ and $v_2 \in G_2$, that is *ii*) is valid.

4. Common positive extensions using an additional set

In the following result we will give a sufficient condition for the existence of a positive linear operator L satisfying the converse inequalities of Theorem 4 *i)b)*. This condition is an implication between two inequalities and next we will simplify the form of the left and respectively of the right member of these inequalities. Note that, instead of majorization of L by a sublinear operator S we will assume the existence of an additional set M and of two maps $h : M \to E$ and $r : M \to F$, obtaining that $L \circ h \leq r$ on M.

Theorem 13. *Let* E_0 *be an ordered vector space,* F *a Dedekind complete ordered vector space, and let* A_1, A_2 *and* M *be arbitrary nonempty sets. Let also* $g_j : A_j \to E_0$, $f_j : A_j \to F$, $j \in \{1,2\}$ *and* $h : M \to (E_0)_+$, $r : M \to F$ *be arbitrary maps, and* $E = span(g_1(A_1) \cup g_2(A_2) \cup h(M)) \subseteq E_0$. *Suppose that:*

$$\sum_{i=1}^{n} \alpha_i g_1(a_{1i}) + \sum_{i=1}^{n} \beta_i g_2(a_{2i}) \leq \sum_{i=1}^{n} h(z_i) \Rightarrow$$

$$\sum_{i=1}^{n} \alpha_i f_1(a_{1i}) + \sum_{i=1}^{n} \beta_i f_2(a_{2i}) \leq \sum_{i=1}^{n} r(z_i)$$

where $n \in \mathrm{N}^*$, *and* $a_{1i} \in A_1$, $a_{2i} \in A_2$, $z_i \in M$, $\alpha_i \in \mathrm{R}$, $\beta_i \in \mathrm{R}$, *for each* $i \in \{1,...,n\}$.

Then, there exists a positive linear operator $L : E \to F$ *such that*

a) $L \circ g_1 \le f_1$ *on* A_1, $L \circ g_2 \le f_2$ *on* A_2, *and*
b) $L \circ h \le r$ *on* M.

Now we will simplify successively the form of the left members in the inequalities which appear in the hypothesis of Theorem 13.

Theorem 14. *Let* E_0 *be an ordered vector space,* F *a Dedekind complete ordered vector space, and let* G_1, G_2 *be two ordered vector spaces and* M *a nonempty set. Let also* $h : M \to (E_0)_+$ *and* $r : M \to F$ *be two maps,* $P_j : G_j \to E_0$ *linear operators and* $T_j : G_j \to F$ *positive linear operators, where* $j \in \{1,2\}$. *Denote by* $E = span(P_1(G_1) \cup P_2(G_2) \cup h(M)) \subseteq E_0$. *Then, the following conditions are equivalent:*
i) There exists a positive linear operator $L : E \to F$ *such that*
a) $L \circ P_j = T_j$ *on* G_j *for* $j \in \{1,2\}$, *and b)* $L \circ h \le r$ *on* M.
ii) $P_1(v_1) + P_2(v_2) \le \sum_{i=1}^{n} h(z_i) \Rightarrow T_1(v_1) + T_2(v_2) \le \sum_{i=1}^{n} r(z_i)$
where $n \in \mathrm{N}^*$, $v_1 \in G_1$, $v_2 \in G_2$ *and* $z_i \in M$ *, for all* $i \in \{1,...,n\}$.

We remark that the form of the left side in the inequalities which appear in (*ii*) can be simplified still, if G_1 and G_2 are two vector subspaces of the ordered vector space E_0.

Theorem 15. *Let* E_0 *be an ordered vector space,* F *a Dedekind complete ordered vector space, and let* G_1, G_2 *be two ordered vector subspaces of* E_0 *and* M *an arbitrary set. Let also* $h : M \to (E_0)_+$, *and* $r : M \to F$ *be two maps, and* $T_1 : G_1 \to F$, $T_2 : G_2 \to F$ *two positive linear operators. Denote by* $E = span(G_1 \cup G_2 \cup h(M)) \subseteq E_0$. *Then, the following conditions are equivalent:*
i) There exists a common positive linear extension L *of* T_1, T_2 *to the space* E *(that is* $L = T_j$ *on* G_j, *for* $j \in \{1,2\}$*) such that* $L \circ h \le r$ *on* M.
ii)
$$v_1 + v_2 \le \sum_{i=1}^{n} h(z_i) \Rightarrow T_1(v_1) + T_2(v_2) \le \sum_{i=1}^{n} r(z_i)$$
for $n \in \mathrm{N}^*$, $v_1 \in G_1$, $v_2 \in G_2$ *and* $z_i \in M$ *, for each* $i \in \{1,...,n\}$.

A new step to simplify the right members of the inequalities that arise in (*ii*) is to choose M an arbitrary subset of $(E_0)_+$ and to take $h = i$, the inclusion of M in E_0.

Theorem 16. *Let* E_0 *be an ordered vector space,* F *a Dedekind complete ordered vector space, and let* G_1, G_2 *be two ordered vector spaces of* E_0 *and* M *an arbitrary subset of* $(E_0)_+$. *Let also* $r : M \to F$ *be a map, and* $T_1 : G_1 \to F$, $T_2 : G_2 \to F$ *two positive linear operators. Denote by* E *, the vector space* $span(G_1 \cup G_2 \cup M) \subseteq E_0$. *Then the following statements are equivalent:*
i) There exists a common positive linear extension L *of* T_1, T_2 *to the space* E *such that* $L \le r$ *on* M.

ii) $$v_1+v_2\le\sum_{i=1}^{n}z_i\Rightarrow T_1(v_1)+T_2(v_2)\le\sum_{i=1}^{n}r(z_i)$$

where $n\in\mathrm{N}^*$, $v_1\in G_1$, $v_2\in G_2$ *and* $z_i\in M$, *for each* $i\in\{1,...,n\}$.

Remark 4. 1) Note that this theorem generalizes a result formulated without proof in [4], and applied in [5]; for the proof, see Theorem 1, p.63 in [6]. Also, Theorem 16 generalizes [7], Theorem 6.4. This result is the consequence of our Theorem 16, obtained taking $G_2=\{0\}$ and $T_2=0$ (the null operator on G_2).

2) If moreover than in the Theorem 16, the cone $(E_0)_+$ is generating and $M=(E_0)_+$, then $E=E_0$ and thus the Theorem 16 gives the existence of a common extension of T_1, T_2 to the whole E_0.

3) We have also $E=E_0$ if E_0 has a positive algebraic basis, chosen instead of M.

Note that we can also simplify the form of the right side in the inequalities which appear in the condition *ii)* in all previous theorems of this section. It suffices to choose the set M a nonempty set closed under addition (in an arbitrary ordered vector space E_1 for Theorem 13 and Theorem 14) and to assume that the maps $-h$ and r are subadditive. So, for example, the corresponding inequalities in the Theorem 13 becomes:

$$\sum_{i=1}^{n}\alpha_i g_1(a_{1i})+\sum_{i=1}^{n}\beta_i g_2(a_{2i})\le h(z)\Rightarrow\sum_{i=1}^{n}\alpha_i f_1(a_{1i})+\sum_{i=1}^{n}\beta_i f_2(a_{2i})\le r(z)$$

for $n\in\mathrm{N}^*$, $z\in M$, and $a_{1i}\in A_1$, $a_{2i}\in A_2$, $\alpha_i\in\mathrm{R}$, $\beta_i\in\mathrm{R}$, for each $i\in\{1,...,n\}$. Also, the corresponding inequalities in Theorem 16 becomes: $v_1+v_2\le z\Rightarrow T_1(v_1)+T_2(v_2)\le r(z)$, where $v_1\in G_1$, $v_2\in G_2$ and $z\in M$.

Remark 5. As consequences of the results included in this section, we obtain respectively Theorems 6.1, 6.2, 6.3 and 6.4 from [7].

Other common positive linear extension results using an additional set can be formulated in the line of some results by Z. Lipecki ([8]) and R. Cristescu ([3]). For example, the following common extension result is in the line of a result of R. Cristescu, concerning the extension of a positive linear operator. This result by R. Cristescu generalizes a result obtained by Z. Lipecki for the extension of a positive linear operator defined on a majorizing vector subspace of an ordered vector space. Note that in the following theorem, F , the range of the operators is an ordered vector space, not necessarily Dedekind complete.

Theorem 17. *Let* E_0 *and* F *be two ordered vector space,* G_1, *and* G_2 *be two vector subspaces of* E_0 *and* $M\subseteq E_0$ *a nonempty set. Let also* $T_1:G_1\to F$, $T_2:G_2\to F$ *be positive linear operators and* $P:E_0\to F$ *a monotone sublinear operator such that* $P=T_1$ *on* G_1 *and* $P=T_2$ *on* G_2. *Denote* $E=span(G_1\cup G_2\cup M)$ *and suppose that*

$$P\left(\sum_{i=1}^{n}z_i\right)=\sum_{i=1}^{n}P(z_i)$$

where $n\in\mathrm{N}^*$ *and* $z_1,...,z_n\in M$.

Then, there exists $L:E\to F$ *a positive linear operator such that*

a) $L=T_1$ *on* G_1, $L=T_2$ *on* G_2, *and*

b) $L=P$ *on* M.

Remember that many results in this paper can be generalized in the line of the Maharam theorem.

References

[1] C. D. Aliprantis and O. Burkinshaw, *Positive Operators*, Academic Press, New-York, London (1985).
[2] R. Cristescu, *Ordered Vector Spaces and Linear Operators*, Ed. Academiei, Bucureşti, Abacus Press, Tunbridge Wells, (1976).
[3] R. Cristescu, *Extension of positive operators*, Topological Ordered Vector Space, Univ. of Bucharest, **6** (1985), 23-42 (in Romanian).
[4] R.-M. Dăneț, *Positive linear extension of linear operators*, Topological Ordered Vector Space, Univ. of Bucharest, **6** (1985), 135-136 (in Romanian).
[5] R.-M. Dăneț, *Some consequences of a theorem about the extension of positive linear operators*, Rev. Roumaine Math. Pures Appl 23, **9** (1988), 721-729.
[6] R.-M. Dăneț, *Some technics for the existence and the extension of positive linear operators*, Publishing House of the Romanian Academy, Buc. (1993) (in Romanian).
[7] N. Dăneț and R.-M. Dăneț, *Existence and extensions of positive linear operators*, Positivity **13**(1)(2009), 89-106.
[8] Z. Lipecki, *Maximal-valued extensions of positive operators*, Math. Nachr. **117** (1984), 51-55.
[9] D. Maharam, *Consistent extensions of linear functionals and of probability measures*, Proc. Sixth Berkley Symp. Math Stat Probab(Berkley), Univ. of California Press, **2**(1972), 127-147.
[10] S. Mazur and W. Orlicz, *Sur les espaces métriques linéaires II*, Studia Math., **13**(1953).
[11] P. Meyer-Nieberg, *Banach lattices*, Springer-Verlag, Berlin (1991).
[12] V. Ptak, *Simoultaneous extensions of two functionals*, Czech Math J. **19**(94), (1969), 553-566.
[13] K. D. Schmidt, *Decomposition and extension of abstract measures in Riesz spaces*, Rend Istit. Math. Univ. Trieste Suppl., **29**(1998), 135-213.
[14] K. D. Schmidt and G. Waldschaks, *Common extensions of order bounded vector measures* (Preprint, Manheim 91-1989), Rend. Circ. Math. Palermo Ser II, vol **28**, Suppl, (1992), 117-124.
[15] K. D. Schmidt and G. Waldschaks, *Common extensions of positive vector measures* (Preprint, Manheim 93-1989), Portugaliae Math **48**(1991), 155-164.
[16] R. M. Short and F. Wehrung, *Common extensions of semi-group valued charges*, Journal of mathematical analysis and applications, **187**(1), San Diego, (1994), 235-238.

COINCIDENCE RESULTS FOR FAMILIES OF MULTIMAPS IN THE FINITE DIMENSIONAL TOPOLOGICAL VECTOR SPACES SETTING AND SOME APPLICATIONS TO EQUILIBRIUM PROBLEMS

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Abstract: We apply some fixed-point theorems, in the finite dimensional topological vector spaces setting, to establish coincidence results for two families of multimaps, some of these multimaps being compact. Also we apply the obtained coincidence theorems to deduce existence results for equilibrium in a generalized abstract economy with two companies (or a generalized abstract game with two families of players). These companies may have different numbers of factories.
Mathematics Subject Classification (2000): 54H25, 91B50.
Key words: multimap, compact multimap, equilibrium point, maximal element.

1. Introduction

This paper is a continuation of [5]. See this paper for the terminology, the definitions and the notation. The starting point of our results is some *fixed-point theorems* from [6] presented for the first time in [7] (see also [4]). The basic framework of the *collectively fixed-point results* used in this paper is the following.

Let I be an index set and for each $i \in I$, let E_i be a finite dimensional topological vector space and X_i a nonempty convex subset of E_i. Let also $X = \prod_{i \in I} X_i$ and let $T_i : X \to 2^{X_i}$, $i \in I$, be a family of nonempty-valued and convex-valued multimaps. In some hypothesis imposed to the space X and to the family $(T_i)_{i \in I}$, it follows the existence of a fixed-point for this family, that is the existence of a point $x = (x_i)_{i \in I} \in X$ such that $x_i \in T_i(x)$, for all $i \in I$. It is possible also to have two families $(S_i)_{i \in I}$ and $(T_i)_{i \in I}$ of multimaps such that for example $\operatorname{co} S_i(x) \subseteq T_i(x)$, for each $i \in I$ and $x \in X$; then imposing some conditions to the family $(S_i)_{i \in I}$, it follows the existence of a fixed-point for the family $(T_i)_{i \in I}$ (see some results in [4] formulated in the line of some interesting results by Q. H. Ansari and J. C. Yao [2] and by L.-J. Lin, Z.-T. Yu, Q. H Ansari and Lai, L.-P. [7]).

Note that some fixed-point results for a family $(T_i)_{i \in I}$ of multimaps were formulated in the literature of the domain *with* or *without* some *compactness* assumptions on the domain and the range sets of multimaps.

The results in this paper use some compactness hypothesis (see also [8]).

A first application of the fixed-point results, in the corresponding literature is in the *coincidence results*. In this paper, we consider such results for two families $(T_i)_{i\in I}$ and $(S_j)_{j\in J}$ of multimaps ($T_i : Y \to 2^{X_i}$, $i \in I$ and $S_j : X \to 2^{Y_j}$, $j \in J$, where $X = \prod_{i\in I} X_i$ and $Y = \prod_{j\in J} Y_j$, with X_i and Y_j nonempty convex sets in some finite dimensional topological vector spaces). We give sufficient conditions for the existence of a point $(\tilde{x}, \tilde{y}) \in X \times Y$, with $\tilde{x} = (\tilde{x}_i)_{i\in I}$ and $\tilde{y} = (\tilde{y}_j)_{j\in J}$ such that $\tilde{x}_i \in T_i(\tilde{y})$ for each $i \in I$ and $\tilde{y}_j \in S_j(\tilde{x})$ for all $j \in J$.

The main results are formulated for compact multimaps in the finite dimensional topological vector spaces setting. In this setting, the convex hull of a compact set is compact too (see, for example, [1] or [3]).

In [5] we have given two collectively fixed-point results for multimaps in the above mentioned setting. Remark that we used one or two families of multimaps, respectively, with compactness assumptions on one of the families.

2. Main Results

In this section, by using the above mentioned results from [5], we will give some *coincidence results* for two families of *compact* multimaps in the finite dimensional topological vector spaces setting.

Theorem 2.1 *Let I and J be two arbitrary index sets and for all $i \in I$ and $j \in J$, let X_i and Y_j be a nonempty convex sets in some finite dimensional topological vector spaces. Let also $X = \prod_{i\in I} X_i$, $Y = \prod_{j\in J} Y_j$ and let $S_j : X \to 2^{Y_j}$ ($j \in J$) and $T_i : Y \to 2^{X_i}$ ($i \in I$) be a nonempty-valued and convex valued multimaps. Suppose that:*

1) for each $j \in J$, $X = \bigcup_{y_j \in Y_j} \operatorname{int} S_j^{-1}(y_j)$ and for each $i \in I$, $Y = \bigcup_{x_i \in X_i} \operatorname{int} T_i^{-1}(x_i)$;

2) for each $i \in I$, T_i is compact and for each $j \in J$, S_j is compact (i.e. there exist the nonempty compact sets $K_i \subseteq X_i$ and $L_j \subseteq Y_j$ such that $T_i(Y) \subseteq K_i$ and $S_j(X) \subseteq L_j$).

Then, there exists a coincidence point $(\tilde{x}, \tilde{y}) \in X \times Y$ (that is $\tilde{x} = (\tilde{x}_i)_{i\in I}$, $\tilde{y} = (\tilde{y}_j)_{j\in J}$) such that $\tilde{x}_i \in T_i(\tilde{y})$ and $\tilde{y}_j \in S_j(\tilde{x})$, for all $i \in I$ and $j \in J$.

In the following theorem, we will consider four families of multimaps, and imposing some conditions on the two of these families, we will also obtain a coincidence result.

Obviously, the following result is a consequence of Theorem 2.1.

Corollary 2.2 *If in Theorem 2.1 we replace the hypothesis "2)" by "2')", the conclusion is still true, where:*

2') for each $i \in I$ and $j \in J$, the sets Y_j and X_i are compact.

Theorem 2.3 *Let I and J be two arbitrary index sets, and for all $i \in I$ and $j \in J$, let X_i and Y_j be nonempty convex sets in some finite dimensional topological vector spaces. Let also $X = \prod_{i\in I} X_i$, $Y = \prod_{j\in J} Y_j$ and let $S_j, R_j : X \to 2^{Y_j}$ $(j \in J)$ and $V_i, T_i : Y \to 2^{X_i}$ ($i \in I$) be* *four families of nonempty-valued multimaps (is suffices that S_j and V_i are nonempty-valued). Suppose that:*

i) for each $j\in J$ and $x\in X$, $\mathrm{co}S_j(x)\subseteq R_j(x)$;

ii) for each $i\in I$ and $y\in Y$, $\mathrm{co}\mathrm{V}_i(y)\subseteq T_i(y)$;

iii) for each $i\in I$, $Y=\bigcup_{x_i\in X_i}\mathrm{int}V_i^{-1}(x_i)$;

iv) for each $j\in J$, $X=\bigcup_{y_j\in Y_j}\mathrm{int}S_j^{-1}(y_j)$;

v) for each $j\in J$, S_j is a compact multimap;

vi) for each $i\in I$, V_i is a compact multimap.

Then there exists $\tilde{x}=(\tilde{x}_i)_{i\in I}\in X$ and $\tilde{y}=(\tilde{y}_j)_{j\in J}\in Y$ such that $\tilde{x}_i\in T_i(\tilde{y})$ and $\tilde{y}_j\in R_j(\tilde{x})$,for all $i\in I$ and $j\in J$ (that is $(\tilde{x},\tilde{y})$ is a coincidence point for the families $(T_i)_{i\in I}$ and $(R_j)_{j\in J}$).

3. Applications in the General Equilibrium Theory

The first result in this section solves the problem of the existence of an equilibrium point for an economy having two companies and compact constraint multimaps C_j and D_j $(j\in J)$. Its proof uses Theorem 2.3 (a coincidence result).

Theorem 3.1 *Let I and J be two arbitrary index sets, $(E_i)_{i\in I}$ and $(F_j)_{j\in J}$ two families of finite dimensional topological vector spaces. For all $i\in I$ and $j\in J$, let $X_i\subset E_i$ and $Y_j\subset F_j$ be nonempty convex sets and let $X=\prod_{i\in I}X_i$, $Y=\prod_{j\in J}Y_j$.*

Consider $\Gamma=(X_i, A_i, B_i, P_i, Y_j, C_j, D_j, Q_j)_{\substack{i\in I\\ j\in J}}$ a generalized abstract economy with two companies (or a generalized abstract game with two families of players) with the constraint multimaps A_i, B_i, C_j, D_j and the preference multimaps P_i, Q_j, where $A_i, B_i, P_i : Y\to 2^{X_i}$ and $C_j, D_j, Q_j : X\to 2^{Y_j}$,for all $i\in I$ and $j\in J$.

Suppose that, for all $j\in J$ and $i\in I$ the following conditions hold:

1) $\mathrm{co}A_i(y)\subseteq B_i(y)$, for all $y\in Y$, and $\mathrm{co}C_j\subseteq D_j(x)$, for all $x\in X$;

2) A_i is a compact multimap;

3) C_j is a compact multimap;

4) $X=\bigcup_{y_j\in Y_j}\mathrm{int}\left(C_j^{-1}(y_j)\cap\left(Q_j^{-1}(y_j)\cup H_j\right)\right)$, where $H_j=\{x\in X \mid C_j(x)\cap Q_j(x)=\varnothing\}$;

5) $Y=\bigcup_{x_i\in X_i}\mathrm{int}\left(A_i^{-1}(x_i)\cap\left(P_i^{-1}(x_i)\cup G_i\right)\right)$, where $G_i=\{y\in Y \mid A_i(y)\cap P_i(y)=\varnothing\}$;

6) for all $\tilde{x}=(\tilde{x}_i)_{i\in I}\in X$ and $\tilde{y}=(\tilde{y}_j)_{j\in J}\in Y$, $x_i\notin\mathrm{co}P_i(y)$ and $y_j\notin\mathrm{co}Q_j(x)$.

Then, there exists $(\tilde{x},\tilde{y})$, an equilibrium point for Γ, that is $\tilde{x}=(\tilde{x}_i)_{i\in I}\in X$ and $\tilde{y}=(\tilde{y}_j)_{j\in J}\in Y$ such that for all $i\in I$ and $j\in J$,

$$\tilde{x}_i\in B_i(\tilde{y}),\ A_i(\tilde{y})\cap P_i(\tilde{y})=\varnothing\ \text{(i.e. } \tilde{y}\in G_i\text{)}$$
$$\tilde{y}_j\in D_j(\tilde{x}),\ C_j(\tilde{x})\cap Q_j(\tilde{x})=\varnothing\ \text{(i.e. } \tilde{x}\in H_j\text{).}$$

The following result gives sufficient conditions to obtain a *maximal element* (or an equilibrium point) for a qualitative game in the finite dimensional topological vector spaces setting. It is a consequence of the Theorem 3.1.

Theorem 3.2 *Let* I, J, $(X_i)_{i\in I}$, $(Y_j)_{j\in J}$, X *and* Y *be like in Theorem 3.1 and let* $\Gamma = (X_i, P_i, Y_j, Q_j)_{\substack{i\in I\\ j\in J}}$ *a qualitative game, where* $P_i : Y \to 2^{X_i}$ *and* $Q_j : X \to 2^{Y_j}$ $(i \in I, j \in J)$ *are preference multimaps. For each* $j \in J$ *and* $i \in I$, *let also* $K_i \subseteq X_i$ *and* $L_j \subseteq Y_j$ *be two nonempty compact subsets. Assume that for all* $i \in I$ *and* $j \in J$, *the following hold:*

1) $Y = \bigcup_{x_i \in \mathrm{co}K_i} \mathrm{int}\left(P_i^{-1}(x_i) \cup G_i\right)$, *where* $G_i = \{y \in Y \mid P_i(y) = \varnothing\}$;

2) $X = \bigcup_{y_j \in \mathrm{co}L_j} \mathrm{int}\left(Q_j^{-1}(y_j) \cup H_j\right)$, *where* $H_j = \{x \in X \mid Q_j(x) = \varnothing\}$;

3) for all $x = (x_i)_{i\in I} \in X$ *and* $y = (y_j)_{j\in J} \in Y$, $x_i \notin \mathrm{co}P_i(y)$ *and* $y_j \notin \mathrm{co}Q_j(x)$.

Then, Γ *has a maximal element (an equilibrium point), that is there exists* $(\tilde{x}, \tilde{y}) \in X \times Y$, $\tilde{x} = (\tilde{x}_i)_{i\in I} \in X$ *and* $\tilde{y} = (\tilde{y}_j)_{j\in J} \in Y$ *such that* $P_i(\tilde{y}) = \varnothing$ *and* $Q_j(\tilde{x}) = \varnothing$, *for all* $i \in I$ *and* $j \in J$.

References

[1] Aliprantis, C. D. and Border, K. C.: *Infinite dimensional analysis, a Hitchhiker's guide*, Third ed. Springer Verlag Berlin Heidelberg, New York, 2006.

[2] Ansari, Q. H. and Yao, J. C.: *A fixed point theorem and its applications to the system of variational inequalities*, Bull. Austral. Math. Soc. **59** (1999), 433–442.

[3] Cristescu, R.: *Notions of Linear Functional Analysis* (in Romanian), Ed. Acad. Rom., Buc, 1998.

[4] Dăneț, R.–M. and Popescu, M.–V.: *Some applications of the fixed point theory in economics*, Creative Mathematics an Informatics, **17**(2008), No. 3, 392-398.

[5] Dăneț, R.–M. and Popescu, M.–V.: *Some fixed-point results for families of multimaps in the finite dimensional topological vector spaces setting and their applications*, Proceedings of 10-th Workshop of Department of Mathematics and Computer Science, Technical University of Civil Engineering, Bucharest, Romania 23 May, 2009, p. 32-36, ISSN 2067-3132.

[6] Dăneț, R.–M., Popovici, I.–M. and Voicu, F.: *Some applications of a collectively fixed–point theorem for multimaps*, Fixed point Theory **10** (2009), No.1, 99–109.

[7] Dăneț, R.–M., Popovici, I.–M. and Voicu, F.: *Various applications of some collectively fixed–point theorems for multimaps*, Fifth International Conference on applied mathematics, North University of Baia Mare, Department of Mathematics and Computer Science, 2006, September 21–24.

[8] Lin, L.–J. And Chen, H. I.: *Coincidence theorems for families of multimaps and their applications to equilibrium problems*, Abstract and Applied Analysis **5**(2003), 295–309.

COMPARISON OF METAHEURISTIC ALGORITHMS WITH APPLICATIONS IN PARAMETER ESTIMATION IN VADOSE ZONE

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Abstract: The Metaheuristic Algorithms (MA) have the advantage over the classical optimization methods that the can be used for the real-world optimization problems when the objective function does not need to be continuous or differentiable because they are not using the gradient or Hessian matrix. Another advantage is that MA's are stochastic optimization methods that generate and use random variables and they can search very large spaces of candidate solutions for finding a global optimum (the best solution), while classical optimization method is finding the solution in the neighborhood of a starting point so it is a local optimum. In order to analyze their similarities and differences we make an analysis of two Metaheuristics: Genetic Algorithms (GA's) and Ant Colony Optimization (ACO). GA's are using techniques inspired by natural evolution principle: the survival of the genetically fittest based on natural selection and genetic inheritance while ACO is based on idea that the foraging ants can find the shortest path between their nest and a food source by marking the paths they follow with pheromone. The MA's are modified and adapted for solving the Inverse Problem (IP) in order to estimate Mualem-van Genuchten unsaturated hydraulic soil parameters.
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Key words: Metaheuristic algorithms, Genetic algorithm, Ant colony optimization, Inverse Problem, modeling unsaturated zone, parameter estimation, Richards equation, Mualem-van Genuchten hydraulic soil parameters

1. Introduction

MA's are Evolutionary Nature-Inspired Stochastic Algorithms, mimicking Natural or Behavioral Phenomena. MA's original purpose was to solve combinatorial problems but because they have the ability to adapt they are now used to solve difficult engineering optimization problems. Metaheuristics ([5]) are (roughly) high-level strategies that combine lower-level techniques for exploration and exploitation of the search space.

MA's represent a new approach for modeling the flow in unsaturated zone. Modeling, simulating and predicting water content in soil is essential for development of the agriculture and for estimating the aquifer recharge. The soil hydraulic properties are given by Mualem - van Genuchten expressions. For the determination of the unknown soil hydraulic parameters we are using MA's because direct measurements of hydraulic properties of soils are tedious and time consuming.

2. Metaheuristic algorithms

MA's are global optimization algorithms that involve the evaluation of the function, usually at a random sample of points in the parameter space, followed by a subsequent manipulation of the sample using probabilistic rules. They guarantee asymptotic convergence to the global optimum. MA's are not tied to any special problem type and are general methods that can be altered to fit the specific problem.

We outline the different components and concepts of two popular MA's ([5]): *Genetic algorithm (GA)* and *Ant colony optimisation (ACO).*

Observations \ Metaheuristics	GA	ACO
Innovator	John Holland (1975)	Marco Dorigo (1991)
Source of Inspiration	Evolution Principle	The foraging behavior of ant colonies
Parameters	Crossover Probability;Mutation Probability;Population size	Pheromone evaporation parameter; Pheromone weighting parameter
Using Memory	Memory less	Using memory to store amount of pheromones
Initial Solution	Random	Random / Local search
Finding Neighbor Solutions	Random search: **mutation operator**	Random search: **random proportional rule**
Finding Local Optimum	Mutation operator	Accumulation pheromone on better solutions
Escaping from Local Optimum	Random search using **crossover operator**	Evaporation mechanism

Table1: Metaheuristics

3. The direct (forward) and the inverse problem for unsaturated flow

The forward problem (FP) refers to: *the mathematical model* and *the numerical model*

- *Mathematical model:*

Governing Equation (Darcy's law + water balance equation) having pressure head h as the dependent variable give *Richards equation* for 1-D vertical flow:

$$C(h)\frac{\partial h}{\partial t}=\frac{\partial}{\partial z}(K(h)(\frac{\partial h}{\partial t}-1));C(h)=\frac{d\theta}{dh}$$

where $h=h(z,t)$ is pressure head; $\theta=\theta(h(z,t))$ is volumetric water content; $K=K(h)$ is unsaturated hydraulic conductivity; $C=C(h)$ is soil water capacity

- *Numerical model*:

Finite difference method (FDM): approximates partial derivatives from 1-D Richards equation with finite differences weight representations. This scheme includes a combined version of classical discretization scheme, unconditionally stable for any time step size, such as Crank-Nicholson scheme for $b=1/2$ or fully implicit for $b=1$

$$C_i^{j+b}\frac{h_i^{j+1}-h_i^j}{\Delta t}=(1-b)\frac{K_{i+1/2}^j\cdot(\frac{h_{i+1}^j-h_i^j}{\Delta z}-1)-K_{i-1/2}^j\cdot(\frac{h_i^j-h_{i-1}^j}{\Delta z}-1)}{\Delta z}+$$
$$b\frac{K_{i+1/2}^{j+1}\cdot(\frac{h_{i+1}^{j+1}-h_i^{j+1}}{\Delta z}-1)-K_{i-1/2}^{j+1}\cdot(\frac{h_i^{j+1}-h_{i-1}^{j+1}}{\Delta z}-1)}{\Delta z}$$

where $h(z_i;t_j)=h_i^j$, $K_{i+1/2}^j=\frac{K_{i+1}^j+K_i^j}{2}$.

After discretization, linearization and simplification it is obtained a tridiagonal linear system that can be solved efficiently by the Thomas algorithm, a simplified form of Gaussian elimination.

The inverse problem estimate model parameters that yields a match between simulated and observed states.

- *Mualem-van Genuchten (MVG) model for unsaturated soil function* ([4]):

$$\theta(h) = \theta_r + \frac{\theta_s - \theta_r}{(1+|\alpha h|^n)^m}; K(h) = \frac{K_s}{(1+|\alpha h|^n)^{\frac{m}{2}}}[1-|\alpha h|^{n-1}(1+|\alpha h|^n)^{-m}]$$

where $\alpha, n, m = 1-\frac{1}{n}$ are empirical parameters of water retention curve shapes; K_s is saturated hydraulic conductivity; θ_s, θ_r are saturated (residual) water content.

- *Estimation of soil parameters is made with a metaheuristic algorithm*

For this purpose the *MA's* are modified to estimate MVG unsaturated hydraulic soil parameters:

Step1: Parameters selection that should be estimated $p = (K_s, n, \theta_r, \theta_s, \alpha)$

Step2: Define objective function by the difference between model predicted and measured values (v-pressure heads and/or water contents)

$$g(p) = \sum_{i=1}^{n}\sum_{j=1}^{nt} w_{iv}^j [(v_i^j(p))^{calc} - (v_i^j)^{obs}]^2$$

where σ is the standard deviation and k is the number of samples.

The inverse problem's aim is to find an optimal vector p^* that minimizes the objective function $g(p)$, so that $g(p^*) = \min g(p)$.

4. Applications of MA's in parameter estimation in unsaturated zone

We will review MA's applicability in vadose zone, with examples and a reference list. The classical way to determine the soil hydraulic functions is to directly measure in the laboratory using soil core samples, but this is a time consuming and expensive procedure. Because in a practical application in a study area is difficult to find the soil hydraulic properties, the study explore the potential of MA's to estimate inversely the soil hydraulic functions in the unsaturated zone.

4.1. Genetic algorithm implementation

GA's use concepts of "Natural Selection" and "Genetic Inheritance" (Darwin 1859) and follow the idea of survival of the fittest: better and better solutions evolve from previous generations until a near optimal solution is obtained. GA's work with a coding of the parameter set, not the parameters themselves: they may encode the variables binary or, for more complex problem, as a floating point, called a chromosome. The initial population is generated randomly. The new generations are produced through selection, crossover, mutation and acceptance.

In order to estimate MVG parameters that minimize the objective function, choosen function is Fitness ([3]):

$$Fitness(p) = \frac{N}{\sum_T \sum_i \sum_j |\theta^{calc} - \theta^{obs}|_{T_{ij}}}$$

where $\theta^{calc}, \theta^{obs}, ET_a^{calc}, ET_a^{obs}$ are soil water content (N observations); T, i, j are index of year, day, soil compartment. The population size is 20 and probability of crossover of 0.5. Considering the ability to match the measured and simulated values, the GA's performance was thought to be excellent, in all cases (12) higher than 91% and in most cases even higher than 99%.

4.2 Ant Colony Optimization implementation

ACO is based on the idea that the foraging ants can find the shortest path between their nest and a food source via pheromone trails ([2]). Each ant moves at random. Pheromone is deposited on path. More pheromone on path increases probability of that path being followed. The ants communicate by stigmergy: indirect communication, without central control, via interaction with environment by depositing pheromones on the ground, the key feature is Self-organization. Ants have an autocatalytic behavior: probability of choosing a branch of a path depends on the total amount of pheromone on the branch. Accumulation it is faster on the shorter path and it is proportional to the number of ants that have used the branch.

In order to find a parameter set (MVG parameters) that minimizes the objective function g:

$$g(p)=\sum_{i=1}^{n}\sum_{j=1}^{nt}w_{ih}^{j}[(h_i^j(p))^{calc}-(h_i^j)^{obs}]^2+\sum_{i=1}^{n}\sum_{j=1}^{nt}w_{i\theta}^{j}[(\theta_i^j(p))^{calc}-(\theta_i^j)^{obs}]^2$$

the classical ACO algorithm is modified and adapted for vadose zone using a sequential fitting ([1]). For the initial population we take a random sampling scheme $S\geq 0.1\prod_{i=1}^{q}m_i$, where the parameter domain $D=\{p_i|a_i\leq p_i\leq b_i\}$ is divided into a number of m_i subintervals of equal length and pathways $u=\overline{1,S}$. Based on the value of the objective function, there is placed a certain amount of trail (pheromone in the case of real ants) on each stratum visited along its pathway. The 'score' for each stratum is calculated similar to the transition probability from classical algorithm. Based on these scores remove from the ends of the parameter intervals the stratum with small or no scores.

Each parameter's interval was divided in 5 stratum, we have six unknown parameters $p=(K_{s1},n_1,,\alpha_1,K_{s2},n_2,\alpha_2)$ for two layers-field profile. The sampling scheme $S=0.2*5^6=3125$ agents. After six iterations, in two hours on a Pentium of 600MHz, the match between the measured and simulated values was bigger than 95%.

5. Conclusion

MA's are powerful tools in inverse modelling compared to other classical methods for parameter estimation based on local gradient optimization techniques. They are very promising for the inverse problem in the unsaturated zone.

Main difference between ACO and GA's is: ACO retains memory of entire colony instead of previous generation only in GA, convergence still being faster than in the case of GA's.
Future work will deal with: the extension to 2D or 3D model, estimate parameters in inverse problem for groundwater hydrology using more metaheuristic algorithms and compare these algorithms and their results, modeling the pollutants transport.

References

[1] Abbaspour, K.C., Schulin, R., van Genuchten, M. Th.: *Estimating unsaturated soil hydraulic parameters using ant colony optimization*, Advances in Water Resources, vol 24, 827-841, 2001

[2] Dorigo, M., Stützle, T.: *Ant colony optimization*. MIT Press, 2004

[3] Ines, A.V.M., Droogers P.: *Inverse modelling in estimating soil hydraulic functions: a Genetic Algorithm approach*, Hydrology and Earth System Sciences, 6(1), *49–65*, 2002

[4] van Genuchten, M. Th.: *A closed-form equation for predicting the hydraulic conductivity of unsaturated soils*, Soil Science Society of America Journal, vol. 4, 892–898, 1980

[5] Silberholz, J., B Golden, B.: *Comparison of Metaheuristics*, Chapter in the Handbook of Metaheuristics (International Series in Operations Research & Management Science), Springer; 2nd ed. Edition, 2010.

ON THE GYRICITY OF THE EARTH

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Abstract: In this paper some aspects of gyricity of the Earth are discussed. Gyricity is important only in a deformable, energy-dissipating like the rotationally unstable Earth. In the absence of external torques, gyricity appears in the form of internal torques. It is shown that gyric torques may be the primary mechanism for plate motion and mantle flow.
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Key words: Gyricity, Earth, Coriolis force, Liouville equation.

1. Introduction

Geodesy uses applied mathematics and satellite measurements to determine and represent the shape of the Earth, its gravity field, Earth tides, and tectonic motion in three-dimensional space through time. Geodesic measurements are very important to describe the kinematics and dynamics of the Earth as a deformable body. In this paper, the influence of the gyroscopic effect on the dynamics of the Earth is analyzed. Accordingly to Pan [1], any instantaneous motion in the Earth is subjected to the effects of giricity. The observed gyricity on the Earth is the Coriolis acceleration. The torsional deformations during the earthquakes and the rotational inertia may be attributed to the Coriolis effect [2]-[5]. Certain authors [6], [7] advance the idea that the observed vertical axis rotation of the crustal blocks may also due to gyricity. On the other hand, Pan [8] suggests that the long-term gyricity done by the secular Coriolis acceleration may be the explanation of the mechanism for plate motion and mantle flow. This paper investigates only the dynamic effects of the long-term gyricity of the Earth.

2. Liouville equation

Let us start with the motion of the gyroscope with respect to the fixed frame of reference can be analyzed by composing the proper rotation of it about the symmetry axis Ox_3, with the constant angular velocity $\overline{\omega}$ (the relative motion), with the motion of precession about the fixed axis Ox_3', with the constant angular velocity ω' (the motion of transportation); we assume that the angle θ between the two axes is constant [9], [10] (Fig.1). An element d*m* of the gyroscope, situated at the point *P*, is subjected to the centrifugal forces (forces of transportation) due to the proper rotation and to the motion of precession, as well as to the Coriolis force.

The Coriolis force $\mathrm{d}F_C = 2v_r \times \omega' \mathrm{d}m$ is given by

$$F_C = 2\int_M \left(\overline{\omega} \times r\right) \times \omega' \mathrm{d}m = -2\omega' \times \left(\overline{\omega} \times \int_M r \mathrm{d}m\right), \tag{1}$$

where $v_r = \overline{\omega} \times r$ is the relative velocity. The resultant moment becomes

$$M_{OC} = 2\int_M \overrightarrow{OP} \times \left[\left(\overline{\omega} \times r \right) \times \omega' \right] \mathrm{d}m . \tag{2}$$

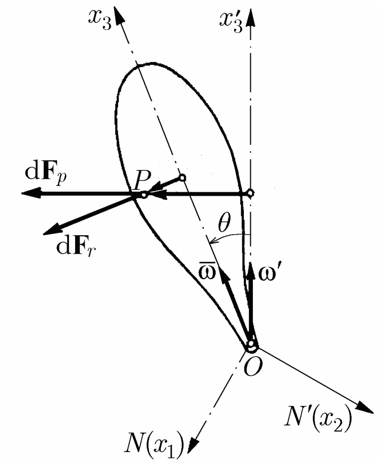

Fig.1. Inertial forces in the motion of regular precession of the gyroscope [9].

The Earth is treated as a rotating system of particles moving among themselves under the actions of central forces according to the law of conservation of angular momentum

By taking account of (1) and (2), the generalized Euler equation lead to the Liouville equation of gyricity which states that the rate of change of the Earth's total angular momentum (internal torques system of the Earth) is zero [1]

$$\dot{I}\omega + I\dot{\omega} + \dot{h} + \omega \times I\omega + \omega \times h = 0 , \tag{3}$$

where I is the inertia tensor of the Earth, ω is the rotation velocity of the Earth, and h is the relative angular momentum due to the mass redistribution in the Earth. Eq. (3) is written in the Earth'axes of instantaneous moments of inertia (x, y, z), in the absence of external torques.

The internal torques system of the Earth represents all the central forces in the system in the absence of external torques [11]. In order to understand the physical meaning of Eq. (3) we shortly explain each term, as follows.

The term $\dot{I}\omega$ characterizes the change in the moment of inertia in response to the angular momentum perturbation; the term $I\dot{\omega}$ is the rotational torque due to the rotational acceleration; the third term $\dot{h}$ is the rate of change of the relative angular momentum; the term $\omega \times I\omega$ arises from the separation of the angular momentum axis of the Earth from the instantaneous axis of rotation; the last term $\omega \times h$ is done by the motion or mass redistribution in the Earth, by rotation.

2. Analysis of terms $\omega \times I\omega$ and $\omega \times h$

By analyzing the terms of Eq. (3), the conclusion is that only the terms $\omega \times I\omega$ and $\omega \times h$ are typical gyric torques, least studied in the literature.

Fig.2 displays the variation of the magnitude of $\omega \times I\omega$ with respect to ω / ω_0. This term $\omega \times I\omega$ is the largest internal torque in the Earth. A comparable magnitude is done by $\dot{h}$, but $\dot{h}$ is transient, while $\omega \times I\omega$ is quasi-periodic. The magnitude of $\omega \times I\omega$ depends on the axial symmetry and triaxialily of the Earth. The interpretation of this gyric torque through the point of view of the conservation of the angular momentum is still an open problem [1].

Fig.3 presents the variation of magnitude of $\omega \times h$ with respect to ω / ω_0.

Pan [8] has analyzed the motion of a tectonic plate and estimates the order of the term $\omega \times h$ to be about 10^{22} erg., and it is toward the center of the Earth (the magnitude of the Coriolis and secular rotational torques have the order of 10^{23} erg.

This estimation is very lower if we compare to the Earth's total rotational energy of about 2.16×10^{36} erg.

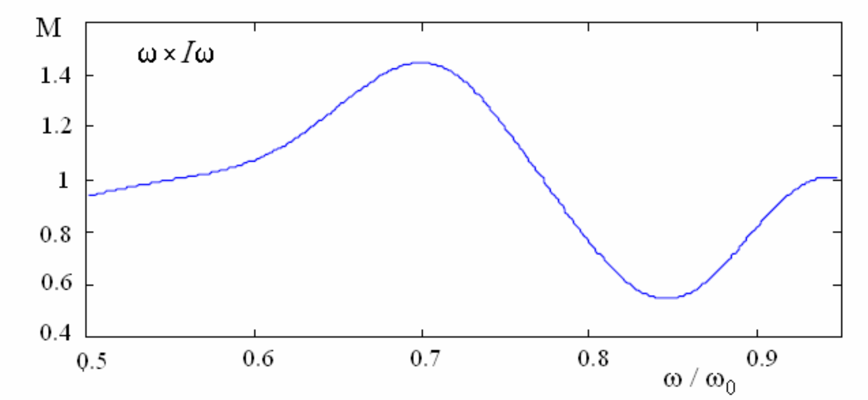

Fig.2. Variation of the magnitude of $\omega\times I\omega$ with respect to ω/ω_0.

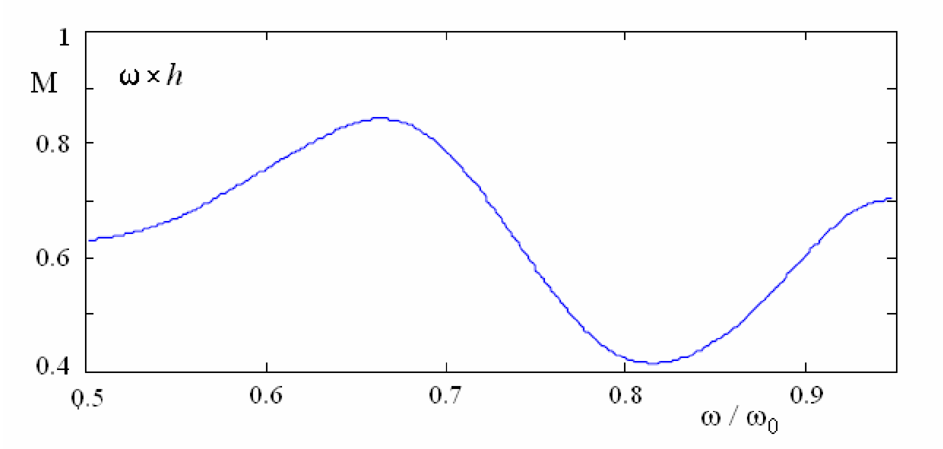

Fig.3. Variation of the magnitude of $\omega\times h$ with respect to ω/ω_0.

The main idea of the article is that the gyric torques are not balanced in Eq. (3). This result was previously obtained and predicted by Pan [1]. The analysis shows that the term $\omega\times I\omega$ is several orders of magnitude greater than any known internal torque in the Earth.
Fig.4 shows the variation of the magnitude of the gyric torques $\omega\times I\omega$ and $\omega\times h$ with respect to angular momentum.
The question is who can balance this gyric torque in the Earth? If the law of conservation of angular momentum is valid, the gyric torque $\omega\times I\omega$ must be balanced in some way into the Earth, we do not know yet how**.** Otherwise, the rotation velocity of the Earth being too high, it is secularly accelerating.
In spite of the fact that only two terms were analyzed in this paper from point of view of the magnitude order, we believe that all motions in the Earth, instantaneous or secular, are influenced by the gyricity.
Finally, we can conclude that dynamic effects of long-term gyricity in the Earth are still unknown. The elasticity and viscoplasticity and other phenomena (fluctuation of the atmosphere, currents in the oceans, flows in the mantle, tectonic motions and mantle convection and so on) in the Earth have to be studied to understand how the gyricity affects the Earth rotation motion.

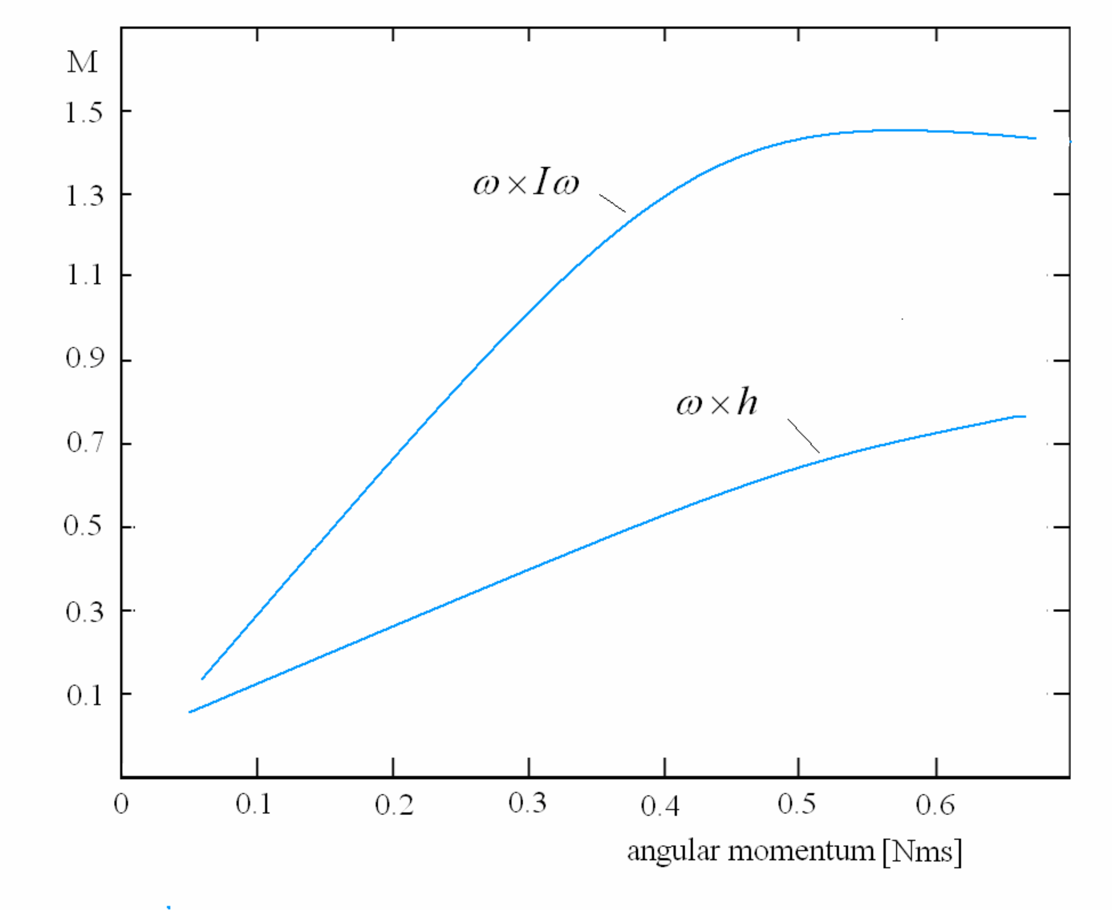

Fig.4. Variation of the magnitude of the gyric torques with respect to angular momentum.

References

[1] Pan, C.: The gyricity of the earth**,** *SLAC-PUB*-6223, 1993.

[2] Rance, H.: Major lineaments and torsional deformation of the Earth, *J. Geophys. Res.*, **72** (1968), 2213-2217.

[3] Rance, H.: Lineaments and torsional deformation of the Earth: Indian Ocean, *J. Geophys. Res.*, **74**(1969), 3271-3272.

[4] Howell, B.F.: Corriolis force and the new global tectonics, *J. Geophys. Res*. **75** (1970), 2769-2772.

[5] Kane, M.F.: Rotational inertia of continents: A proposed link between polar wandering and plate tectonics, *Science* **175** (1972), 1355-1357.

[6] Hudnut, K.W.: Geodetic evidence for the rotation of crustal blocks EOS, *Trans. Am. Geophys*. **72**(1991).

[7] Argus, W. and Asaro, F.: An extraterrestrial impact, *Scien. Am.*, **263**(1990), 4, 74-84.

[8] Pan, C., *Polar instability, plate motion, and geodynamics of the mantle*, J. Phys. Earth, **33** (1985), 411-434.

[9] Teodorescu, P.P.: *Mechanical Systems, Classical Models*, **2**, Springer, 2007.

[10] Donescu, Şt. and Munteanu, L.: *The effect of damping on the stability of dynamical systems*, Topics in Applied Mechanics, vol.2, 85-117, Publishing House of Romanian Academy, Bucharest (eds. V.Chiroiu, T.Sireteanu), 2004.

[11] Becker, R.A.: *Introduction to theoretical mechanics*, McGraw-Hill, New York, 1954.

A CONSTRUCTION OF A SEQUENCE OF ADMISSIBLE PARTITIONS FOR INCOMPLETE STATE-INFORMATION SEMI-MARKOV DECISION PROCESSES

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Abstract. It is well-known that in practice it is rather difficult to calculate the minimal value function and optimal policies for semi-Markov decision processes. Thus, in general, approximation procedures are investigated. The concept of admissibility for the decomposition of the state space of a Markov decision processes was introduced by Hahnewald-Busch and Nollau (1978) and generalized for incomplete state-information semi-Markov decision processes by Dragut M. (1990). An approximation procedure based on the concept of admissibility for a numerical solution of a finite horizon incomplete state-information semi-Markov decision process with countable state space, finite action and signal spaces was introduced by Dragut M. (2001).
In this paper, the construction of a sequence of admissible partitions is presented. We prove that there exists a sequence of admissible partitions for our decision process and also there exists a measure of cohesion which in an index on the set of admissible partitions. An algorithm for the construction of a sequence of admissible partitions is also presented. This algoritm is a clustering algoritm similar to the one presented in Dragut A.B. and Nichitiu (2004).
Mathematics Subject Classification (2000): 60J99
Key words: semi-Markov decision processes

1. INTRODUCTION

Semi-Markov processes have become the mathematical foundation for much current work in reinforcement learning, decision theoretic planning, information retrieval, speech recognition, active vision, robot navigation,and in general in modeling the availability, reliability and maintainability of a complex system. A number of successful applications of reinforcement learning to large scale problems such as robot navigation, board games, dispatch problems is due to the development of hierarchical decision making models, where the semi-Markov processes are the preferred language for modeling temporally extended actions. However, the semi-Markov approach with or without complete state information suffers even in the discrete case from the fact that for a large number of states it is impractical to calculate the optimal value function and optimal policies. There are numerous applications in which the number of states can be assumed to be a countable state space and general approximation procedures have to be investigated. With few exceptions, the existing main approaches for "removing irrelevant detail" (i.e. , state aggregation/ decomposition and temporal abstraction) are mainly developed for much simpler case of Markov processes. State decomposition methods typically represent states as collections of factored variables or simplify them by eliminating "useless" states. Temporal abstraction mechanism, for example in reinforcement learning encapsulate lower- level observation or action sequences into a single unit at a more abstract level. A unified algebraic treatment of abstraction of Markov decision processes that covers both spatial and temporal abstraction can be found for the first time in Hahnewald- Busch and Nollau, [4]. In this paper we use the clustering concepts and techniques for large data sets by adequate grouping of objects.

2. THE EXISTENCE OF A SEQUENCE OF ADMISSIBLE PARTITIONS

Definition 1. Let $\{\pi^a\}_{a\in A}$ a probability measures family on $P(N^*)$, with the property $\pi^a(\Delta) > 0$ for all $\Delta \in P(N^*)$ where $P(N^*)$ is the set of all subsets of N^*. Let $\{\sigma_k\}_{k\in N}$ be a sequence of natural numbers, $\{\varepsilon_k\}_{k\in N}$ be a strictly decreasing sequence of non-negative numbers which tends to zero and $\{\Xi^{(k)}\}_{k\in N}$ be a complete system of σ_k disjoint subsets of N^*. The sequence $\{\Xi^{(k)}\}_{k\in N}$ is called admissible with respect to the sequence $\{\varepsilon_k\}_{k\in N}$ if and only if $\pi^a(\Delta)-\pi^a(\{i\}) \leq \varepsilon_k$ for every $i\in N^*$, $a\in A$ and $\Delta\in \Xi^{(k)}$. We call an admissible tuple, the tuple $\{\{\Xi^{(k)}\}_{k\in N}, \{\sigma_k\}_{k\in N}, \{\varepsilon_k\}_{k\in N}\}$.

Definition 2. Let Æ be a subset of $P(N^*)$. A function τ : Æ $\rightarrow [0,\infty)$ is called a measure of cohesion on Æ if $\tau(\Delta) \geq 0$ for all $\Delta\in$ Æ and $\tau(\Delta)=0$ if and only if $|\Delta|=1$ where $|\Delta|$ is the notation for the cardinal of the set Δ.

Definition 3. Let Æ be a subset of $P(N^*)$. A function τ : Æ $\rightarrow [0,\infty)$ is called an index on Æ if $\tau(\Delta) \geq 0$ for all $\Delta\in$ Æ and $\tau(\Delta)\leq \tau(\Gamma)$ for all Δ and $\Gamma \in$ Æ and $\Delta\subset\Gamma$.

Proposition 1. *There exists a measure of cohesion which is an index on Æ.*

Proof. If the set A in Definition 1 has only one element ,i.e. $\{\pi^a\}$ $a\in A=\{\pi\}$ where π is a probability measure on $P(N^*)$, $\pi(\{i\})>0$ for any $i\in N^*$, we define

$$\tau(\Delta)= \pi(\Delta) - \max_{i\in\Delta} \pi(\{i\})$$

for all $\Delta\in$ Æ . From the definition of $\pi(\Delta)= \sum_{i\in\Delta} \pi(\{i\})$ and from the property $\pi(\{i\}) >0$ for any $i\in N^*$ we have $\tau(\Delta) \geq 0$ for all $\Delta\in$ Æ and $\tau(\Delta) = 0$ if $|\Delta|=1$. In order to prove the property $\tau(\Delta)\leq \tau(\Gamma)$ for all Δ and $\Gamma \in$ Æ and $\Delta\subset\Gamma$, let us define $F=\Delta - \Gamma$. We observe that

$$\tau(\Gamma)= \tau(\Delta UF)= \pi(\Delta UF)- \max_{i\in\Delta uF} \pi(\{i\})$$
$$\geq \pi(\Delta)-\max_{i\in\Delta} \pi(\{i\})+\pi(F)- \max_{i\in F} \pi(\{i\})$$

which conclude the proof of this case. ◊

Now let $\{\pi^a\}$ $a\in A$ be a family of probability measures on $P(N^*)$ with the property $\pi^a(\{i\}) >0$ for any $i\in N^*$ and Æ $\subset P(N^*)$. We define

$$\tau(\Delta) = \max_{a\in A} [\pi^a(\Delta) - \max_{i\in\Delta} \pi^a(\{i\})].$$

Using the same arguments as in the case $\{\pi^a\}$ $a\in A=\{\pi\}$ we have that $\tau(\Delta)$ is a measure of cohesion which is an index on Æ.

Definition 4. We call dissimilarity index on a set N a function $d : N\text{x}N \rightarrow R$ with the following properties:

(1) $d(x,y) \geq 0$ for all $x,y \in N$ and

(2) $d(x,y)=0$ if and only if $x=y$.

Remark. One can construct a dissimilarity index on Æ $\subset P(N^*)$ based on the cohesion measure from Proposition 1: $\tau(\{x\},\{y\})=\min_\Delta\{ \tau(\Delta) | x,y \in\Delta\}$ for all $\{x\},\{y\} \in$ Æ.

Proposition 2. Let $\{\varepsilon_k\}_{k\in N}$ be a strictly decreasing sequence of non-negative numbers which tends to zero and let $\{\pi^a\}_{a\in A}$ be a finite family of probability measures on $P(N^*)$ having the property $\pi^a(\{i\})> 0$ for any $i\in N^*$ and $a \in A$. Then there exists a sequence $\{\Xi^{(k)}\}_{k\in N}$ admissible with respect to the sequence $\{\varepsilon_k\}_{k\in N}$.

Proof. If $\{\pi^a\}_{a\in A}= \{\pi\}$ for every $k \in N^*$ we define the index $i_k \in N^*$ such that $i_k= \min_{j\in N}$

$\{j \mid \sum^{j}_{i=1} \pi(\{i\}) > I-\varepsilon_k\}$. Let us define $A^k = \{ i \in N^* \mid i \geq i_k\}$ and $B^k = \{ i \in N^* \mid i < i_k\}$. The sequence $\{\Xi^{(k)}\}_k = \{\{i\}, i < i_k, A_k\}_k$ is admissible with respect to the sequence $\{\varepsilon_k\}_{k \in N}$.
The statement holds true also in the case of a finite family of probability which contains more than one element. With this purpose one has to define $\tau(\Delta) = \max_{a \in A} [\pi^a(\Delta) - \max_{i \in \Delta} \pi^a(\{i\})]$ and

$$i_k = \max_{a \in A} \min_{j \in N} \{j \mid \sum^{j}_{i=1} \pi(\{i\}) > I-\varepsilon_k\}. \Diamond$$

3. THE CONSTRUCTION OF A SEQUENCE OF ADMISSIBLE PARTITIONS

We further provide an algorithm to construct a sequence $\{\Xi^{(k)}\}_{k \in N}$ admissible with respect to the sequence $\{\varepsilon_k\}_{k \in N}$. But before , let us clear some details. We have to start with an initial discrete partition P^0 of B_k, which is a finite set. In order to obtain the partition in linear time (i.e. when B^k has M elements, this process takes M time units) without random access to the whole set B^k , we will use a clustering algorithm similar to the one presented in Dragut A.B. and Nichitiu (2004), using as a measure of cohesion the measure τ constructed in Proposition 1.

The algorithm reads one by one the $M= \mid B^k \mid$ elements, and, in constant time (depending only on K) , places them in one of the K classes, where $K \leq M$ is fixed beforehand. Smaller values for K determine smaller dimensions for the matrices of the complete state information process associated to the incomplete state information semi-Markov decision process , however , at the cost of some loss of precision. If none of the K classes seems close enough to the current point, then the closest two classes are merged and a new class is created with this point. This way, the algorithm builds a classification tree T which holds in its root the links to the K classes of our classification. At the end of the M-step process, the K classes can be easily recovered into P^0 by recursively descending in the tree T and collecting all points assigned to the class through insertions and successive merges.

Step 0.0.0 $k \leftarrow 1$.

0.1 Initialize B^k, and $K \leq M = \mid B^k \mid$. Also $T \leftarrow$ the tree having as root the set of the first K elements of B^k and as K sons the singletons with these elements. Denote by V_j with $1 \leq j \leq M$ each of these elements.Also $i \leftarrow K+1$.
0.2 while $i \leq M$ do
0.3 $X \leftarrow$ ith element of B^k; $\mu \leftarrow \min_{1 \leq j \leq K}\{\tau(X,V_j)\}$; $m \leftarrow \arg\min_{1 \leq j \leq K}\{\tau(X,V_j)\}$
0.4 $\eta \leftarrow \min_{1 \leq r,s \leq K} \tau\{(V_r,V_s)\}$; $(p,q) \leftarrow \arg\min_{1 \leq r,s \leq K} \tau\{(V_r,V_s)\}$
0.5 $TX \leftarrow [\{X\},\square]$; $H \leftarrow T$
0.6 if $\mu \leq \eta$ then
0.7 $TS \leftarrow [H.son_m .root\ U\{X\},[H.son_m, TX]]$; $n \leftarrow [TS.root]$
0.8 $T \leftarrow [H.root\ U\{X\}, [H.son_1, \ldots, H.son_{m-1},TS, H.son_{m+1},\ldots, H.son_K]]$
0.9 $V_m \leftarrow (V_m(m-1)+X):n$
0.10 else
0.11 $TP \leftarrow [H.son_p .root\ U\ H.son_q .root,[H.son_p , H.son_q]]$
0.12 $a \leftarrow [H.son_p .root]$; $b \leftarrow [H.son_q .root]$
0.13 $T \leftarrow [H.root\ U\{X\}, [H.son_1, \ldots, H.son_{p-1},TP, H.son_{p+1},\ldots,H.son_{q-1},TX, H.son_{q+1}, \ldots, H.son_K]]$
0.14 $V_p \leftarrow (V_p.a+V_q.b):(a+b)$; $Vq \leftarrow X$
0.15 end if
0.16 $i \leftarrow i+1$
0.17 end while
0.18 $P^0 \leftarrow$ partition from classification tree T.

0.19

Now P^0 is a discrete partition of B^k. Let C^0= $P(B^k)\backslash P^0$, γ_0=0.

Step s≥1.

1 Determine $\gamma_s=\min\{\tau(\Delta) \mid \Delta \in C_{s-1}\}$

2 If $\gamma_s > \varepsilon_k$ then go to 13

3 $D_s=\{\Delta \in C_{s-1} \mid \tau(\Delta)=\gamma_s\}$

4 Choose $\Delta \in D_s$ {with minimal index}, $D=D_s-\{\Delta\}$, Es=Δ

5 If D=Φ then go to 11,

6 Choose $\Gamma \in D$ {with minimal index}. $F=E_s$.

7 If $\Phi=\phi$ then let $E_s=E_s U \{\Gamma\}$, $D=D\backslash\{\Gamma\}$ and go to 5.

8 Choose $\Phi \in F$ {with minimal index]

9 If $\Gamma \cap \Phi=\phi$ then let $F=F\backslash\{\Phi\}$ and go to 7.

10 If $\Gamma \cap \Phi=\phi$ then let $E_s=E_s\backslash\{\Phi\}$, $F=F\backslash\{\Phi\}$ and then go to 7.

If $\Gamma \cap \Phi \neq \phi$ then let $D=D\backslash\Gamma$ and go to 5.

11 $P^s=E_s U\{\Delta \in P^{s-1} \mid \Delta \cap \Gamma=\phi$ for any $\Gamma \in E_s\}$

12 $C_s=\{\Delta \in C_{s-1} \mid \Delta \cap \Gamma \in \{\phi, \Gamma\}, \Delta \neq \Gamma$ for any $\Gamma \in P^s\}$.

13 If $C_s=\phi$, $L=\{P^0,...,P^s\}$. STOP. If $C_s \neq \phi$ then we execute another cycle (we start in 0.1 with k=k+1).

We notice that from a practical point of view, since the sets B^k are included in one another as k increases, for all the main cycles for which we maintain the same desired K number of clusters, the algorithm can be applied incrementally: that is, the classification tree from the last step for B^{k-1} can be used as input when processing the next new element (if any) of B^k.

Remark. However, in the algorithm one can obtain as centroids of the clusters rational numbers. Thus, the dissimilarity index will be extended to rational numbers in the following way:

□τ ({a/b},{c/d})=τ({ │a/b│ },{ │c/d│ })+τ({□a +1},{ □c +1})+ τ({□b +1},{ □d +1})

where a,b,c,d, □a , □b, □c, □d are natural numbers, □a/ □b =a/b-[a/b] and □c/ □d =c/d-[c/d] , and 0≤ □a ≤ □b, 0≤ □c≤ □d, the greatest common divisor of □a and □b is 1 and also the greatest common divisor of □c and □d is 1. By convention, if a/b is a natural number, then □a =0 and □b=1. This way □τ coincides with τ for natural numbers and is indeed an extention of it.

References

[1] Dragut, A.B. and Nichitiu, C.M. : A monotonic on-line linear algorithm for hierarchical agglomerative classification, *Information Tehnology* & *Management Journal* **31 (** 2004), 115-124.

[2] Dragut, M. : A sufficient statistic for semi-Markov decision processes with incomplete state-information, *Studii si Cercetari Matematice*, **42** (1990), 9-1.

[3] Dragut, M. : An approximation procedure for finite horizon semi-Markov decision processes with incomplete state information, *Mathematical Reports* **3 (53). No 3** (2001), 234-241.

[4] Hahnewald- Busch, A. and Nollau,V.: An approximation procedure for stochastic dynamic programmind in countable state-space, *Math. Operationforsch. Statist. Ser. Optimization*, **9** (1978) , 10 9-117.

FLOWS ON WARPED PRODUCTS WITH TIME SCALES FACTOR: A GEOMETRICAL STUDY

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Abstract: The present paper presents an extension of space-time models by considering the time scale approach. Starting from the Friedmann Robertson Walker cosmological model [5], we show that if the scale function is defined on a time scale in the sense of [3], [1], instead of being defined on an interval, then the model can encompass unions of intervals and discrete points. Thus we obtain a generalized Friedmann model which can account for periods of discontinuity in the information, alternating with continuous periods.
Mathematics Subject Classification (2000): 34K34, 53Z99, 83F05
Key words: semi-Riemannian manifolds, warped products, Robertson Walker space-times, Friedmann Robertson Walker cosmological model, time scales calculus

1. Introduction

In the search for an adequate representation of our universe, mathematical models for the study of the associated spacetime rely on the solutions to Einstein's field equations [6], $R_{ab} - \frac{1}{2} R g_{ab} + \Lambda g_{ab} = k_0 T_{ab}$. Among them, Robertson Walker (RW)-spacetimes [5] and the associated flow provide a geometrical model (the Friedmann Robertson Walker cosmological model) which can be adapted, under supplementary assumptions, to characterize our universe. In order to obtain a model which can accommodate periods for which the time evolution is completely understood as well as *gaps* in the information, the time scale approach [1],[3], [4] offers the necessary frame.

We recall first some definitions and results characterizing the Friedmann cosmological model as well as those from the theory of time scales which are used in this paper. Then we introduce the notion of a time scale warped product (TSWP) and specialize the time scale to suit our formerly mentioned mixture of continuous and discontinuous periods. We consider a generalized Friedmann cosmological model, for which the scale function is completely delta differentiable on $\mathbf{T} - \{0\}$, so 0 is a physical singularity.

2. Warped products and cosmological models

The FRW models are cosmological models in which the geometrical structure is that of a warped product. Let S be a connected three-dimensional Riemannian manifold of constant curvature $k \in \{-1,0,1\}$, let $I \subseteq \mathbf{R}_1^1$ be an open interval, where $\mathbf{R}_1^1$ is the semi-Euclidean space $\mathbf{R}^1$ endowed with the line element $-dt^2$. Let $f : I \to \mathbf{R}$ be a smooth function, with $f(x) > 0$ for any $x \in I$. A *Robertson Walker spacetime* (RW) is the warped product

$$M(k,f)=I\times_f S$$

the product manifold $I\times S$ is endowed with the line element $-dt^2+f^2(t)d\sigma^2$, $d\sigma^2$ is the lift to $I\times S$ of the line element of S and $\pi: I\times S\to I$, $\sigma: I\times S\to S$ are the usual projections.
The following results from [5] will be used in the sequel.

Proposition. For a Robertson-Walker spacetime $M(k,f)$ with flow vector field $U=\partial_t$:

i) the Ricci curvature is given by

$$Ric(U,U)=-3\frac{f''}{f},\quad Ric(U,X)=0,\ Ric(X,Y)=[2(\frac{f'}{f})^2+2\frac{k}{f^2}+\frac{f''}{f}]<X,Y>,\text{ if } X,Y\perp U$$

ii) the scalar curvature is $S=6[(\frac{f'}{f})^2+\frac{k}{f^2}+\frac{f''}{f}]$

iii) (U,ρ,μ) is a perfect fluid with energy density ρ and pressure μ given by

$$\frac{8\pi\rho}{3}=(\frac{f'}{f})^2+\frac{k}{f^2},\quad -8\pi\mu=2\frac{f''}{f}+(\frac{f'}{f})^2+\frac{k}{f^2}$$

A *Friedmann Robertson Walker* (FRW) cosmological model consists of a Robertson-Walker spacetime such that the following two conditions are satisfied: i) the galactic fluid is a dust ($\mu=0$ and $\rho>0$); ii) there exists $t_0\in I$ such that $H(t_0)=\frac{f'}{f}(t_0)>0$.

Proposition. Let $M(k,f)=I\times_f S$. If $M(k,f)$ is a FRW cosmological model, then:

i) $\rho f^3=M$, a positive constant; ii) Friedmann's equation is valid $f'^2+k=\frac{8\pi M}{3f}>0$.

Corollary. Let $M(k,f)=I\times_f S$ be a FRW cosmological model. Then the solution to Friedmann's equation for the Euclidean space $\mathbf{R}^3$ is given by $f=Ct^{2/3}$ and $C^3=\frac{9A}{4}$

3. A time scale for a gravitational model

Along with the basic definitions and results recalled from [3], we shall introduce a particular time scale which we shall use to describe a cosmological model with missing information. .
A *time scale* $\mathbf{T}$ is an arbitrary nonempty closed subset of $\mathbf{R}$.

Let $a,b>0$, $n\in\mathbf{N}, n\geq 3$ and define $h=\frac{b}{n}$. We introduce the following time scale

$$\mathbf{T}_{csm}=\mathbf{T}_{a,b,h}=\bigcup_{k=-\infty}^{\infty}[k(a+b),k(a+b)+a]\cup\bigcup_{k=-\infty}^{\infty}\{k(a+b)+a+jh,1\leq j\leq n-1\}=\mathbf{T}_{a,b}\cup\mathbf{A}_h$$

The following real valued functions characterize a time scale [3]:
the forward jump operator $\sigma(t)=\inf\{s\in\mathbf{T}\mid s>t\}$, the backward jump operator $\rho(t)=\sup\{s\in\mathbf{T}\mid s<t\}$, the graininess function $\mu(t)=\sigma(t)-t$

Corresponding to different possible values of these functions, the subsets of a time scale are:

i) the set of left-dense and right-dense points $T_{dd}=\{t\in T\mid\sigma(t)=t=\rho(t)\}$

ii) the set of left-scattered and right-dense points $T_{sd}=\{t\in T\mid\rho(t)<t=\sigma(t)\}$

iii) the set of left-dense and right-scattered points $T_{ds}=\{t\in T\mid\rho(t)=t<\sigma(t)\}$

iv) the set of left-scattered and right-scattered points $T_{ss}=\{t\in T\mid\rho(t)<t<\sigma(t)\}$

The following subset is introduced in [3]: $\mathbf{T}^K=\begin{cases}\mathbf{T}\setminus(\rho(\sup\mathbf{T}),\sup\mathbf{T}],\sup\mathbf{T}<\infty\\ \mathbf{T},\sup\mathbf{T}=\infty\end{cases}$

Lemma. For the time scale $\mathbf{T}_{csm}$ the following characterizations are valid: $\mathbf{T}_{csm}^K=\mathbf{T}_{csm}$,

$\mathbf{T}_{dd} = \bigcup_{k=-\infty}^{\infty} (k(a+b), k(a+b)+a)$, $\mathbf{T}_{sd} = \bigcup_{k=-\infty}^{\infty} \{k(a+b)\}$, $\mathbf{T}_{ds} = \bigcup_{k=-\infty}^{\infty} \{k(a+b)+a\}$, $\mathbf{T}_{ss} = \mathbf{A}_h$,

Proof. Standard calculations show that for $\mathbf{T}_{csm}$

$$\sigma(t) = \begin{cases} t, t \in \mathbf{T}_{dd} \cup \mathbf{T}_{sd} \\ t+h, t \in \mathbf{T}_{ds} \cup \mathbf{A}_h \end{cases} \text{ and } \rho(t) = \begin{cases} t, t \in \mathbf{T}_{dd} \cup \mathbf{T}_{ds} \\ t-h, t \in \mathbf{T}_{sd} \cup \mathbf{A}_h \end{cases}$$

The function $f : \mathbf{T} \to \mathbf{R}$ is *delta differentiable* on $\mathbf{T}^K$ if for every $t_0 \in \mathbf{T}^K$ [3] there exists a number A such that for every $\varepsilon > 0$ there exists a δ neighborhood $U_\delta(t_0)$ with

$$\left| f(\sigma(t_0)) - f(t) - A(\sigma(t_0) - t) \right| \le \varepsilon \left| \sigma(t_0) - t \right|$$

for all $t \in U_\delta(t_0)$. The constant A is denoted by $f^\Delta(t_0)$. The function $f : \mathbf{T} \to \mathbf{R}$ is *completely delta differentiable* on $\mathbf{T}^K$ if for every $t_0 \in \mathbf{T}^K$ [2] there exists a number A, a δ neighborhood $U_\delta(t_0)$ and functions $\alpha = \alpha(t_0, t), \beta = \beta(t_0, t) : U_\delta(t_0) \to \mathbf{R}$ such that

$$f(t_0) - f(t) = A(t_0 - t) + \alpha(t_0 - t), \; f(\sigma(t_0)) - f(t) = A(\sigma(t_0) - t) + \beta(\sigma(t_0) - t)$$

for all $t \in U_\delta(t_0)$ and $\alpha(t_0, t_0) = 0, \beta(t_0, t_0) = 0, \lim_{t \to t_0} \alpha(t_0, t) = 0, \lim_{t \to t_0} \beta(t_0, t) = 0$.

For a completely delta differentiable function $f : \mathbf{T} \to \mathbf{R}$, define

$$(\Delta f)(t) = f^\Delta(t) = \begin{cases} \dfrac{df}{dt}(t), t \in \mathbf{T}_{dd} \cup \mathbf{T}_{sd} \\ \dfrac{f(\sigma(t)) - f(t)}{\mu(t)}, t \in \mathbf{T}_{ds} \cup \mathbf{T}_{ss} \end{cases}$$

The function $f : \mathbf{T} \to \mathbf{R}$ has a second order delta derivative on $\mathbf{T}^{K^2} = (\mathbf{T}^K)^K$ if [3] f^Δ is differentiable on $\mathbf{T}^{K^2}$ with derivative $f^{\Delta\Delta} = (f^\Delta)^\Delta : \mathbf{T}^{K^2} \to \mathbf{R}$.

Proposition. Let $f : \mathbf{T}_{csm} \to \mathbf{R}$ be given by $f(t) = t^{\frac{2}{3}}$. Then f is completely delta differentiable at every $t_0 \in \mathbf{T}_{csm} - \{0\}$. Moreover, $f(t) > 0$ for every $t \in \mathbf{T}_{csm} - \{0\}$.

Proof. The second statement follows from the definition of f. For the first statement, we prove it on each subset of $\mathbf{T}_{csm}$:

i) if $t_0 \in \mathbf{T}_{dd} \cup \mathbf{T}_{sd}$, then $A = f'(t_0) = \frac{2}{3} t_0^{-\frac{1}{3}}$ and $\alpha(t_0, t) = \beta(t_0, t) = \dfrac{t_0^{\frac{1}{3}} + t^{\frac{1}{3}}}{t_0^{\frac{2}{3}} + t_0^{\frac{1}{3}} t^{\frac{1}{3}} + t^{\frac{2}{3}}} - \frac{2}{3} t_0^{-\frac{1}{3}}$

ii) if $t_0 \in \mathbf{T}_{ds}$, $A = \dfrac{(t_0 + h)^{\frac{2}{3}} - t_0^{\frac{2}{3}}}{h}$, $\alpha(t_0, t)$ as in i),but $\beta(t_0, t) = \dfrac{(t_0 + h)^{\frac{1}{3}} + t_0^{\frac{1}{3}}}{(t_0 + h)^{\frac{2}{3}} + (t_0 + h)^{\frac{1}{3}} t_0^{\frac{1}{3}} + t_0^{\frac{2}{3}}} - \frac{2}{3} t_0^{-\frac{1}{3}}$

iii) if $t_0 \in \mathbf{A}_h$, then $A = \dfrac{(t_0 + h)^{\frac{2}{3}} - t_0^{\frac{2}{3}}}{h}$ and $\beta(t_0, t) = 0$.

We recall from [2] that a continuous function defined on time scale $\gamma : \mathbf{T} \to \mathbf{R}$ has a *delta tangent line* L_0 at the point $P_0 = P(t_0, \gamma(t_0))$ where $t_0 \in \mathbf{T}^K$ if L_0 passes through the points P_0, $P_0^\sigma = P^\sigma(\sigma(t_0), \gamma(\sigma(t_0)))$ and if $t_0 \notin \mathbf{T}_{ss}$ then $\lim_{P \to P_0, P \ne P_0} \dfrac{d(P, L_0)}{d(P, P_0)} = 0$ where $P(t, \gamma(t))$ is a point on the curve and $d(,)$ are the corresponding distances.

We shall modify the definition of RW spacetime by assuming $\mathbf{T}_{csm} \subseteq \mathbf{R}_1^1$. Then $M(k, f)$ is a *time scale warped product* (TSWP). We shall denote by U the lift of Δ to $\mathbf{T}_{csm} \times S$, i.e. $\mathrm{U}(f) = \Delta(f \circ \pi)$.

Theorem. Let $M(k, f)$ be a TSWP. Let $\rho(t)$ and $p(t)$ be defined by

$$\frac{8\pi\rho}{3}(t) = [\frac{f^{\Delta}(t)}{f(t)}]^2 + \frac{k}{f^2(t)}, \quad -8\pi p(t) = 2\frac{f^{\Delta\Delta}(t) f(t) - [f^{\Delta}(t)]^2}{f(t) f(\sigma(t))} + \frac{[f^{\Delta}(t)]^2}{[f(t)]^2} + \frac{k}{f^2(t)}$$

Then (U, ρ, p) is a perfect fluid on $M(k, f) = \mathbf{T}_{csm} \times_f S$.

The proof is obtained in a similar way to the corresponding result from [5] and relies on the following two lemmas.

Lemma. Let $M(k, f)$ be a TSWP, let $X, Y \in \mathrm{V}$ (the lifts of tangent vectors to S). Then for every $(t, p) \in M(k, f)$

$$(D_{\mathrm{U}} X)_{(t,p)} = (D_X \mathrm{U})_{(t,p)} = (\Delta f)(t) X_p$$

Proof. For $t \in \mathbf{T}_{dd} \cup \mathbf{T}_{sd}$ the result follows from [5], while for $t \in \mathbf{T}_{ds} \cup \mathbf{A}_h$

$$U < X, Y >_{(t,p)} = \Delta[f^2(t)(X, Y)_p] = 2 f(t) f^{\Delta}(t)(X, Y)_p = 2\frac{f^{\Delta}(t)}{f(t)} < X, Y >_{(t,p)}$$

Lemma. Let $M(k, f)$ be a TSWP, let $X, Y, Z \in \mathrm{V}$. Then

$$R_{X\mathrm{U}} Y_{(t,p)} = (\frac{f^{\Delta\Delta}(t) f(t) - [f^{\Delta}(t)]^2}{f(t) f(\sigma(t))} + \frac{[f^{\Delta}(t)]^2}{[f(t)]^2}) < X, Y > \mathrm{U}_{(t,p)}$$

$$R_{XY} Z_{(t,p)} = [(\frac{f^{\Delta}(t)}{f(t)})^2 + (\frac{k}{f^2(t)})][< X, Z > Y - < Y, Z > X]_{(t,p)}$$

Proof. For $t \in \mathbf{T}_{dd} \cup \mathbf{T}_{sd}$ the results follow from [5], while for $t \in \mathbf{T}_{ds} \cup \mathbf{A}_h$ they are a direct consequence of the preceding lemma and of the fact that

$$\Delta(\frac{f^{\Delta}(t)}{f(t)}) = \frac{f^{\Delta\Delta}(t) f(t) - [f^{\Delta}(t)]^2}{f(t) f(\sigma(t))}$$

Remark. The model obtained in the preceding theorem is indeed a generalized Friedmann cosmological model, since the time scale cannot be reduced to an open interval as in the original model [5]. Nevertheless, the result for $f = Ct^{2/3}$ and $C^3 = \frac{9A}{4}$ stays true, while similar results for $k = \pm 1$ rely on proving that the parametric functions involved are completely delta differentiable (see also [5]).

References

[1] Agarwal, R., Bohner, M., O'Reagan, D., Peterson, A.: Dynamic equations on time scales: a survey, *J.Computational and Applied Mathematics* **141** (2002), 1-26

[2] Bohner, M., Guseinov, G.: Partial differentiation on time scales, *Dynamic Systems and Applications* **13** (2004), 351-379

[3] Bohner, M. and Peterson, A.(eds): *Advances in Dynamic Equations on Time Scales*, Birkhäuser Boston, 2003

[4] Hilger, S.: Analysis on measure chains – a unified approach to continuous and discrete calculus, *Results Math*. **18** (1990), 18-56

[5] O'Neill, B.: *Semi-Riemannian Geometry with Applications to Relativity*, Academic Press, 1983

[6] Stephani, H., Kramer, D., Maccallum, M., Hoenselaers, C. and Herlt, E.: *Exact Solutions of Einsteins's Field Equations*, Cambridge University Press, 2003

ADDITIVE INTEGRAL FUNCTIONS IN VALUED FIELDS

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Abstract
The additive integral functions with the coefficients in a complete non-archimedean algebraically closed field of characteristic $p \neq 0$ are studied.

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Keywords: non-archimedean absolute value, additive integral function

1. Introduction

Let $(K,|\,|)$ be a valued field of characteristic $p \neq 0$, where $|\,|$ is a non-trivial *non-archimedean absolute value* defined on *K*, that is a mapping $|\,|: K \to [0,\infty)$ such that, for every $x, y \in K$,

(i) $|x| = 0$ if and only if $x = 0$;
(ii) $|xy| = |x||y|$;
(iii) $|x + y| \leq \max\{|x|,|y|\}$;
(iv) there exists a non-zero $x \in K$ such that $|x| \neq 1$.

For $x, y \in K$, define $d(x,y) = |x - y|$ and thus (K,d) is an ultrametric space. A formal power series

$$f(X) = \sum_{k=0}^{\infty} a_k X^k \in K[[X]] \qquad \textbf{(1)}$$

is called an *integral function* with coefficients in *K* if, for every $x \in K$, the sequence

$$S_n(x) = \sum_{k=0}^{n} a_k x^k \qquad \textbf{(2)}$$

is a Cauchy sequence. It follows easily that $H(K)$, the set of all integral functions with coefficients in *K*, is a $K-$algebra with the respect to the ordinary addition and multiplication of integral functions. An integral function *f* with coefficients in *K* is called *additive* if, for every $x, y \in K$,

$$f(x + y) = f(x) + f(y). \qquad \textbf{(3)}$$

Suppose now that K is an algebraically closed field of characteristic $p \neq 0$. Then, for every positive integer r, we may consider the Galois field $GF(p^r)$ as a subfield of K. In this case an additive integral function f is called *$GF(p^r)-$linear*, if for every $x \in K$ and $\alpha \in GF(p^r)$,

$$f(\alpha x) = \alpha f(x). \qquad \textbf{(4)}$$

Since, for every $\alpha \in GF(p^r)$, it follows that $\alpha^{p^r} = \alpha$ (see, for example, [3], p. 83), by (3) it follows that an additive integral function is $GF(p)-$linear.

This paper follows the ideas of Nicolae Popescu who conjectured that the additive integral functions have similar properties as the additive polynomials (see[1]).

2. Representation and zeros of additive integral functions

The following result gives a representation of the $GF(p^r)-$linear integral functions.

Theorem 1. *Let K be an algebraically closed field of characteristic $p \neq 0$ which is complete with respect to a non-archimedean absolute value and let f be an integral function with coefficients in K. Then f is $GF(p^r)$ - linear, where r is fixed, if and only if*

$$f(X) = \sum_{i=0}^{\infty} a_i X^{p^{ir}}, \textbf{ with } a_i \in K. \qquad \textbf{(5)}$$

Proof. Since $(x+y)^p = \sum_{i=0}^{p} \binom{p}{i} x^{p-i} y^i$ and $\binom{p}{i} \equiv 0 \pmod p$ it follows that $(x+y)^p = x^p + y^p$. Hence $(x+y)^{p^i} = x^{p^i} + y^{p^i}$ which implies that the integral function f given by (5) is additive. Because, for every $\alpha \in GF(p^r)$, $\alpha^{p^r} = \alpha$, by (5) we obtain that f is a $GF(p^r)-$linear function.

Conversely, we suppose that the integral function $f(X) = \sum_{j=0}^{\infty} b_j X^j$ is a $GF(p^r)-$linear function. We use the formal derivative $f'(X) = \sum_{j=1}^{\infty} j b_j X^{j-1}$. It is easily to see that this operation satisfies the standard rules of differentiation. Since $f(x+y) - f(x) - f(y) = 0$, for every $x, y \in K$, because the zeros of an integral function are isolated, by taking two arbitrary sequences $\{x_n\}_{n \in \mathbf{N}}$, $\{y_n\}_{n \in \mathbf{N}}$ of elements of K which converge to zero, it follows that $f(X+Y) = f(X) + f(Y)$. Hence, for every $y \in K$, $f'(y) = \frac{d}{dX} f(X+y)\Big|_{X=0} = \frac{d}{dX}\big(f(X) + f(y)\big)\Big|_{X=0} = f'(0) = b_1$. Because $f(0)=0$ we obtain that

$$f(X) = c_0 X + \sum_{j=1}^{\infty} c_j X^{n_j}, \textbf{ with } c_0 = b_1, \tag{6}$$

where $n_j > 1$ and $n_j \equiv 0 (\mathrm{mod}\, p)$. We write

$$f(X) = f_1(X) + f_2(X), \tag{7}$$

where

$$f_1(X) = c_0 X + \sum_{j \in I_1} c_j X^{n_j}, \; f_2(X) = \sum_{j \in I_2} c_j X^{n_j}, \tag{8}$$

$I_1 = \{j : n_j \text{ is a power of } p^r\}$ and $I_2 = \{j : n_j \text{ is not a power of } p^r\}$. We shall prove that $f_2 = 0$. Since f and f_1 are $GF(p^r)-$linear integral functions it follows that f_2 is a $GF(p^r)-$linear integral function. Because *K* is an algebraically closed field it follows that the mapping $\tau_p : K \to K$ given by $\tau_p(x) = x^p$ is an automorphism of *K*. Hence $\tau_{p^e} : K \to K$ defined by $\tau_{p^e}(x) = x^{p^e}$ is also an automorphism of *K* and we obtain that

$$f_2(X) = f_3^{p^e}(X), \tag{9}$$

where p^e is the largest power of *p* dividing all n_j, $j \in I_2$ and *e* is not divisible by *r*. Then, because τ_{p^e} is an automorphism of *K* it follows that f_3 is an additive integral function. Moreover, if there exists $\alpha \in GF(p^e)$ and $x \in K$ such that $f_3(\alpha x) \neq \alpha f_3(x)$ it follows that $\alpha^{p^e} \neq \alpha$, a contradiction which implies that f_3 is a $GF(p^e)-$linear integral function. Thus by using the form of f_2, because $1 = p^{0r}$ we obtain as above that $f_3'(y) = 0$, for every $y \in K$. This implies that $f_3(X) = \sum_{j=1}^{\infty} d_j X^{pm_j}$, with $d_j \in K$ and m_j a positive integer. Hence, because p^e is the largest power of *p* dividing all n_j, we obtain that $f_2 = 0$ which implies the theorem. □

Since every additive integral function is a $GF(p)-$linear integral function, by Theorem 1 we obtain the following result.

Corollary 1. *Under the hypotheses of Theorem 1 f is an additive function if and only if*

$$f(X) = \sum_{i=0}^{\infty} a_i X^{p^i}, \textbf{ with } a_i \in K. \tag{10}$$

Theorem 2. *Let K be an algebraically closed field of characteristic* $p \neq 0$ *which is complete with respect to a non-archimedean absolute value and let f be an integral function with coefficients in K having infinitely many distinct roots. If* $G = \{\alpha_i\}_{i \geq 0}$, *where* $\alpha_0 = 0$, *is* *the set of all the roots of f, then* f *is* $GF(p^r)-$*linear* *if and only if G is a* $GF(p^r)-$*linear subspace of K and there exists a chain of* $GF(p^r)-$*linear subspaces*

$$G_{n_1} \subset G_{n_2} \subset ... \subset G_{n_s} \subset ... \qquad \textbf{\textit{(11)}}$$

of G such that the order of G_{n_j} *is equal to* n_j *and* p^r *divides* n_j*, for every j.*

Proof. Let f be an additive integral function. If $\alpha_i, \alpha_j \in G$, then, because *f* is an additive integral function, it follows that $f(\alpha_i - \alpha_j) = f(\alpha_i) - f(\alpha_j) = 0$ and for every $\alpha \in GF(p^r)$, $f(\alpha\alpha_i) = \alpha f(a_i) = 0$. Hence we obtain that *G* is a $GF(p^r)-$ linear subspace of *K*.

Now we consider the critical radius of *f* (see [2], p. 291) $r_1 < r_2 < ... < r_k < ...$. Then inside the ball $B_j = \{x \in K : |x| \le r_j\}$, f has n_j roots (the proof of Theorem 1 of [2], p. 307 is the same in this case). Since $|\alpha| = 1$, for every non-zero $\alpha \in GF(p^r)$, it follows that B_j is a $GF(p^r)-$linear subspace of *G*. Hence $G_{n_j} = \{\alpha \in B_j : f(\alpha) = 0\}$, *j*=1,2,…, is a finite $GF(p^r)-$linear subspace of *K*, p^r divides n_j and (11) holds.

Conversely, if *f* is an integral function, by Theorem of [2], p. 314, it follows that $f = CX \prod_{j=1}^{\infty} \left(1 - \frac{X}{\alpha_j}\right)$. Suppose that *G* is a $GF(p^r)-$linear subspace of *K* and there a chain of $GF(p^r)-$linear subspaces G_{n_j} of *G*, of orders n_j, such that (11) holds. We consider the polynomials $P_j = CX \prod_{\substack{\alpha_j \in G_{n_j} \\ \alpha_j \ne 0}} (X - \alpha_j) = C_j Q_j$, where $Q_j = CX \prod_{\substack{\alpha_j \in G_{n_j} \\ \alpha_j \ne 0}} \left(1 - \frac{X}{\alpha_j}\right)$ and $C_j = \prod_{\substack{\alpha_j \in G_{n_j} \\ \alpha_j \ne 0}} \alpha_j$. By Corollary 1.2.2 from [1] it follows that P_j is a $GF(p^r)-$linear polynomial. Hence Q_j is a $GF(p^r)-$linear polynomial and similarly as in the proof of Theorem of [2], p. 314 we obtain that $\lim_{j \to \infty} Q_j = f$. This implies the theorem. □

References

[1] Goss, D.: *Basic Structures of Function Field Arithmetic*, Springer Verlag, Berlin, 1996.

[2] Robert, A.: *A Course in p-adic Analysis*, Springer Verlag, New York, 2000.

[3] Zariski, O., Samuel, P.: *Commutative Algebra*, vol. I, Springer Verlag, New York, 1958.

APPLICATIONS OF LEVEL FUNCTIONS IN LORENTZ SPACES

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Abstract: We find the best constant in the triangle inequality in Lorentz sequence spaces with an increasing weight.
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Key words: weighted Lorentz sequence spaces, normability, equivalent norm, level function, level sequence.

1. Introduction

Let $1 < p < \infty$. For a sequence $x = (x_n) \in c_0$ (the space of null sequences) the decreasing rearrangement x^* of x is obtained by rearranging x, $w = (w_n)$ will be called weight sequence. Without loss of generality we may suppose that $w \notin l^1(\mathbb{N})$. We recall the definition of the weighted Lorentz sequence spaces

$$d(w,p) = \{x : \|x\|_{p,w} := \left(\sum_{n=1}^{\infty}(x_n^*)^p w_n\right)^{\frac{1}{p}} < \infty\}$$

It is proved in [4] that $\|\cdot\|_{p,w}$ is a norm if and only if w is a decreasing sequence. Also $d(w,p)$ is equivalently normable if and only if

$$\sum_{k=0}^{n}\left(\frac{1}{W_k}\right)^{\frac{1}{p}} \leq C\frac{n+1}{W_n^{\frac{1}{p}}}, \tag{1}$$

$n = 0, 1, ...$ The condition (1) characterize the boundedness of the discrete Hardy operator

$$A_d(x_n) = \frac{1}{n+1}\sum_{k=0}^{n} x_n,$$

$n \in \mathbb{N}$ from $d(w,p)$ to $l^p(w)$. In the inequality (1), $W_n = \sum_{k=0}^{n} w_k$. In the recent papers [1] and [2] the authors considered estimates between a dual norm (defined in terms of Kothe duality), decomposition norm and the usual norm in continuous case. The main reason for these consideration was that the equivalent norm defined in terms of maximal function (or Hardy operator) does not give the best constant in the triangle inequality. In [1] it was considered the case of classical Lorentz spaces $L^{p,s}$ with $p < s$ while in [2] similar results were proved for the weighted function. Using other techniques the same relations between norms on “discrete” Lorentz spaces $l^{p,s}$ were proved in [3]. In this paper we extend some of the results proved in [3] to the case of more general Lorentz sequence spaces with an increasing weight. Since we need the space $d(w,p)$ to be normable we have to assume that the weight satisfies the condition (1). Here and in the sequel we denote by $\tilde{w} = w^{1-p'}$. Also p' will denote the conjugate of p, namely p' is such that $\frac{1}{p} + \frac{1}{p'} = 1$. Observe that $\tilde{w}$ is a decreasing weight sequence. In the second section we review the notion of level sequence and in the last section we present the main result of the paper. Also we give some ideas about the proof of the main result.

2 Level sequences

The level function was introduced in the early 1950's by I. Halperin and developed by G. G. Lorentz and also studied more recently, in conection to weighted norm inequalities, by G. Sinnamon. The concept of level sequence with respect to another sequence was used in [3] in analogy with the level function for the study of the best constants in Lorentz sequence spaces. The unique sequence $x^\circ = (x_n^\circ)$ in the next theorem is called the level sequence of $x = (x_n)$ with the respect to $\varphi = (\varphi_n)$.

Theorem 1 ([3]). Let $\varphi = (\varphi_n)$ be a sequence of positive numbers, $\Phi_n = \sum_{k=0}^n \varphi_k$, $n \geq 0$ and let $x = (x_n)$ be a sequence of positive numbers such that

$$\lim_{n\to\infty} \frac{\sum_{i=0}^n x_i}{\sum_{i=1}^n \varphi_i} = 0.$$

Then, there exists a unique nonnegative sequence $x^\circ = (x_n^\circ)$ satisfying the following conditions:

(a) $\left(\frac{x_n^\circ}{\varphi_n}\right)$ is decreasing;

(b) $x \prec x^\circ$;

(c) The set $\{n : x_n^\circ \neq x_n\} = \cup_{k=1}^\infty I_k$, where $\{I_k\}$ are finite sets such that $\sum_{i\in I_k} x_i = \sum_{i\in I_k} x_i^\circ$ and $\frac{x_i^\circ}{\varphi_i} = \lambda_k$ for all $i \in I_k$.

3 Triangle inequality

The main result of the paper is the following:

Theorem 2. Let $1 < p < \infty$ and w satisfying the condition (1) be an increasing weight. Assume that $x^{(k)} \in d(p,w)$, $(k = 1, 2, \ldots, N)$. Then

$$\left\| \sum_{k=1}^N x^{(k)} \right\|_{p,w} \leq C_{p,w}^* \sum_{k=1}^N \|x^{(k)}\|_{p,w}$$

and the constant is optimal. Here $C_{p,w}^* = \sup_{n\in\mathbb{N}^*} \left(\frac{1}{n}\sum_{k=1}^n w_k\right)\left(\frac{1}{n}\sum_{k=1}^n w_k^{1-p'}\right)^{p-1}$.

Proof. Using the ideeas from [3] we can show that $\left\| \sum_{k=1}^N x^{(k)} \right\|_{p,w} \leq C_{p,w} \sum_{k=1}^N \|x^{(k)}\|_{p,w}$, where $C_{p,w} = \sup_{m,n\in\mathbb{N}^*} \left(\frac{1}{n}\sum_{k=m}^{m+n-1} w_k\right)\left(\frac{1}{n}\sum_{k=m}^{m+n-1} w_n^{1-p'}\right)^{p-1}$. Now using Karamata inequality and the techniques from [2] we derive that $C_{p,w}^* = C_{p,w}$. To show that $C_{p,w}^*$ is the best constant is sufficient to consider the sequence

$$x_n = \begin{cases} 1 \text{ if } n \leq k_0 \\ 0 \text{ otherwise } , \end{cases}$$

for a fixed $k_0 \in \mathbb{N}^*$. Computing the norms we obtain equality, hence the theorem is proved.

References

[1] Barza, S., Kolyada, V. and Soria, J.: Sharp constants related to the triangle inequality in Lorentz spaces, *Trans. Amer. Math. Soc.* **361**, (2009), 5555-5574.

[2] Barza, S. and Soria, J.: Sharp constants between equivalent norms in weighted Lorentz spaces, *J. Aust. Math. Soc.* **88,** (2010), 19-27.

[3] Barza, S, Marcoci, A. M. and Persson, L. E.: Best constants between equivalent norms in Lorentz sequence spaces, *J. Funct. Spaces Appl.*, to appear.

[4] Carro, M. J., Raposo, J. A. and Soria, J.: Recent Developments in the Theory of Lorentz Spaces and Weighted Inequalities, *Mem. Amer. Math. Soc.* **187,** Providence, RI, (2007).

A DUALITY RESULT ON A SCHATTEN TYPE SPACE

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Abstract: Let $B_p(\ell^2)$ denote the Besov-Schatten space of all upper triangular matrices A such that

$$\|A\|_{B_p(\ell^2)} = \left[\int_0^1 (1-r^2)^{2p}\|A''(r)\|_{C_p}^p d\lambda(r)\right]^{\frac{1}{p}} < \infty.$$

In this paper we present and prove a duality result between $B_p(\ell^2)$ and $B_q(\ell^2)$, $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$.

Mathematics Subject Classification (2000): 47B10, 47L20, 47B35, 30H25.
Key words: Besov-Schatten spaces, Schur multipliers.

1.Introduction

Analytic Besov spaces first found a direct application in operator theory in Peller's book [6]. A comprehensive account of the theory of Besov spaces is given in Peetre's book [5]. In what follows we consider the Besov-Schatten spaces in the framework of matrices e.g. infinite matrix valued functions. The extension to the matriceal framework is based on the fact that there is a natural correspondence between Toeplitz matrices and formal series associated to 2π-periodic functions (see e.g. [1], [2], [3] and [9]). In particular the Besov-Schatten space $B_1(\ell^2)$ can be used to prove that the associated Hankel operator is nuclear. We use the powerfull device Schur multipliers and its characterizations in the case of Toeplitz matrices to prove some of the main results.

The *Schur product* (or *Hadamard product*) of matrices $A = (a_{jk})_{j,k\geq 0}$ and $B = (b_{jk})_{j,k\geq 0}$ is defined as the matrix $A * B$ whose entries are the products of the entries of A and B:

$$A * B = (a_{jk}b_{jk})_{j,k\geq 0}.$$

If X and Y are two Banach spaces of matrices we define Schur multipliers from X to Y as the space

$$M(X,Y) = \{M : M * A \in Y \text{ for every } A \in X\},$$

equipped with the natural norm

$$\|M\| := \sup_{\|A\|_X \leq 1} \|M * A\|_Y.$$

In this paper we follow the notation from [4], [7] and [8].

2. THE DUALITY THEOREM

The main theorem of the paper is the following:

Theorem 1

Under the pairing

$$< A, B >= \int_0^1 tr(V(A)[V(B)]^*)d\lambda(r)$$

we have the following dualities:

(1) $B_p(\ell^2)^* \approx B_q(\ell^2)$ if $1 < p < \infty$ and $\frac{1}{p}+\frac{1}{q}=1$;

(2) $\mathcal{B}_{0,c}(D,\ell^2)^* \approx B_1(\ell^2)$ and $B_1(\ell^2)^* \approx \mathcal{B}(D,\ell^2)$.

Proof:
Since V is an embedding from $\mathrm{B}_p(\ell^2)$ into $L_p(D,\ell^2)$ for all $1 \le p < \infty$, Holder's inequality shows that $B_q(\ell^2) \subset B_p(\ell^2)^*$ for $1 \le p < \infty$ and $B_1(\ell^2) \subset \mathcal{B}^*_{0,c}(D,\ell^2)$.

Suppose that F is a bounded linear functional on the Besov-Schatten space $B_p(\ell^2)$ with $1 \le p < \infty$. Then $F \circ V^{-1} : VB_p(\ell^2) \to \mathbb{C}$ can be extended to a bounded linear functional on $L_p(D,\ell^2)$. Thus there exists $C(\cdot) \in L_p(D,\ell^2)$ such that $\|C(\cdot)\|_{L_p(D,\ell^2)} = \|F \circ V^{-1}\|$ and

$$(F \circ V^{-1})(B) = \int_0^1 tr(B(r))[C(r)]^* d\lambda(r), \quad B(\cdot) \in L_p(D,\ell^2).$$

In particular, if $B(\cdot) = V(A)$ with $A \in B_p(\ell^2)$, then

$$F(A) = \int_0^1 tr((VA)(r))[C(r)]^* d\lambda(r).$$

Let $B = P(C)$; then $B \in B_q(\ell^2)$ and it is easy to check that

$$F(A) = \int_0^1 tr((VA)(r)[(VB)(r)]^*) d\lambda(r), \quad A \in B_p(\ell^2),$$

with $\|B\|_{B_q(\ell^2)} \le \|C(\cdot)\|_{L_p(D,\ell^2)} = \|F \circ V^{-1}\| \le \|V^{-1}\|\|F\|$. This proves the duality $B_p(\ell^2)^* \approx B_q(\ell^2)$ for $1 \le p < \infty$. For the second part we follow the techniques from [4], and the theorem is proved.

References

[1] Arazy, J: Some remarks on interpolation theorems and the boundedness of the triangular projection in unitary matrix spaces, *Integral Equations Operator Theory* **1** (1978), no. 4, 453-495.

[2] Barza, S, Persson, L. E. and Popa, N: A Matriceal Analogue of Fejer's theory, *Math. Nach*. **260** (2003), 14-20.

[3] Marcoci, A. N. and Marcoci, L. G.: A new class of linear operators on ℓ^2 and Schur multipliers for them, *J. Funct. Spaces Appl*. **5** (2007), 151-164.

[4] Marcoci, L. G., Persson, L. E., Popa, I and Popa, N.: A new characterization for a Bergman-Schatten spaces and a duality result, *J. Math. Anal. Appl*. **360** (2009), No.1, 67-80.

[5] Peetre, J.: *New thoughts on Besov spaces*, Duke Univ. Press., Durham, N.C.,1976.

[6] Peller, V: *Hankel Operators and Their Applications* , Springer- Verlag, New York, 2003.

[7] Popa, N: *Matriceal Bloch and Bergman-Schatten spaces*, Rev. Roumaine Math. Pures Appl. **52** (2007), 459-478.

[8] Popa, N.: Schur multipliers between Banach spaces of matrices, *Proc. of the Sixth Congress of Romanian Mathematicians*, Edit. Academiei Romane, Bucharest, 2009, 373-380.

[9] Shields, A. L.: An analogue of a Hardy-Littlewood-Fejer inequality for upper triangular trace class operators, *Math. Z.* **182** (1983), 473-484.

A PROPERTY OF DUALITY MAPPINGS ON A SOBOLEV SPACE WITH A VARIABLE EXPONENT

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Abstract: We will show that any duality mapping on the Sobolev space with variable exponent $W_0^{1,p(\cdot)}(\Omega)$ endowed with a suitable norm, is a mapping of type (S_+).
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Key words: duality mapping, Sobolev space with variable exponent, mapping of type (S_+).

Let $\varphi : \mathbf{R}_+ \to \mathbf{R}_+$ be a *gauge function*, that is a continuous, strictly increasing function satisfying $\varphi(0) = 0$ and $\varphi(r) \to \infty$ as $r \to \infty$.
If $(X, \|\cdot\|_X)$ is a real Banach space, by *duality mapping* corresponding to the gauge function φ one understands the multivalued mapping $J_\varphi : X \to 2^{X^*}$ defined as follows:

$$J_\varphi 0_X = \{0_{X^*}\},$$
$$J_\varphi x = \varphi(\|x\|_X) \cdot \{x^* \in X^* | \langle x^*, x \rangle = \varphi(\|x\|_X) \cdot \|x\|_X, \|x^*\|_{X^*} = 1\}, \; x \neq 0_X .$$

According to an Asplund's result, at any $u \in X$,

$$J_\varphi u = \partial F(u), \text{ where } F(u) = \int_0^{\|u\|_X} \varphi(t) \mathrm{d}t$$

and ∂F stands for the subdifferential of F in the sense of convex analysis.
If $\|\cdot\|_X$ is Gâteaux differentiable, the duality mapping corresponding to the gauge function φ is a single valued operator $J_\varphi : X \to X^*$ defined as follows

$$J_\varphi 0_X = 0_{X^*},$$
$$J_\varphi x = \varphi(\|x\|_X) \cdot \|\cdot\|_X' (x), \; x \neq 0_X .$$

Let Ω be a bounded and smooth domain in $\mathbf{R}^N$, $N \geq 2$, and let $p \in L^\infty(\Omega)$ be such that

$$1 \leq p^- := ess \inf_{x \in \Omega} p(x) \leq p^+ := ess \sup_{x \in \Omega} p(x).$$

The *Lebesgue space* $L^{p(\cdot)}(\Omega)$ *with variable exponent* $p(\cdot)$ is defined as

$$L^{p(\cdot)}(\Omega) = \left\{ u : \Omega \to \mathbf{R} \,; u \text{ is } \mathrm{d}x\text{ - measurable and } \rho_{p(\cdot)}(u) = \int_\Omega |u(x)|^{p(x)} \mathrm{d}x < \infty \right\}$$

and it is endowed with the norm

$$\|u\|_{p(\cdot)} = \inf \left\{ \lambda > 0 \;\; \rho_{p(\cdot)}\left(\frac{u}{\lambda} \right) \leq 1 \right\}.$$

The following result will be useful:

Proposition 1. ([3]) *Let* $p \in L^{\infty}(\Omega)$ *and* $u \in L^{p(\cdot)}(\Omega)$. *Then, if* $u \neq 0$, *we have that* $\|u\|_{p(\cdot)} = a$ *if and only if* $\rho_{p(\cdot)}\left(\frac{u}{a}\right) = 1$.

Given a function $p \in L^{\infty}(\Omega)$ that satisfies $p^{-} \geq 1$, the *Sobolev space* $W^{1,p(\cdot)}(\Omega)$ *with variable exponent* $p(\cdot)$ is defined as

$$W^{1,p(\cdot)}(\Omega) = \left\{u \in L^{p(\cdot)}(\Omega) \;\; \partial_i u \in L^{p(\cdot)}(\Omega), 1 \leq i \leq N\right\},$$

where for each $1 \leq i \leq N$, ∂_i denotes the distributional derivative operator with respect to the i-th variable. This space may be equivalently defined as

$$W^{1,p(\cdot)}(\Omega) = \left\{u \in L^{p(\cdot)}(\Omega) \;\; |\nabla u| \in L^{p(\cdot)}(\Omega)\right\},$$

where $\nabla u(x) = (\partial_1 u(x), \ldots, \partial_N u(x))$. It is a Banach space with respect to the norm

$$\|u\| = \|u\|_{p(\cdot)} + \||\nabla u|\|_{p(\cdot)}.$$

We consider the Sobolev space $W_0^{1,p(\cdot)}(\Omega)$, the closure of $C_0^{\infty}(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$. This space can be equivalently renormed via Poincaré inequality ([1]) with the norm

$$\|u\|_{1,p(\cdot)} = \||\nabla u|\|_{p(\cdot)}.$$

In [1] and [2] is established that any duality mapping corresponding to the gauge function φ on $\left(W_0^{1,p(\cdot)}(\Omega), \|\cdot\|_{1,p(\cdot)}\right)$ has the form

$$\langle J_{\varphi} u, h\rangle = \frac{\varphi\left(\|u\|_{1,p(\cdot)}\right) \cdot \int_{\Omega_{0,u}} p(x) \dfrac{|\nabla u|^{p(x)-2} \nabla u \cdot \nabla h}{\|u\|_{1,p(\cdot)}^{p(x)-1}} \, \mathrm{d}x}{\int_{\Omega} p(x) \dfrac{|\nabla u|^{p(x)}}{\|u\|_{1,p(\cdot)}^{p(x)}} \, \mathrm{d}x}, \quad h \in W_0^{1,p(\cdot)}(\Omega).$$

Here $\Omega_{0,u} = \{x \in \Omega \,; |\nabla u(x)| \neq 0\}$.
The main result is

Theorem 1. *Any duality mapping corresponding to the gauge function* φ *on* $\left(W_0^{1,p(\cdot)}(\Omega), \|\cdot\|_{1,p(\cdot)}\right)$ *is a mapping of type* (S_+).

Proof. We recall that, if X is a real Banach space, the operator $T : X \to X^*$ is a mapping of type (S_+) if and only if, as $n \to \infty$, the following holds:

$$u_n \xrightarrow{w} u \text{ and } \limsup_{n\to\infty} \langle Tu_n, u_n - u\rangle = 0 \text{ implies } u_n \xrightarrow{s} u.$$

Let $(u_n)_n \subset W_0^{1,p(\cdot)}(\Omega)$ be a sequence such that $u_n \xrightarrow{w} u$ as $n \to \infty$ and

$$\overline{\lim_{n\to\infty}} \langle J_{\varphi} u_n, u_n - u\rangle \leq 0.$$

J_{φ} being monotone, it follows that

$$\lim_{n\to\infty} \langle J_{\varphi} u_n - J_{\varphi} u, u_n - u\rangle = 0.$$

But

$$p^{-} \leq \int_{\Omega} p(x) \frac{|\nabla u|^{p(x)}}{\|u\|_{1,p(\cdot)}^{p(x)}} \mathrm{d}x \leq p^{+} \text{ for any } u \in W_0^{1,p(\cdot)}(\Omega).$$

(Proposition 1).

Consequently, since $\lim_{n\to\infty}\langle J_\varphi u, u_n - u\rangle = 0$, one obtains

$$\lim_{n\to\infty}\varphi\left(\|u\|_{1,p(\cdot)}\right)\cdot\int_{\Omega_{0,u}} p(x)\frac{|\nabla u|^{p(x)-2}\nabla u\cdot\nabla(u_n-u)}{\|u\|_{1,p(\cdot)}^{p(x)-1}}\,\mathrm{d}x = 0$$

and

$$\lim_{n\to\infty}\langle J_\varphi u_n, u_n - u\rangle = 0,$$

therefore

$$\lim_{n\to\infty}\varphi\left(\|u_n\|_{1,p(\cdot)}\right)\cdot\int_{\Omega_{0,u_n}} p(x)\frac{|\nabla u_n|^{p(x)-2}\nabla u_n\cdot\nabla(u_n-u)}{\|u_n\|_{1,p(\cdot)}^{p(x)-1}}\,\mathrm{d}x = 0.$$

But

$$\left|\int_{\Omega_{0,u_n}} p(x)\frac{|\nabla u_n|^{p(x)-2}\nabla u_n\cdot\nabla(u_n-u)}{\|u_n\|_{1,p(\cdot)}^{p(x)-1}}\,\mathrm{d}x\right| \le p^+\int_{\Omega_{0,u_n}}\frac{|\nabla u_n|^{p(x)-1}\left(|\nabla u_n|+|\nabla u|\right)}{\|u_n\|_{1,p(\cdot)}^{p(x)-1}}\,\mathrm{d}x \le$$

$$\le p^+\cdot\left(\|u_n\|_{1,p(\cdot)}\cdot\rho_{p(\cdot)}\left(\frac{|\nabla u_n|}{\|u_n\|_{1,p(\cdot)}}\right)+C\cdot\left\|\frac{|\nabla u_n|^{p(x)-1}}{\|u_n\|_{1,p(\cdot)}^{p(x)-1}}\right\|_{p'(\cdot)}\cdot\left\||\nabla u|\right\|_{p(\cdot)}\right)=$$

$$= p^+\cdot\left(\|u_n\|_{1,p(\cdot)}+C\cdot\left\||\nabla u|\right\|_{p(\cdot)}\right),$$

since

$$\rho_{p'(\cdot)}\left(\frac{|\nabla u_n|^{p(x)-1}}{\|u_n\|_{1,p(\cdot)}^{p(x)-1}}\right)=\rho_{p(\cdot)}\left(\frac{|\nabla u_n|}{\|u_n\|_{1,p(\cdot)}}\right)=1.$$

Therefore, since the gauge function φ is continuous and $\|u_n\|_{1,p(\cdot)}\to\|u\|_{1,p(\cdot)}$ as $n\to\infty$, it follows that

$$\lim_{n\to\infty}\varphi\left(\|u_n\|_{1,p(\cdot)}\right)\cdot\int_{\Omega_{0,u_n}} p(x)\frac{|\nabla u_n|^{p(x)-2}\nabla u_n\cdot\nabla(u_n-u)}{\|u_n\|_{1,p(\cdot)}^{p(x)-1}}\,\mathrm{d}x =$$

$$=\lim_{n\to\infty}\varphi\left(\|u\|_{1,p(\cdot)}\right)\cdot\int_{\Omega_{0,u_n}} p(x)\frac{|\nabla u_n|^{p(x)-2}\nabla u_n\cdot\nabla(u_n-u)}{\|u_n\|_{1,p(\cdot)}^{p(x)-1}}\,\mathrm{d}x.$$

Consequently, one has

$$\lim_{n\to\infty}\varphi\left(\|u\|_{1,p(\cdot)}\right)\cdot$$

$$\cdot\left(\int_{\Omega_{0,u_n}} p(x)\frac{|\nabla u_n|^{p(x)-2}\nabla u_n\cdot\nabla(u_n-u)}{\|u_n\|_{1,p(\cdot)}^{p(x)-1}}\,\mathrm{d}x-\int_{\Omega_{0,u}} p(x)\frac{|\nabla u|^{p(x)-2}\nabla u\cdot\nabla(u_n-u)}{\|u\|_{1,p(\cdot)}^{p(x)-1}}\,\mathrm{d}x\right)=0.$$

Therefore

$$\lim_{n\to\infty}\left(\int_{\Omega_{0,u_n}} p(x)\frac{|\nabla u_n|^{p(x)-2}\nabla u_n\cdot\nabla(u_n-u)}{\|u_n\|_{1,p(\cdot)}^{p(x)-1}}\,\mathrm{d}x-\int_{\Omega_{0,u}} p(x)\frac{|\nabla u|^{p(x)-2}\nabla u\cdot\nabla(u_n-u)}{\|u\|_{1,p(\cdot)}^{p(x)-1}}\,\mathrm{d}x\right)=0.$$

The proof continue as in [3].

The result is important in the study of the existence of solutions **for** operator equations involving duality mappings.

References

[1] Dincă, G. and Matei, P.: Geometry of Sobolev Spaces with Variable Exponents and a Generalization of the p-Laplacian, *Analysis and Applications* vol. **7**, no. **4** (2009), 373-390.
[2] Dincă, G. and Matei, P.: Geometry of Sobolev spaces with variable exponent: smoothness and uniform convexity, *C. R. Math. Acad. Sci. Paris* **347** (2009), no. 15-16, 885-889.
[3] Fan, X.-L. and Zhang, Q.-H.: Existence of solutions for $p(x)$-Laplacian Dirichlet problem, *Nonlinear Analysis* **52** (2003), 1843-1852.

SECOND ORDER CONDITIONS OF QUASIINVEXITY VIA PREQUASIINVEXITY AND APPLICATION FOR QUADRATIC FUNCTIONS

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Abstract. We establish necessary and sufficient conditions of quasiinvexity via prequasiinvexity notion. Particularly,is obtained a criterion for the quasiinvexity of the quadratic functions.A numerical example is given.
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Keywords : generalized convexity, invexity, quasiinvexity.

1. Introduction and preliminaries

The first section (1) of the paper is an introduction where we present the definitions of invex, quasiinvex, preinvex and prequasiinvex functions. In section 2 we present three twice differentiable necessary conditions and three twice differentiable sufficient conditions for the prequasiinvexity of the real functions. According to the equivalence relation *prequasiinvex* $\Leftrightarrow$ *quasiinvex* [6],[7] in section 3, from the twice differentiable conditions of prequasiinvexity established in section 2 we infer second order conditions for quasiinvex functions. In section 4 we establish a criterion for the quasiinvexity of the quadratic functions. A numerical example is given.

$\nabla f(x)$ is the gradient of f, $Hf(x)$ is the Hessian matrix and $\tilde{H}(x,\lambda) = Hf(x) + \lambda \nabla f(x) \nabla' f(x)$ is the extended Hessian matrix, all at the point *x*.

Let X be a nonempty set in $\mathbf{R}^n$, a mapping $\eta : X \times X \to \mathbf{R}^n \backslash 0$ and a function $f : X \to \mathbf{R}$ and a point $u \in X$.

Definition 1. X is called *invex set at u* if there exists a mapping $\eta \neq 0$ such that for every $x \in X$ and $\lambda \in [0,1]$, $u + \lambda \eta(x,u) \in X$.

X is called *invex set* if it is invex at each point $u \in X$ with respect the same mapping η.

In these condition we say that X is an η-invex set. In the following we consider X a η-invex set.

Definition 2(Hanson [3], Craven [2],1981).**A.** The function *f* is called *invex* (I) at u if there is a mapping $\eta \neq 0$ such that

(I) $$\forall x \in X : f(x) - f(u) \geq \eta'(x,u) \nabla f(u).$$

B. The function f is called *incave* at *u* if - f is *invex* at *u.*

C. The function *f* is called *invex* or *incave on X*, if it is *invex* or *incave* at each point $u \in X$.

Definition 3(Hanson [3],Craven[2]1981).**A.** A function f is called *pseudoinvex* (PI) at u if there is a vector function $\eta \neq 0$ such that

(PI) $$\forall x \in X : \eta'(x,u) \nabla f(u) \geq 0 \Rightarrow f(x) \geq f(u).$$

B. The function f is called *pseudoincave* at *u* if - f is pseudoinvex at *u.*

C. The function *f* is called *pseudoinvex* or *pseudoincave on X*, if it is *quasiinvex* or *quasiincave* at each point $u \in X$.

Definition 4(Hanson [3],Craven[2]1981).**A.** A function f is called *quasiinvex* (QI) at u if there is a vector function $\eta \neq 0$ such that

(QI) $\forall x \in X : f(x) \leq f(u) \Rightarrow \eta'(x,u)\nabla f(u) \leq 0.$

B. The function f is called *quasiincave* at u if - f is quasiinvex at u.

C. The function f is called *quasiinvex* or *quasiincave on X*, if it is *quasiinvex* or *quasiincave* at each point $u \in X$.

At the point u, the following implications are true : *invex*$\Rightarrow$*pseudoinvex*$\Rightarrow$*quasiinvex*[2],[3].

Definition 2' (Weir, Mond[10],1988). **A.**The function f is called [*strictly*] *preinvex* (pI) at u if there is a mapping $\eta \neq 0$ such that

(pI) $\forall x \in X, [x \neq u], \forall \lambda \in [0,1]: f(u + \lambda\eta(x,u))[<] \leq \lambda f(x) + (1-\lambda) f(u).$

B. The function f is called [*strictly*] *preincave* at u if - f is *strictly*[*preinvex*] at u.

C. The function f is called [*strictly*] *preinvex* or [*strictly*] *preincave on X*, if it is [*strictly*] *preinvex* or [*strictly*] *preincave* at each point $u \in X$, respectively.

Definition 3'(Pini[9], Mititelu[4]).**A** A function f is called *prepseudoinvex* (pPI) on X if there is a vector function $\eta \neq 0$ such that

(pPI) $\forall x \in X, \forall \lambda \in (0,1): f(u + \lambda\eta(x,u)) \geq f(u) \Rightarrow f(x) \geq f(u).$

B. The function f is called [*strictly*] *prequasiincave* at u if - f is *strictly* [*prequasiinvex*] at u.

C. The function f is called [*strictly*] *prepseudoinvex* or [*strictly*] *prepseudoincave on X*, if it is [*strictly*] *prepseudoinvex* or [*strictly*] *prepseudoincave* at each point $u \in X$, respectively.

Definition 4'(Pini[9]).**A** A function f is called *prequasiinvex* (pQI) on X if there is a vector function $\eta \neq 0$ such that

(pQI) $\forall x \in X, \forall \lambda \in [0,1]: f(x) \leq f(u) \Rightarrow f(u + \lambda\eta(x,u)) \leq f(u).$

B. The function f is called [*strictly*] *prequasiincave* at u if - f is *strictly* [*prequasiinvex*] at u.

C. The function f is called [*strictly*] *prequasiinvex* or [*strictly*] *prequasiincave on X*, if it is [*strictly*] *prequasiinvex* or [*strictly*] *prequasiincave* at each point $u \in X$, respectively.

The following implication is true : *preinvex*$\Rightarrow$*prepseudoinve* $\Rightarrow$*prequasiinvex*[6], [7] .

Mititelu[6],[7] established the following equivalences:

(E) *preinvex* $\Leftrightarrow$ *invex; prepseudoinvex* $\Leftrightarrow$ *pseudoinvex; prequasiinvex* $\Leftrightarrow$ *quasiinvex.*

2.Twice differentiable conditions of prequasiinvexity for functions

Theorem 2.1(Necessity). *Let f be a real function, twice differentiable on open* η*-invex set* $X \subseteq \mathbf{R}^n$ *and* $u \in X$. *Suppose that* $Hf(x) \neq 0, \forall x \in X$.*Then each of the following equivalent conditions*

(i) $\forall x \in X : \eta'(x,u)\nabla f(u) = 0 \Rightarrow \eta'(x,u)Hf(u)\eta(x,u) \geq 0,$

(ii) *the equation*

$$\begin{vmatrix} Hf(u) - \lambda I & \nabla f(u) \\ \nabla' f(u) & 0 \end{vmatrix} = 0$$

(of degree $n-1$) *has only positive roots* $(\lambda \geq 0)$,

(iii) $\eta'(x,u)\tilde{H}f(u,\lambda)\eta(x,u) \geq 0, \forall x \in X$ *for a* $\lambda \in [0,\infty)$,

is necessary for f to be prequasiinvex (*but not prequasiincave*) *at u with respect to* η.

Proof. For equivalence see (i)$\Leftrightarrow$(ii) and (i)$\Leftrightarrow$(iii) from the theorem 4.1 in [7].

Now we prove the implication

$$< f \text{ prequasiinvex at } u > \Rightarrow \text{(i)}.$$

We suppose, by absurdum that, although f is prequasiinvex at point u, the condition (i) does not hold. Then there is $x' \in A$ such that

$$\eta^t(x',u)\nabla f(u) = 0 \Rightarrow \eta^t(x',u)Hf(u)\eta(x',u) < 0.$$

Due to the fact that $f \in C^2(A)$, there is $\varepsilon > 0$ such that we have:

$$\eta^t(x',u)Hf(u+\lambda\eta(x',u))\eta(x',u) < 0, \quad \forall \lambda \in (-\varepsilon, \varepsilon).$$

Consider now a function

$$F : (-\varepsilon, 1] \to R,\ F(\lambda) = f(u+\lambda\eta(x',u)).$$

Because F is twice differentiable, we get

$$F'(\lambda) = \eta^t(x',u)\nabla f(u+\lambda\eta(x',u)),$$

$$F''(\lambda) = \eta^t(x',u)Hf(u+\lambda\eta(x',u))\eta(x',u),$$

$$F'(0) = \eta^t(x',u)\nabla f(u) = 0,$$

$$F''(0) = \eta^t(x',u)Hf(u)\eta(x',u) < 0.$$

It follows that $\lambda = 0$ is a local maximum point of F and then

$$\exists \lambda \in (0,1): \quad F'(0) = 0 \Rightarrow F(\lambda) < F(0).$$

or, equivalently,

(2.1) $$\exists \lambda \in (0,1): \ \eta^t(x',u)\nabla f(u) = 0 \Rightarrow f(u+\lambda\eta(x',u)) < f(u).$$

But f being prequasiinvex at point u we find

$$f(u+\lambda\eta(x',u)) \le \max\{f(x'), f(u)\}.$$

Taking account of relation (2.1) it follows

$$\max\{f(x'), f(u)\} = f(u),$$

if and only if $f(x') \le f(u)$. From here it results

(2.2) $$\eta^t(x',u)\nabla f(u) = 0 \Rightarrow f(x') \le f(u).$$

But from hypothesis, f is not η-prequasiincave, therefore nor η-prepseudoincave $\equiv$ η-pseudoincave. In this case relation (2.2) is false. Then, the implication (i) is true.

Theorem 2.2 (Sufficient conditions). *Let f be a real function of class C^2 on open set $X \subseteq$ $\boldsymbol{R}^n$, where $Hf(x) \neq 0, \forall x \in X$. Then each of the following equivalent conditions is sufficient for the prequasiinvexity of the function f at the point $u \in X$ with respect to mapping $\eta : X \times X \to \mathbf{R}^n$:*

(i') $\forall x \in X, \eta^t(x,u)\nabla f(u) = 0 \Rightarrow \eta^t(x,u)Hf(u)\eta(x,u) > 0,$

(ii') *the equation*

$$\begin{vmatrix} Hf(u) - \lambda I & \nabla f(u) \\ \nabla^t f(u) & 0 \end{vmatrix} = 0$$

has strictly positive roots ($\lambda > 0$).

(iii') $\eta^t(x,u)\tilde{H}f(u,\lambda)\eta(x,u) > 0, \forall x \in X,$ *for λ great enough.*

Proof. The equivalence of conditions (i') - (iii') is establish in a similar manner as the one of conditions (i)-(iii) from Theorem 2.1.

Now we prove that (i')$\Rightarrow \langle f$ prequasiinvex at $u \rangle$.

Let us suppose that, although (i') is true, f is not prequasiinvex function at point *u*. Then there is a point $x' \in A$, such that

$$\max_{\lambda \in [0,1]} f(u+\lambda\eta(x',u)) > \max\{f(x'), f(u)\}.$$

Due to the continuity of f there is $\lambda_0 \in (0,1]$, where the maximum of f is atteind and then we have

$$f(u+\lambda_0\eta(x',u)) > \max\{f(x', f(u)\}.$$

The continuity of f implies that there is $\varepsilon > 0$, small enough, such that

$$f(u+\lambda_0\eta(x',u)) \ge f(u+(\lambda_0+\lambda)\eta(x',u)) > \max\{f(x'), f(u)\},$$

$\forall \lambda \in (-\varepsilon, \varepsilon)$ $(u + (\lambda_0 + \varepsilon)\eta(x',u) \in A$).

Denote by $a = u + \lambda_0\eta(x',u)$ and we get

(2.3) $$f(a+\lambda\eta(x',u)) \le f(a), \forall \lambda \in (-\varepsilon, \varepsilon).$$

Let us consider the function

$$F:(-\varepsilon,1]\to\mathbf{R},\ F(\lambda)=f(a+\lambda\eta(x',u)).$$

Relation (2.3) becomes $F(\lambda)\le F(0),\ \forall\lambda\in(-\varepsilon,\varepsilon).$ Therefore $\lambda=0$ is a maximum point for function $F(\lambda),$ of class C^2. We obtain $F'(0)=0$, $F''(0)\le 0,$ which become, respectively

$$\eta^t(x',u)\nabla f(a)=0,\ \eta^t(x',u)Hf(a)\eta(x',u)\le 0,$$

or

(2.4) $$\begin{cases}\eta^t(x',u)\nabla f(u+\lambda\eta(x',u))=0\\ \eta^t(x',u)Hf(u+\lambda\eta(x',u))\eta(x',u)\le 0, \quad \forall\lambda\in(-\varepsilon,\varepsilon).\end{cases}$$

Because $f\in C^2(A),$ for $\lambda\to 0$, relations (2.4), become, respectively:

(2.5) $$\eta^t(x',u)\nabla f(u)=0,$$

(2.6) $$\eta^t(x',u)Hf(u)\eta(x',u)\le 0.$$

But, according to hypothesis (i'), (2.6) implies:

(2.7) $$\eta^t(x',u)Hf(u)\eta(x',u)>0.$$

We remark that relations (2.6) and (2.7) are contradictory. It follows that the above made assumption is false. Therefore, function f is prequasiinvex.

3. Consequences : second order conditions for quasiinvexity

According to the equivalences (E), Theorems 2.1 and 2.2 get the next forms, respectively:

Theorem 3.1(Necessity). *Let f be a real function, twice differentiable on open η-invex set $X\subseteq\mathbf{R}^n$ and a point $u\in X$. Suppose that $Hf(x)\ne 0,\forall x\in X$. Then each of the following equivalent conditions*

(i) $\forall x\in X:\eta'(x,u)\nabla f(u)=0\Rightarrow\eta'(x,u)Hf(u)\eta(x,u)\ge 0,$

(ii) *the equation*

$$\begin{vmatrix}Hf(u)-\lambda I & \nabla f(u)\\ \nabla' f(u) & 0\end{vmatrix}=0$$

(of degree $n-1$) *has only positive roots* $(\lambda\ge 0),$

(iii) $\eta'(x,u)\tilde{H}f(u,\lambda)\eta(x,u)\ge 0,\forall x\in X$ *for a* $\lambda\in[0,\infty),$

is necessary for f to be quasiinvex (*but not quasiincave*) *at u with respect to* η.

Theorem 3.2 (Sufficient conditions). *Let f be a real function of class C^2 on open set $X\subseteq\mathbf{R}^n$, where $Hf(x)\ne 0,\forall x\in X$. Then each of the following equivalent conditions is sufficient for the quasiinvexity of the function f at the point $u\in X$ with respect to mapping* $\eta:X\times X\to\mathbf{R}^n$:

(i') $\forall x\in X,\eta^t(x,u)\nabla f(u)=0\Rightarrow\eta^t(x,u)Hf(u)\eta(x,u)>0,$

(ii') $\forall x\in X$, *equation*

$$\begin{vmatrix}Hf(u)-\lambda I & \nabla f(u)\\ \nabla^t f(u) & 0\end{vmatrix}=0$$

has strictly positive roots ($\lambda>0$).

(iii') $\eta^t(x,u)\tilde{H}f(u,\lambda)\eta(x,u)>0,\forall x\in X,$ *for* λ *great enough.*

We can take η defined be

$$\eta(x,u)=\begin{cases}\dfrac{[f(x)-f(u)]\nabla f(u)}{\|\nabla f(u)\|^2}, \text{ if } \nabla f(u)\ne 0\\ 0, \qquad \text{if } \nabla f(u)=0.\end{cases}$$

Indeed, we suppose that $f(x)\le f(u)$. Then

1) if $\nabla f(u) \neq 0,$ we have $\left[\frac{[f(x)-f(u)]\nabla' f(u)}{\|\nabla f(u)\|^2}\right]\nabla f(u) \leq 0 \Rightarrow \eta'(x,u)\nabla f(u) \leq 0.$

2) if $\nabla f(u) = 0$, we have $\eta'(x,u)\nabla f(u) = 0.$

Consequently, f is η-quasiinvex at *u*.

4. Application: the quasiinvexity of quadratic functions

Consider the quadratic function $Q(x) = (1/2)\, x'Qx + q'x$ where Q is a $n \times n$ real symmetric matrix and $q \in \mathbf{R}^n$.To give a criterion of quasiinvexity for quadratic function we use the following part (extract) of Theoren 3.2:

Corollary 4.1(Sufficient condition). *Let f be a real function of class C^2 on open set $A \subseteq \mathbf{R}^n$, where $Hf(x) \neq 0, \forall x \in X$. Then the following condition is sufficient for the quasiinvexity of the function f at the point $u \in X$ with respect to mapping $\eta : X \times X \to \mathbf{R}^n$:*

(i') $\quad \forall x \in X, \eta^t(x,u)\nabla f(u) = 0 \Rightarrow \eta^t(x,u)Hf(u)\eta(x,u) > 0$.

We have $\nabla Q(x) = Qx + q$ and $HQ(x) = Q.$ Then (i') becomes

(i'') $\quad \forall x \in X : \eta'(x,u)(Qx+q) = 0 \Rightarrow \eta'(x,u)Q\eta(x,u) > 0.$

Criterion 4.2. *Suppose that the quadratic function $Q(x)$ is not positive semidefinite. Then the next statements hold:*

a) *If $q \geq 0$, then $Q(x)$ is quasiinvex on the set* $\{x \in \mathbf{R}^n \mid x \leq 0\}$.

b) *If $q \leq 0$, then $Q(x)$ is quasiinvex on the set* $\{x \in \mathbf{R}^n \mid x \geq 0\}$.

c) $Q(x)$ *is quasiinvex on the half-space* $\{x \in \mathbf{R}^n \mid q'x \leq 0\}$.

Proof. For simplicity, we put η instead of $\eta(x,u)$ and use (i''). From $\eta'(Qx+q) = 0$, successsively we obtain:

$\eta'Qx + \eta'q = 0$ (transpose),

$x'Q'\eta + q'\eta = 0$ ($Q' = Q$, multiply at the left by η),

$\eta(x'Q\eta) + (\eta\eta')q = 0$,

$(\eta x')Q\eta + \|\eta\|^2 q = 0$,

(4.1) $\quad (x\eta')Q\eta + \|\eta\|^2 q = 0$.

Finally, from (4.1) we obtain

(4.2) $\quad x(\eta'Q\eta) + \|\eta\|^2 q = 0$

and from (4.2) we obtain too $x'x(\eta'Q\eta) + \|\eta\|^2 x'q = 0$, or equivalently

(4.3) $\quad \|x\|^2(\eta'Q\eta) + \|\eta\|^2 q'x = 0.$

For to have $\eta'Q\eta > 0,$ as in (i'') :

a) according to (4.2), if $q \geq 0$ it must $x \leq 0$

and b) according to (4.2) if $q \leq 0$ it must $x \geq 0.$

c) Also $\eta'Q\eta > 0$ if, according to (4.3), we have $q'x \leq 0.$

Numerical example. Consider the quadratic function $Q(x_1, x_2) = -x_1x_2 + x_1 - x_2$. We have $Q = \frac{1}{2}\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, q = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and the eigenvalues $\lambda_1 = 1, \lambda_2 = -1.$ $Q(x_1, x_2)$ is not convex, neiter concave [1],[7] , but it is quasiinvex on the half-plane $\{ (x_1, x_2) \mid x_1 - x_2 \leq 0 \}$.

$Q(x_1, x_2)$ is a quasiinvex function at $u =$ (0,0) with respect to η defined be

$$\eta(x_1, x_2, 0, 0) = \frac{1}{\sqrt{2}}(-x_1 x_2 + x_1 - x_2, x_1 x_2 - x_1 + x_2)'$$

because $\nabla f(0, 0) = (1, -1)' \neq 0$.

References

[1] A. Cambini and L. Martein, *Generalized Convexity and Optimization*, Springer, 2009.

[2] B. D. Craven, *Invex functions and constrained local minima*, Bull. Austral. Math. Soc., **24** (1981), 357-366.

[3] M. A. Hanson, *On sufficiency of the Kuhn-Tucker conditions*, J.Math. Anal.Appl.,**8**(1964), 84-89.

[4] Şt. Mititelu, *Generalized invexities and global minimum properties,* Balkan J. Geom. Appl., **2**(1997), *1*, 61-72.

[5] Şt. Mititelu, *Invex functions,* Rev. Roumaine Math. Pures, Appl., **49**(2004), *5-6*, 529-544.

[6] Şt. Mititelu, *Nonsmooth invex functions. The preinvex case,* Math.Reports, **8(58)**(2006), *4*, 435-452.

[7] Şt. Mititelu, *Generalized Convexities*, Fair Partners Pub., Bucharest, 2011.

[8] Şt. Mititelu and M. Postolache, *Nonsmooth invex functions via upper directional derivative of Dini,* J. Adv. Math. Stud., **4**(2011), *1*, 57-76.

[9] R. Pini, *Invexity and generalized convexity*, Optimization, **22**(1991), 513-525.

[10] T. Weir and B. Mond, *Pre-invex functions in multiple objective optimization,* J. Math. Anal.Apl., **136**(1988), *1*, 29-38.

VARIATIONS ON THE WEIERSTRASS APROXIMATION THEOREMS

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Abstract : This is a survey about the Weiestrass approximation theorems and some of their generalizations. The Weierstrass approximation theorems (W.A.T.) are two theorems published in 1885, when Weierstrass was 70 years old. The first theorem states the density of algebraic polynomials in the space of continuous real-valued functions on a finite interval, and the second, the density of trigonometric polynomials in the space of 2π-periodic continuous real-valued functions on $\mathbb{R}$,both, in the uniform norm. In the first part of the paper we present some various proofs of the first Weierstrass approximation theorem, following the recent survey of A. Pinkus ([1]).
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Key words: algebraic polynomials, trigonometric polynomials, density, uniform, convergence, positive operator, monotone operator, Chebyshev polynomials.

1. FIRST WEIERSTRASS APROXIMATION THEOREM

There are many proofs of the first W.A.T. We can divide these proofs into three groups: The first group contains the proofs based on singular integral (Weierstrass, Feyér, Landau and de la Valleé Pousin).

The second group is based on the idea of approximating a particular function (Runge/Phragmén, Lebesque, Mittag-Leffler and Lerch).

The third group contains the proofs which do not quite belong to either of the above groups (Lerch, Volterra and Bernstein). All this proofs appeared prior to 1913. There are other proofs developed, after 1913. Among this proofs, the Huhn's proof (1964) is very elegant and simple.

The original Weierstrass's proof is based on the following two facts: first, he states that if f is a continuous and bounded function on $\mathbb{R}$, then

$$f(x)=\lim_{k\to 0_+}\frac{1}{k\sqrt{\pi}}\int_{-\infty}^{\infty} f(u)e^{-\left(\frac{u-x}{k}\right)^2}du.$$

If we denote by

$$F(x,k)=\frac{1}{k\sqrt{\pi}}\int_{-\infty}^{\infty} f(u)e^{-\left(\frac{u-x}{k}\right)^2}du,$$

then, $F(\cdot,k)$ is an entire function for any $k>0$. Thus, for every continuous and bounded function f on $\mathbb{R}$, there exists a sequence of entire functions $F(x,k)$ such that

$$\lim_{k\to 0_+}F(x,k)=f(x).$$

Next, Weierstrass proves that on a finite interval, the convergence is not only pointwise, but also uniform. Furthermore, since $F(\cdot,k)$ is entire, the truncated power series of $F(\cdot,k)$

uniformly converges to $F(\cdot,k)$ on any finite interval, so, there exists a sequence of algebraic polynomials which converges (uniformly) to f.

From the second group of proofs, we present Lebesque's proof (1898, when he was a 23 years old student at the École Normale Supérieure). The proof is based on the following steps:

1. Every continuous real-valued function on $[a,b]$ can be uniformly approximated by a polygonal functions (a spline function of the first order) which interpolated f in the equidistant knots
$$x_i = a + i\frac{b-a}{n},\ i=\overline{0,n}.$$
2. Every spline function of the first order has the form:
$$s(x) = \alpha x + \beta + \sum_{i=1}^{n} C_i |x - x_i|.$$
3. The absolute value function $|x|$ can be uniformly approximated on the interval $[-1,1]$ by an algebraic polynomial. Thus,
$$|x| = \sqrt{1-(1-x^2)} = 1 + \frac{1}{2}(1-x^2) - \frac{1}{2\cdot 4}(1-x^2) + \ldots.$$

Let Q be the polynomial with the property:
$$\left||x| - Q(x)\right| < \varepsilon,\ \forall x \in [-1,1].$$

4. For any $M > 0$ we have:
$$\left|M\cdot Q\left(\frac{x}{M}\right) - |x|\right| = M\left|Q\left(\frac{x}{M}\right) - \left|\frac{x}{M}\right|\right| < M\cdot\varepsilon = \varepsilon',\ \forall x \in [-M, M].$$

From the third group of proofs, we remember the Berstein's proof (1912). For every continuous real-valued function on $[0,1]$, S. Berstein constructs the algebraic polynomial:
$$B_n(f)(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) C_n^k x^k (1-x)^{n-k},\ x \in [0,1],$$
and proves that the sequence $B_n(f)$ converges uniform to f on $[0,1]$. A similar result is true on an arbitrary interval $[a,b]$, by a change of variables.

2. STONE-WEIERSTRASS TYPE THEOREMS

A broad generalization of Weierstrass approximation theorem, is obtained by Marchal H. Stone in 1937. His result is known as the Stone-Weierstrass theorem. The $S-W$ theorem generalizes the Weierstrass approximation theorem in two directions: instead of the real interval $[a,b]$, an arbitrary compact Hausdorff space X is considered, and instead of the algebra of polynomial functions, approximation with elements from more general subalgebras of $C(X)$ is investigated. The $S-W$ theorem is a vital result in the study of the algebra of continuous functions on a compact Hausdorff space. Now we recall the $S-W$ theorem for the real case.

Theorem 1. *Let X be a compact Hausdorff space and let A be a subalgebra of $C(X,\mathbb{R})$ with the properties:*
A contains the constant functions;

A separates the points of X i.e., for any $x, y \in K, x \neq y$ there is an $a \in A$ such that $a(x) \neq a(y)$. Then A is dense in $C(X,\mathbb{R})$ ($\overline{A} = C(X,\mathbb{R})$).

Many proofs of this theorem have evolved since its discovery by M. H. Stone. From these proofs, we remind only the proof of L. de Branges (1959) which makes use of the means of functional analysis (the Hahn-Banach theorem and the Krein-Milman theorem) and the "elementary" proof of Bruno Brosowski and Frank Deutsch (1981), which depends only of the definition of compactness and of the Bernoulli inequality.

The conclusion of Theorem 1 is false in the complex case. Indeed, let $D = \{z \in \mathbb{C}; |z| \leq 1\}$ and let H be the algebra of all $h \in C(D,\mathbb{C})$ which are holomorphic in the interior of D. Obviously, H fulfills the assumptions of Theorem 1, but $H \neq C(K;\mathbb{C})$, because, for example, the function $f(z) = \operatorname{Re} z, z \in D$ is not in H.

The conclusion of Theorem 1 still holds in the complex case, if A is self-adjoint, i.e., $f \in A$ implies $\overline{f} \in A$.

In 1961 Erett Bishop extended the $S-W$ theorem for the algebras which are not self-adjoint. In order to present the Bishop's approximation theorem we recall some definitions and results.

Definition 1. *A subset S of K is called antisymmetric with respect to the subalgebra A of $C(K,\mathbb{C})$ if every $f \in A$ which is real-valued on S, is constant on S.*

Thus, for a real algebra A, an antisymmetric set is just a set of constancy for A. For a complex algebra, we present the following example. Let K be a compact set in $\mathbb{C}$, and let A be the algebra of all $f \in C(K,\mathbb{C})$ which is holomorphic in the interior $\overset{0}{K}$ of K. Then, every component S of $\overset{0}{K}$ is an antisymmetric set with respect to A.

Let K be a compact Haussdorff space and let A be a subalgebra of $C(K,\mathbb{C})$ containing the constants. The family $\mathcal{S}$ of all A- antisymmetric subsets of K has the following properties:

(i). For any $x \in K$, it results that $\{x\} \in \mathcal{S}$.

(ii). If $S_i \in \mathcal{S}$, $i = 1,2$, and $S_1 \cap S_2 = \phi$, then $S_1 \cup S_2 \in \mathcal{S}$.

(iii). $S \in \mathcal{S}$ implies that the closure $\overline{S} \in \mathcal{S}$.

Now, it is easy to see that every $x \in K$ belongs to a maximal A- antisymmetric set S_x and if $x \neq y$ then $S_x = S_y$ or $S_x \cap S_y = \phi$.

For any $S \subset K$ and $f \in C(X,\mathbb{C})$ we denote by $f|S$ the restriction of f to S. Obviously $A|S = \{a|S; a \in A\}$.

Theorem 2 (Bishop's approximation theorem)

Let A be a closed subalgebra of $C(K,\mathbb{C})$ that contains the constant function. Then:

(i) The family $\widetilde{\mathcal{S}}$ of all maximal A-antisymmetric subset of X forms a pairwise disjoint partition of K.

(ii) A function $f \in C(K,\mathbb{C})$ belongs to A iff $f\Big|S \in \overline{A|S}$ for any $S \in \widetilde{\mathcal{S}}$.

$A|S$ is closed in $C(S,\mathbb{C})$ for any $S \in \widetilde{\mathcal{S}}$.

Bishop's original proof is difficult to understand, because it is based on transfinite induction. Irving Glicksberg is the first who has given in 1962 a reasonable proof of Theorem 2. Some "elementary" proofs of Bishop's theorem were given by R. B. Burckel (1984) and T. J. Ranseford (1984).

Remark 1 If A is self-adjoint and A separates the points of K, then every A-antisymmetric set is a singleton, and thus, every $f \in C(K,\mathbb{C})$ trivially satisfies the condition $f\big|S \in \overline{A|S}$ for any $S \in \tilde{S}$, no $f \in \overline{A}$; that is $\overline{A} = C(K,\mathbb{C})$. Therefore Bishop's theorem is a generalization of the Stone-Weierstrass theorem for the complex case.

In 1978, the author of the present paper generalized Bishop's theorem, considering instead of subalgebras of $C(K,\mathbb{C})$, the vector subspaces of $C(K,\mathbb{C})$ (see [6]).

D. Feyel and A de la Pradelle expended in 1984 Bishop's approximation theorem for a convex cone. A version of the $S-W$ theorem is also true when X is only locally compact. An important generalization of the $S-W$ theorem for weighted spaces is given by L. Nachbin in 1965 (see [4]).

Some Bishop's type approximation theorem for real or complex locally convex lattices of (M) – type are given in [8] and [9]. Also, in [9] we show how various well known approximation theorems for weighted spaces , follows from this results.

3. KOROVKIN THEOREM

A very nice generalization of the Weierstrass approximation theorem is obtained by Pavel Korovkin in 1953.

Definition 2. *Let $C(K)$ be a Banach algebra of all real-valued functions on a compact K. An operator $U : C(K) \to C(K)$ is called positive if $f \geq 0$ implies $U(f) \geq 0$ and it is called monotone if $f \geq g$ implies $U(f) \geq U(g)$. If U is linear then U is positive if it is monotone.*

Theorem 3. (Korovkin 1953)

Let $K = [a,b]$, $f_0(x) = 1$, $f_1(x) = x$, $f_2(x) = x^2$, $\forall x \in [a,b]$ and let be $L_n : C[a,b] \to C[a,b]$ a sequence of positive linear operators. If $L_n(f_i) \xrightarrow{u} f_i, \forall i = 0,1,2$, then $L_n(f) \xrightarrow{u} f, \forall f \in C[a,b]$.

A similar result is true for the space of 2π - periodic continuous real-valued functions on $\mathbb{R}$. In this case the test functions are $f_0(1) = 1$, $f_1(x) = \cos x$ and $f_2(x) = \sin x$, $x \in [a,b]$.

The proof is based on the following idea. For every $f \in C([a,b])$ and $\forall t \in [a,b]$, there are two polynomial functions:

$$q_t^-(x) = a_1(t)x^2 + a_2(t)x + a_3(t),$$
$$q_t^+(x) = b_1(t)x^2 + b_2(t)x + b_3(t),\ x \in [a,b]$$

such that

$$q_t^- < f < q_t^+ \text{ and } \left|q_t^+(t) - q_t^-(t)\right| = 2\varepsilon .$$

Since L_n is monotone operator we have:

$$L_n\left(q_t^-\right) < L_n(f) < L_n\left(q_t^+\right)$$

According to the hypothesis it results that

$$L_n\left(q_t^-\right) \to q_t^- \text{ and } L_n\left(q_t^+\right) \to q_t^+ \text{, and thus}$$

$$q_t^-(t)-\varepsilon < L_n(f)(t) < q_t^+(t)+\varepsilon, \quad \forall t\in[a,b]$$

and every *n* sufficiently large. On the other hand we have:

$$q_t^-(t) < f(t) < q_t^+(t)$$

Hence, for any $t\in K$, $|L_n(f)(t)-f(t)| < 4\varepsilon, \forall t$, therefore $L_n(f)\to f$.

Remark 2. Bernstein's operators

$$B_n(f)(x)=\sum_{k=0}^{n} f\left(\frac{k}{n}\right)C_n^k x^k(1-x)^{n-k}, x\in[0,1],$$

are positive and linear on $C([0,1])$.

Moreover, we have

$$B_n(f_0)=f_0;\ B_n(f_1)=f_1 \text{ and } B_n(f_2)=\frac{n-1}{n}f_2+\frac{1}{n}f_1.$$

Since $B_n(f_i)\xrightarrow{u} f_i, \forall i=0,1,2$ it follows by Korovkin theorem that

$$B_n(f)\xrightarrow{u} f, \forall f\in C[0,1]$$

hence, every continuous function on [0,1] an be uniformly approximated by a algebraic polynomial (i.e. Weierstrass- approximation theorem).

Theorem 4. (Bohman-Korovkin [1957])

Let K compact and L_n a sequence of positive linear operators on $C(K)$. Assume that there exist two finite sets (a_i), $(f_i)\in C(K)$ such that

$$p_t(x)=\sum_{i=1}^{m} a_i(t)\,f_i(x)\ge 0,$$

with equality iff $x=t$. If $L_n(f_i)\to f_i, \forall i=\overline{1,m}$, then $L_n(f)\to f, \forall f\in C(K)$.

Two classical examples are:

$$K=[a,b],\ f_0=1,\ f_1=x,\ f_2=x^2,\ p_t(x)=(x-t)^2,$$

$$K=[0,2\pi],\ f_0=1,\ f_1=\cos x,\ f_2=\sin x,\ p_t(x)=1-\cos(x-t).$$

4. BORWEIN THEOREM

The Chebysev polynomials are defined by:

$$\mathrm{T_n}(\mathrm{x})=\cos[\mathrm{n}\ \mathit{arccosx}],\ \mathrm{x}\in[-1,1].$$

We have $T_0(x)=1$ and $T_1(x)=x$.

The T_n satisfy the recurrence relation:

$$T_{n+1}(x)=2xT_n(x)-T_{n-1}(x), \quad n=2,3,...$$

We compute easily

$$T_2(x) = 2x^2 - 1,\ T_3(x) = 4x^3 - 3x,\ T_4(x) = 8x^4 - 8x^2 + 1, \ldots$$

The degree of T_n is n, $\|T_n\| = 1$ and $T_n(x) = 2^{n-1}x^n + \ldots, \quad x \in [-1,1]$.

The zeros of T_n are precisely the points

$$z_k = \cos\frac{(2k+1)\pi}{2n}, \quad k = 1, 2, \ldots, n-1$$

and the extrema of $T_n(x)$ in $[-1,1]$ are attained precisely at the points:

$$\xi_k = \cos\frac{k\pi}{4},\ k = 0, 1, \ldots, n.$$

Observe that the zeros of T_n and T_{n+1} interlace, as do the extrema. More,

$$|z_i - z_{i-1}| = \left| -2\sin i \cdot \frac{\pi}{n} \cdot \sin\frac{\pi}{2n} \right| \le \frac{\pi}{n}, \quad \forall i$$

The nth Chebyshev polynomial has the following equi-oscillation property on [-1, 1]. There exist $(n+1)$ points $\xi_i \in [-1,1]$ with $-1 = \xi_n < \xi_{n-1} < \ldots < \xi_0 = 1$ such that:

$$T_n(\xi_j) = (-1)^{n-j} \|T_n\|_{[-1,1]} = (-1)^{n-j}, \quad j = 0, 1, \ldots, n$$

Definition 3. (Chebyshev System). *Let $A \subset [a,b]$ a compact with $(n+1)$ distinct points, and $f_0, f_1, \ldots, f_n$ a set at continuous real-valued funtions on A. The set $(f_0, f_1, \ldots, f_n)$ is called a real Chebyshev system of dimension $(n+1)$ on A if span$(f_0, \ldots, f_n)$ is an $(n+1)$ dimensional subspace of $C(A)$ and any element of space span$(f_0, \ldots, f_n)$ that has $(n+1)$ distint zeros in A is identically zero.*

A sequence $f_0, f_1, \ldots, f_m, \ldots$ is a Markov system on A if $f_0, f_1, \ldots, f_m$ is a Chebyshev system an A, for any $m = 0, 1, 2, \ldots$.

Examples of Markov systems:

a). $1, x, x^2, \ldots, x^m, \ldots$

b). $1, \cos x, \sin x, \ldots, \cos mx, \sin nx, \ldots$

Suppose that $f_0, f_1, \ldots, f_n$ is a Chebyshev system on A.

We can define the generalized Chebyshev polynomial $T_n = T_n[f_0, f_1, \ldots, f_n; A]$ by the following properties:

1). $T_n \in Span(f_0, f_1, \ldots, f_n)$;

2). There exists an alternating sequence $x_0 < x_1 < \ldots < x_n$ for T_n on A that is,

$sign(T_n(x_{n+1})) = -sign(T_n(x_n)) = \pm\|T_n\|_A, \quad i = \overline{0, n-1}$;

3). $\|T_n\|_A = 1$ with $T_n(\max A) > 0$.

One can prove the existence and uniqueness of such a T_n.

Let $(f_0, f_1, \ldots, f_n, \ldots)$ be a Markov system on $[a,b]$ and let $T_n = T_n\{f_0, f_1, \ldots, f_n; [a,b]\}$ be the generalized Chebyshev associated polynomials.

Denote by $(a \le) x_1 < x_2 < ... < x_n (\le b_n)$,thezeros of T $_n$. Let $x_0 = a$ and $x_{n+1} = b$.

The mesh of T_n is defined by

$$M_n = \max_{1 \le i \le n+1} |x_i - x_{i-1}|.$$

Theorem 5. (Borwein)

Suppose $M = \{1, f_1, f_2, ...\}$ *is an infinite Markov system on* $[a,b]$ *with each* $f_i \in C^1[a,b]$. *Then* Span$M$ *is dense in* $C[a,b]$ *iff* $\lim_{n \to \infty} M_n = 0$.

Corollary 1. *The algebra of polynomials is dense in* $C([-1,1])$.

Indeed, $M = \{1, x, x^2, ...\}$ is an infinite Markov system of C^1 functions on $[-1,1]$.

The associated Chebyshev polynomials are just the usual Chebyshev polynomials T_n and

$$M_n \le \frac{\pi}{n}, \quad n = 1, 2, ...$$

It fallows that $\lim_{n \to \infty} M_n = 0$,hence $span\, M$ is dense in $C[-1,1]$.

References

[1] Pinkus Allan: *The Weierstrass Approximation Theorems, Surveys in Approximation Theory*, Vol. 1, 2005, pp. 1-37;

[2] Borwein P.B., *Zeros of Chebyshev Polynomials in Markov systems, J. Approx. Theory* 63 (1990), 56-64.

[3] Bishop E., *A generalization of the Stone-Weierstrass theorem,* Pacific J. Math., 11(1961), 777-783.

[4] Nachbin L., *Weighted approximation for algebras and modules of continuous functions, real and self-adjoint complex cases,* Ann. Of Math., 81 (1965), 289-302.

[5] Korovkin P. P., *On convergence of linear positive operators in the space of continuous functions*, Dokl. Akad. Nauk. SSSR (N. S.), 90 (1953) pp 961-964 (In Russian)

[6] Păltineanu G., *A generalization of the Stone-Weierstrass theorem for weighted spaces,* Rev. Roum. Math. Pures Appl. 7 (1978), 1065-1068.

[7] Păltineanu G. , Vuza D. T., *A generalization of the Bishop approximation theorem for locally convex lattices of (AM)-type*, Rend. Circ. Mat. Palermo (2) Suppl. Mo.52. Vol II (1998), 687-694.

[8] Kravvaritis D., Păltineanu G., *A density theorem for locally convex lattices,* Abstract and Applied Analysis, 5(2004), 387-393.

[9] Kravvaritis D., Păltineanu G., *Some density theorems for complex locally convex lattices*, Rev. Roum. Math. Pures Appl., 53 (2008), 2-3, 167-179.

[10] Vuza D. T., *Elements of the theory of modules over ordered rings: Order Structures in Functional Analysis,* Vol 2, pp 172-283. Ed. Academiei Române, Bucureşti, 1989.

SERII TEMPORALE ALE POLUĂRII AERULUI ÎN CENTRUL BUCUREȘTIULUI

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Abstract
Using time series of NO, CO and PM 2.5 at the center of Bucharest we tried to predict the evolution of those pollutants using the techniques of the dynamic of chaos. The hourly measurements were made near the Military Club in downtown Bucharest and refer to the period 01/ian/2007-31/dec/2008. For the calculations we used TISEAN software package version 3.0. Some predictions were found astonishing goods.

1. Preliminarii teoretice

Analiza seriilor temporale de date măsurate în ideea predicției evoluției unui fenomen s-a făcut din multe puncte de vedere: statistic, cu rețele neuronale [6], prin metode de analiză a sistemelor dinamice cu comportări haotice [3]. În lucrarea de față adoptăm metoda sistemelor haotice, aşa cum este descrisă în [1], [2], [3]. O implementare practică a acestor metode se găseşte în [4] sau [5]. Ideea de bază a metodei este că datele urmează o evoluție deterministă

$$\bar{x}' = \bar{X}(x) \tag{1}$$

cu $\bar{x}$ într-un anumit spațiu normat finit sau infinit dimensional E, dupa o anumita perioada de timp traiectoria sistemului dinamic (1) evoluează în jurul unui atractor A. Dinamica în A poate fi surprinsă efectiv de aproape orice funcție $s : E \to R$.
Fie $\Phi(t, x)$ fluxul lui (1) pentru o valoare fixată a lui t=τ. Fie $F(\bar{x}) = \Phi(\tau, \bar{x})$, $F^{-1}(\bar{x}) = \Phi(-\tau, \bar{x})$, şi fie $s : U \to R$ definită într-o vecinatate a lui A. Atunci, conform unei teoreme a lui F. Takens (vezi [1], [2]), în condiții generale pentru s şi pentru un $m \in N$ suficient de mare, aplicația

$$\begin{aligned} &S : A \to R^m \\ &S(\bar{x}) = \left(s\left(F^{-(m-1)}(\bar{x})\right),\ s\left(F^{-(m-2)}(\bar{x})\right), \ldots s\left(F^{-1}(\bar{x})\right), \bar{x}\right) \end{aligned} \tag{2}$$

este o scufundare a lui A în R^m.
Pe imaginea atractorului $S(A) \subset R^m$ avem

$$S(\Phi(n\tau, \bar{x}_0)) = \bar{s}_n = \left(s_{n-(m-1)}, s_{n-(m-2)} \ldots s_{n-1}, s_n\right) \tag{3}$$

Dinamica lui F de pe atractorul A este exprimată pe imaginea sa S(A) prin

$$\bar{s}_n \to G(\bar{s}_n) = \bar{s}_{n+1}$$

ca în figura următoare:

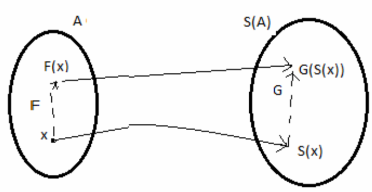


Fig.1

O uşoară modificare a lui (3) introducând o anumită întârziere este $d \in N^*$ conduce la

$$\bar{s}_n = \left(s_{n-(m-1)d}, s_{n-(m-2)d} \ldots s_{n-d}, s_n\right) \quad (4)$$

Intârzierea optima sugerată de Fraser şi Swinney pentru (4) este recomandată ca fiind primul minim al informației mutuale empirice (vezi [1], [2]):

$$H = \sum_{i,j} p_{i,j}(d) \ln\left(\frac{p_{i,j}(d)}{p_i \; p_j}\right) \quad (5)$$

Probabilitatile p_i și $p_{i,j}(d)$ în (5) sunt obținute prin partiționarea datelor $(s_n)_{1\le n\le T}$ într-un numar de subintervale al valorilor datelor măsurate: p_i este probabilitatea ca funcția s să ia valori în subintervalul i iar $p_{i,j}(d)$ este probabilitatea de tranziție de la subintervalul i la subintervalul j după d unități.

Pentru a determina m, dimensiunea de scufundare, folosim metoda celui mai apropiat vecin fals (vezi [1], [2], [3]). Ideea este că dacă $\bar{s}_j$ este cel mai apropiat vecin al lui $\bar{s}_i$ în R^m atunci $\bar{s}_{i+1} = G(\bar{s}_i)$ şi $\bar{s}_{j+1} = G(\bar{s}_j)$ sunt de asemenea apropiați și raportul $R_i = \| \bar{s}_{i+1} - \bar{s}_{j+1} \| / \| \bar{s}_i - \bar{s}_j \|$ este mai mic decât un prag euristic Rt. In caz contrar punctul $\bar{s}_j$ este marcat ca vecin fals. Dacă procentul de puncte cu un cel mai apropiat vecin fals este prea mare, consideram ca m este prea mică.

2. Metode de predicție

Predicția de ordin zero este data de

$$s_{n+k} = \frac{1}{card(U_n)} \sum_{\bar{s}_j \in U_n} s_{j+k} \quad (6)$$

unde U_n este o mică vecinătate a lui $\bar{s}_n$ în R^m. Predicția este făcută pentru $n+k>T$, ultima măsuratoare.

Predicția local liniară este dată de formula

$$s_{n+1} = \bar{a}_n \bar{s}_n + b_n \quad (7)$$

unde coeficienții $\bar{a}_n \in R^m$, $b_n \in R$ sunt determinați astfel încât

$$\sigma^2 = \sum_{\bar{s}_j \in U_n} \left(s_{j+1} - \bar{a}_n \bar{s}_j - b_n\right)^2 \quad (8)$$

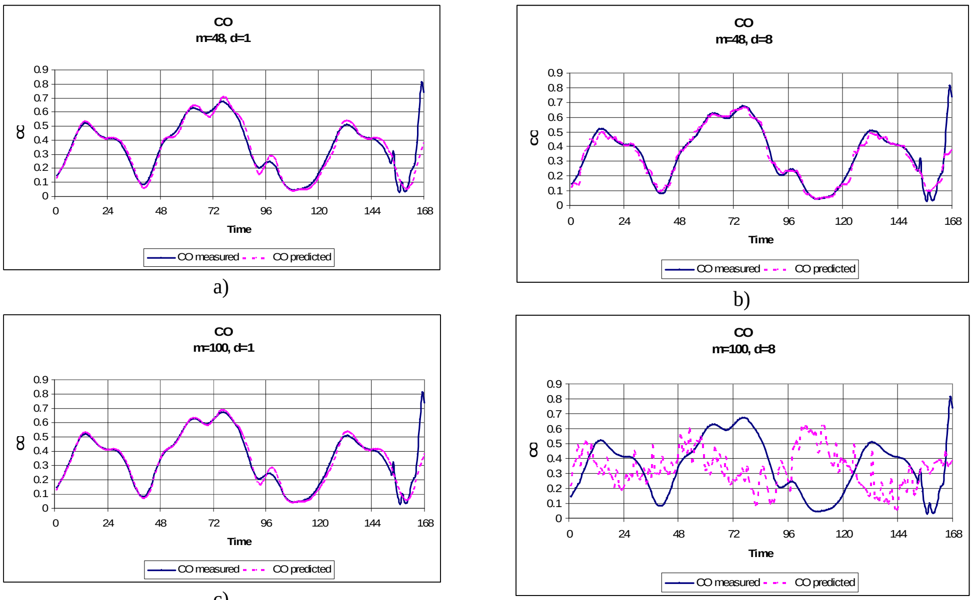
să fie minimă. Suma este efectuată după toți $\bar{s}_j$ dintr- o mică vecinătate U_n a lui $\bar{s}_n$ dar destul de mare încât să conțină suficiente puncte pentru a asigura ca problema celor mai mici patrate (8) să fie nesingulară.

3. **Rezultate**

Datele orare ale poluantilor provin de la stația din centrul oraşului Bucureşti, de lângă Cercul militar, din perioada 1/1/2007 01:00 - 12/30/2008 24:00. Ca date de lucru am utilizat esantionul dintre 1/1/2007 01:00 - 6/30/2008 24:00 şi am încercat să prezicem poluarea pentru următoarele câteva zile. Pentru efectuarea calculelor am folosit pachetul software TISEAN version 3.0 disponibil publicului pe http://www.mpipks-dresden.mpg.de/ ~ tisean (Institutul Max Planck pentru Fizica schemelor complexe, Dresda).

Pentru datele NO am gasit întârzierea optima d = 8 având aici fracțiunea de puncte cu cel mai

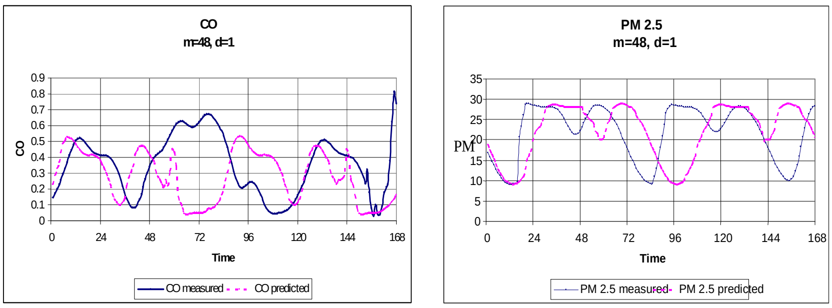
apropiat vecin fals sub 6,8% dacă $m \geq 30$. Pentru datele CO este optim d=8 şi acelaşi m ca pentru NO. Pentru datele PM 2.5 întârzierea optima este d = 21. Pentru dimensiunea de scufundare m am folosit $m = 48 \geq 30$, cât şi m = 100.
Experimental am constatat că vecinătatea U_n trebuie să fie suficient de mică, astfel ca, în (6), media să fie efectuată pentru $k \approx 10$ puncte. Prea multe puncte fac graficul valorilor prezise prea neted şi departe de valorile măsurate.
Reprezentările grafice ale predicțiilor în comparație cu cele ale valorilor măsurate este mai sugestivă şi o adoptăm în cele ce urmează:



a) b) c) d)

Fig. 2. Predicția individuală pentru CO este excelentă pentru d = 1, dar pentru întârzierea optimă d = 8 predicția scade pentru valori m mari (Fig. 2, d).

Predicții simultane ale poluanților sunt mai slabe ca cele individuale. Se pare că m=48 si d=1 sunt alegeri bune pentru predicția din locația aleasă.



a) **b)**

Fig. 3 Cele mai bune predictii simultane sunt obținute pentru d = 1 şi sunt semnificative numai pentru CO şi PM2.5 pentru un interval de timp mai mic de 20 de ore

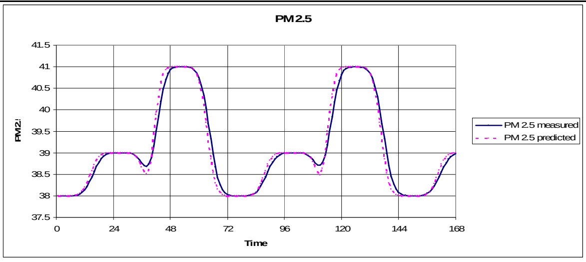
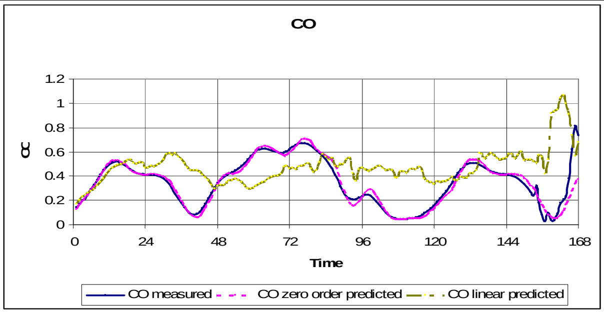
	

Fig. 4. Predictia individuala a PM2.5 folosind predictia local liniara	Fig.5. Pentru predicția local liniară, este necesar să se adapteze numărul minim de puncte în vecinătatea astfel încât minimizarea (8) sa nu fie cu matrice singulara. Am constatat experimental că este suficient un număr de puncte de patru ori numărul de parametri estimati.

4. **Concluzii**

1. Întârzierea d = 1 este o alegere bună, uneori mai bună ca întârzierea la care informația mutuală empirică ajunge la un minim local.
2. O alegere bună pentru k, numărul de puncte în formula (6) se află experimental. Pentru predicția local liniară o alegere bună a numărului de puncte în formula (8) este de cel puțin patru ori numărul de parametri de estimat.
3. Alegerea dimensiunii de scufundare astfel încât fracțiunea de puncte având cel mai apropiat vecin fals este mai mica de 6% , dă rezultate bune. De asemenea, este indicat să alegem m, astfel ca vectorul $\bar{s}_n$ să cuprindă datele unei perioade în cazul în care se suspectează că datele au o amumită periodicitate.
4. Este posibil ca datele să nu respecte o lege deterministă. În acest caz, orice alegere a lui m şi d nu este o alegere bună.
5. Previziunile pe termen scurt sunt în general mai bune. Pentru CO predicția bună pentru o săptămână (Fig.2, a, b, c) este surprinzător de bună.

Bibliografie

[1] Eckmann, J.P., D. Ruelle, 1985. *Ergodic theory of chaos and strange attractors*, Rev. Mod. Phys. 57, pp. 617-656

[2] H. D. I. Abarbanel, Reggie Brown, John J. Sidorowich, and Lev Sh. Tsimring, 1993. *The analysis of observed chaotic data in physical systems*, Rev M Phys, 65, pp. 1331-1392.

[3] Holger Kantz, Thomas Schreiber, 1997. Nonlinear Time Series Analysis, Cambridge University Press, Cambridge.

[4] Rainer Hegger, Holger Kantz, and Thomas Schreiber, 1999. *Practical implementation of nonlinear time series methods: The TISEAN package*, Chaos 9,.pp. 413-435.

[5] Rainer Hegger, Holger Kantz, and Thomas Schreiber, TISEAN 3.0.1, http://www.mpipks-dresden.mpg.de/~tisean/Tisean_3.0.1/index.html.

[6] Adriana Coman, Anda Ionescu, Yves Candau, 2008. *Hourly ozone prediction for a 24-h horizon using neural networks*, Environmental Modelling & Software, Vol 23, 12, Elsevier Amsterdam, pp. 1407-1421

REZOLVAREA FORMALĂ A SISTEMELOR DE ECUATII DIFERENTIALE CU COEFICIENTI CONSTANTI IN MATHCAD

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Abstract

We present an algorithm for determination of the fundamental solution of a system of linear differential homogenous equations with constant coefficients using Mathcad.

1. Preliminarii teoretice

Rezolvarea formală a ecuațiilor diferențiale liniare este un proces simplu care se învață în primul an la o universitate tehnică. Rezolvarea automată cu un program de calcul formal presupune anumite cerințe pentru acest program. In ce priveşte softul Mathcad abia de la versiunea 14 au fost vectorizate toate operațiile ceea ce a permis scrierea unui program de rezolvare formală a ecuațiilor diferențiale liniare cu coeficienți constanți.

Soluția sistemului de ecuații diferențiale

$$\begin{cases} y' = Ay \\ y(0) = V \end{cases} \qquad (1)$$

unde A este o matrice constantă nxn, iar V este un vector din R^n, este dată de formula

$$y(x) = e^{xA} \cdot V \qquad (2)$$

unde

$$e^{xA} = I + \frac{x}{1!}A + \frac{x^2}{2!}A^2 + \ldots + \frac{x^n}{n!}A^n + \ldots \qquad (3)$$

In relația (3), I este matricea identitate nxn.

Matricea nxn $R(x) = e^{xA}$ se numeşte soluția fundamentală a sistemului (1) şi este caracterizată de

$$\begin{cases} \frac{d}{dx}R(x) = A \cdot R(x) \\ R(0) = I \end{cases} \qquad (4)$$

Dacă $H(x)$ este o matrice nxn care verifică doar

$$\frac{d}{dx}H(x) = A \cdot H(x) \qquad (5)$$

atunci avem

$$R(x) = H(x) \cdot H^{-1}(0) \qquad (6)$$

Determinarea unei funcții matriceale H(x) care să verifice (5) cu condiția inițială

$$H(0) = C \qquad (7)$$

se poate face uşor dacă C este o matrice nesingulară care are pe coloane vectorii proprii generalizați ai matricei A. Dacă multiplicitatea valorii proprii λ este k şi V este un vector propriu generalizat al lui A ce corespunde acestei valori proprii (echivalent $(A - \lambda I)^k V = \overline{0}$), atunci

$$y_V(x) = e^{xA} \cdot V = e^{\lambda x}\left(I + \frac{x}{1!}A + \frac{x^2}{2!}A^2 + ... + \frac{x^{k-1}}{(k-1)!}A^{k-1}\right)V \qquad (8)$$

Spre deosebire de formula (3) în (8) suma este finită. Prin alegerea a n vectori proprii generalizați independenți care să formeze o matrice nesingulară C se rezolvă problema (5)+(7) şi de aici prin (6) rezultă o soluție fundamentală a ecuației omogene.

2. Programul în Mathcad

ORIGIN≡ 1

Avem nevoie de matricea identitate si de vectorul zero.

```
Id(N) :=  for i ∈ 1..N                zero(N) :=  for i ∈ 1..N
            for j ∈ 1..N                            m_i ← 0
              m_{i,j} ← if(i = j, 1, 0)             m
          m
```

Rezolvarea sistemului depinde de posibilitatea de determinare formală a valorilor proprii. Pentru matricea următoare acest lucru este posibil.

$$A := \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \qquad N := cols(A)$$

$$z0 := zero(N) \rightarrow \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Polinomul caracteristic $\quad |A - x \cdot Id(N)| \rightarrow x^4 - 4 \cdot x^3 + 16 \cdot x - 16$

Valorile proprii ale matricei A

$$vp := eigenvals(A) \rightarrow \begin{pmatrix} 2 \\ 2 \\ 2 \\ -2 \end{pmatrix}$$

Strângem într-o matrice cu două coloane valorile proprii şi multiplicitățile lor. Rutina următoare face acest lucru

```
fvpm(vp) := | gr ← 1
            | val ← vp_1
            | co ← 1
            | for i ∈ 2.. N
            |   | if vp_i = val
            |   |   | co ← co + 1
            |   |   | continue
            |   | m_{gr,1} ← val
            |   | m_{gr,2} ← co
            |   | val ← vp_i
            |   | co ← 1
            |   | gr ← gr + 1
            | m_{gr,1} ← val
            | m_{gr,2} ← co
            | m
```

vpm := fvpm(vp)

$$vpm = \begin{pmatrix} 2 & 3 \\ -2 & 1 \end{pmatrix}$$

In rutina următoare se determină coloanele nule ale unei matrice. Acest lucru este necesar deoarece prin determinarea vectorilor proprii ai matricei $(A-\lambda I)^k$, vectorii proprii ce corespund valorii proprii zero nu apar într-o poziție standard şi pentru determinarea lor din matricea $vp = eigenvecs((A-\lambda I)^k)$ se utilizează comanda poz0(vp).

```
poz0(m) := | p1 ← 0
           | co ← 0
           | for j ∈ 1.. cols(m)
           |   if m^⟨j⟩ = z0
           |     | co ← co + 1
           |     | p1 ← j  if  p1 = 0
           | p2 ← p1 + co − 1
           | poz_1 ← p1
           | poz_2 ← p2
           | poz
```

Vecorii proprii extraşi sunt introduşi în formula (8) şise obțin k soluții ale ecuației (1) dacă multiplicitatea valorii proprii este λ. Aceste k soluții formează tot atâtea coloane ale matricei H(x) din (5)+(7). Realizarea acestei etape se face în procedura următoare

$$
sol(x) := \left|
\begin{array}{l}
ms \leftarrow z0 \\
\text{for } i \in 1..rows(vpm) \\
\quad \left|
\begin{array}{l}
\lambda \leftarrow vpm_{i,1} \\
mi \leftarrow vpm_{i,2} \\
B \leftarrow A - \lambda \cdot identity(N) \\
Bker \leftarrow B^{mi} \\
vpi \leftarrow eigenvecs(Bker) \\
test \leftarrow Bker \cdot vpi \\
poz \leftarrow poz0(test) \\
spi \leftarrow submatrix\left(vpi, 1, N, poz_1, poz_2\right) \\
ni \leftarrow max(1, mi - 1) \\
six \leftarrow e^{x \cdot \lambda} \cdot \left[\sum_{i=0}^{ni} \left(\frac{x^i}{i!} B^i \right) \right] \cdot spi \\
ms \leftarrow augment(ms, six)
\end{array}
\right. \\
ms \leftarrow submatrix(ms, 1, N, 2, cols(ms)) \\
ms
\end{array}
\right.
$$

Soluția fundamentală este

$$
R(x) := sol(x) \cdot sol(0)^{-1} \text{ simplify } \rightarrow
\begin{bmatrix}
\frac{e^{-2 \cdot x} \cdot \left(3 \cdot e^{4 \cdot x} + 1\right)}{4} & \frac{sinh(2 \cdot x)}{2} & \frac{sinh(2 \cdot x)}{2} & \frac{sinh(2 \cdot x)}{2} \\
\frac{sinh(2 \cdot x)}{2} & \frac{e^{-2 \cdot x} \cdot \left(3 \cdot e^{4 \cdot x} + 1\right)}{4} & -\frac{sinh(2 \cdot x)}{2} & -\frac{sinh(2 \cdot x)}{2} \\
\frac{sinh(2 \cdot x)}{2} & -\frac{sinh(2 \cdot x)}{2} & \frac{e^{-2 \cdot x} \cdot \left(3 \cdot e^{4 \cdot x} + 1\right)}{4} & -\frac{sinh(2 \cdot x)}{2} \\
\frac{sinh(2 \cdot x)}{2} & -\frac{sinh(2 \cdot x)}{2} & -\frac{sinh(2 \cdot x)}{2} & \frac{e^{-2 \cdot x} \cdot \left(3 \cdot e^{4 \cdot x} + 1\right)}{4}
\end{bmatrix}
$$

Bibliografie

[1] Gavriil Păltineanu, Pavel Matei – *Ecuații Diferențiale şi Ecuații cu Derivate Parțiale cu Aplicații*, Matrix, Bucureşti, 2007
[2] L. S. Pontriaguine - *Équations Différentielles Ordinaires*, Mir, 1969
[3] PTC – *Mathcad 14 Professional*

STABILITY ANALYSIS OF A HYSTERETIC STRUCTURAL SYSTEM

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Abstract: A structural isolation scheme, modeled as single degree-of-freedom system, with a restoring force characterizing a highly asymmetric hysteretic behavior is considered. Stability of the zero solution is analyzed for a switched non-linear system.
Mathematics Subject Classification (2000): 93D05, 47J40
Key words: Dynamic response, hysteresis, switched system, common quadratic Lyapunov function.

1. Introduction

The Bouc-Wen model, widely used in structural and mechanical engineering, gives an analytical description of a smooth hysteretic behavior. It was introduced by Bouc and extended by Wen, who demonstrated its versatility by producing a variety of hysteretic characteristics. In practice, hysteresis loops of structural systems may exhibit highly asymmetric shape due to asymmetry in geometry boundary conditions, or material properties [4]. The study of stability of equilibria in switched systems of differential equation is an active area of research in recent years (see Liberzon, 2003; Shorten et al., Margaliot, 2006). Halanay and Ursu [2] proved that existence of a common quadratic Lyapunov function (CQLF) for some lower dimensional linear systems is sufficient to ensure local uniform stability of the zero solution of the switched non-linear system and a regular asymptotic behavior. Zhu, Cheng and Qin construct CQLF for a class of stable matrices. They combine the Lie algebra structures with Narendra Balakrishnan (N-B) structure to enlarge the applicability of N-B structures [5].

2. The mathematical model

Considering a base isolation scheme, modeled as a single degree-of-freedom characterizing a hysteretic behavior of the isolator material, the system is described by the second order differential equation

$$m\ddot{x} + c\dot{x} + \Phi(x,t) = f(t) \tag{2.1}$$

$$\Phi(x,t) = \alpha k x + (1-\alpha) k z$$

$$\dot{z} = \dot{x}(A - |z|^n \Psi(x,\dot{x},z))$$

with mass $m>0$, viscous damping $c>0$ and a restoring force $\Phi(x,t)$, assumed to be described by the generalized Bouc-Wen model for highly asymmetric hysteresis. In real applications, the base isolation devices are designed to dissipate the energy introduced in the structure by external perturbations. In the absence of disturbances $f(t)=0$, the structures is in free motion so that when its initial conditions are not zero, the structure dissipates the energy due to the initial condition and keeps at rest asymptotically [3]. Next are introduced the notations $c/m = 2\zeta\omega,\ \ k/m = \omega^2$ and equation (2.1) becomes

$$\ddot{x} + 2\zeta\omega\dot{x} + \alpha\omega^2 x + (1-\alpha)\omega^2 z = 0\,,\ \ \dot{z} = \dot{x}(A - |z|^n \Psi(x,\dot{x},z)) \tag{2.2}$$

It is desirable develop a shape-control function Ψ that can assume different values for all the phases of a full cycle as determined by the signs of x, $\dot{x}$ and z. The following shape-control function is proposed by Der Kiureghian and Song [4]

$$\Psi(x,\dot{x},z)=\beta_1\,\mathrm{sgn}(\dot{x}z)+\beta_2\,\mathrm{sgn}(x\dot{x})+\beta_3\,\mathrm{sgn}(xz)+\beta_4\,\mathrm{sgn}(\dot{x})+\beta_5\,\mathrm{sgn}(z)+\beta_6\,\mathrm{sgn}(x) \quad (2.3)$$

Taking in (2.2) $n=1$, the state variables as $x_1:=x,\ x_2:=\dot{x}$, $x_3:=z$ the canonical first order system derived from equations (2.2) splits into six systems

$$\begin{cases}\dot{x}_1=x_2\\ \dot{x}_2=-2\zeta\omega x_2-\alpha\omega^2x_1-(1-\alpha)\omega^2x_3,\\ \dot{x}_3=Ax_2-x_2x_3\Psi_1\end{cases}\quad \begin{cases}\dot{x}_1=x_2\\ \dot{x}_2=-2\zeta\omega x_2-\alpha\omega^2x_1-(1-\alpha)\omega^2x_3,\\ \dot{x}_3=Ax_2-x_2x_3\Psi_2\end{cases} \quad (2.4)$$

$$\begin{cases}\dot{x}_1=x_2\\ \dot{x}_2=-2\zeta\omega x_2-\alpha\omega^2x_1-(1-\alpha)\omega^2x_3,\\ \dot{x}_3=Ax_2+x_2x_3\Psi_3\end{cases}\quad \begin{cases}\dot{x}_1=x_2\\ \dot{x}_2=-2\zeta\omega x_2-\alpha\omega^2x_1-(1-\alpha)\omega^2x_3,\\ \dot{x}_3=Ax_2+x_2x_3\Psi_4\end{cases}$$

$$\begin{cases}\dot{x}_1=x_2\\ \dot{x}_2=-2\zeta\omega x_2-\alpha\omega^2x_1-(1-\alpha)\omega^2x_3,\\ \dot{x}_3=Ax_2+x_2x_3\Psi_5\end{cases}\quad \begin{cases}\dot{x}_1=x_2\\ \dot{x}_2=-2\zeta\omega x_2-\alpha\omega^2x_1-(1-\alpha)\omega^2x_3\\ \dot{x}_3=Ax_2-x_2x_3\Psi_6\end{cases}$$

3. Stability analysis of the zero solution

Consider the switched system

$$\begin{aligned}\dot{\xi}&=D(\mu_k)\xi+F_\mu(y,\xi)\\ \dot{y}&=Y_\mu(y,\xi)\end{aligned} \quad (3.1)$$

where $\xi\in\Re^n$, $y\in\Re$, $D:\Omega\to M_n(\Re)$ is continuous, $\Omega\in\Re^p$, $F_\mu(y,0)=Y_\mu(y,0)=0$

THEOREM 3.1 Suppose $D(\mu_k)$ is Hurwitz and there exists $P=P^t>0$ such that

$$D(\mu)^tP+PD(\mu)\le -cI<0,\quad \forall\mu\in\Omega \quad (3.2)$$

for some positive c. Then the zero solution of (3.1) is uniformly stable. Moreover, There exists $\delta>0$ such that if $\|(y(0),\xi(0))\|<\delta$, then $\lim_{t\to\infty}\xi_i(t)=0,\ \ \forall i=\overline{1,n}$ whenever (y,ξ) is a solution of (3.1).

Define $\Psi_{\min}=\min(\Psi_1,\Psi_2,\Psi_6)$, $\Psi_{\max}=\max(\Psi_3,\Psi_4,\Psi_5)$ and let x_0 be a reference input such that

$$-\frac{A(1-\alpha)+\alpha}{\alpha\Psi_{\min}}<x_0<\frac{A(1-\alpha)+\alpha}{\alpha\Psi_{\max}} \quad (3.3)$$

$$\hat{x}_1=x_0,\ \hat{x}_2=0,\ \hat{x}_3=-\frac{\alpha}{(1-\alpha)}x_0 \quad (3.4)$$

Then (3.4) are equilibria for (2.5). Translate these equilibria into zero by

$$y_1=x_1-\hat{x}_1,\ y_2=x_2,\ y_3=x_3-\hat{x}_3, \quad (3.5)$$

The systems (2.4) are transformed into

$$\begin{cases}\dot{y}_1=y_2\\ \dot{y}_2=-2\zeta\omega y_2-\alpha\omega^2(y_1+x_0)-(1-\alpha)\omega^2(y_3-\frac{\alpha}{1-\alpha}x_0),\\ \dot{y}_3=Ay_2-y_2(y_3-\frac{\alpha}{1-\alpha}x_0)\Psi_i\end{cases} \quad (3.6)$$

Let A_1,..,A_6 the Jacobian matrices calculated in zero for (3.6), respectively

$$A_i = \begin{bmatrix} 0 & 1 & 0 \\ -\alpha\omega^2 & -2\zeta\omega & -(1-\alpha)\omega^2 \\ 0 & A + \dfrac{\alpha}{1-\alpha} x_0 \Psi_1 & 0 \end{bmatrix},. \tag{3.7}$$

It is easy to see that the characteristic polynomials of A_1-A_6, denotes as P_1-P_6, respectively, have zero as a root so $P_i(\lambda) = \lambda P_{i1}(\lambda)$, $i = 1..6$,

THEOREM 3.2. Suppose P_{i1} in are stable polynomials. Then the zero solution of (3.6) is simply stable by Lyapunov. Moreover, $\forall \varepsilon > 0$, $\exists \delta_\varepsilon > 0$ such that if $\|y(0)\| < \delta_\varepsilon$ then $\lim_{t\to\infty} |y_i(t)| < \varepsilon$, thus, in (2.5) if $\|x(0) - \hat{x}\| < \delta_\varepsilon$, $\lim_{t\to\infty} |x_i(t) - \hat{x}_i| < \varepsilon$.

The proof of Theorem 3.2 relies on the classical Lyapunov-Malkin theorem. In the same vein, Theorem 2.1 will be used to prove a similar result for the switched system (2.5). We follow several stages to bring systems (3.6) to the canonical form to which Theorem 3.1 will be applied.

Write systems as

$$\begin{cases} \dot{\xi} = D_i \xi + F^{(i)}(y,\xi) \\ \dot{y} = \tilde{Y}^{(i)}(y,\xi) \end{cases} \tag{3.10}$$

where $\xi = (\xi_1, \xi_2)$ and $F^{(i)}(y,0) = 0$

$$D_i = \begin{bmatrix} 0 & 1 \\ -\omega^2[A(1-\alpha) + \alpha + \alpha\Psi_i x_0] & -2\zeta\omega \end{bmatrix} \tag{3.11}$$

From the hypothesis that P_{11}- P_{16} are stable polynomials, it follows that $D_1..D_6$ are Hurwitz matrices.

THEOREM 3.3 In the setting of Theorem 3.1, suppose that there exists a CQLF for the family of systems

$$\dot{\xi} = D_i \xi \tag{3.12}$$

Then the zero solution of the switched system (3.12) is uniformly stable.

Theorem 3.3 relies entirely on the hypothesis on the existence of a CQLF for (3.12).

The N-B method for constructing CQLF has been generalized in [5] to the case where the set of stable matrices are not commutative.

THEOREM 3.4 Consider the set A of stable matrices and choose any $A_i \in A$, denote it as A_N $[A_N, A_j] = C_{N,j}$, $1 \le j \le N$. Choose any $P_{N-1} > 0$, set $P_N A_N + A_N^T P_N = -P_{N-1}$ and define $P_{i,j} = P_j A_i + A_i^t P_j$, $i = 1..N$, $j = N-1, N$. If $P_{i,N-1} + P_N C_{N,i} + C_{N,i}^T P_N < 0,\ i = 1..N-1$ then the P_N constructed as

$$P_N = \int_0^\infty e^{A_N^t t_N} \int_0^\infty e^{A_{N-1}^t t_{N-1}} ... \int_0^\infty e^{A_1^t t_1} P_0 e^{A_1 t_1} dt_1 e^{A_2 t_2} dt_2 ... e^{A_N t_N} dt_N \tag{3.13}$$

is a CQLF of ***A***.

4. Numerical results

Considering

$$\begin{aligned} &\omega = 2\pi,\ A = 1,\ \beta_1 = 1,\ \beta_2 = -0.7,\ \beta_3 = 0.5,\ \beta_4 = -0.1, \\ &\beta_5 = -0.1,\ \beta_6 = -0.1, \alpha = 0.1, \zeta = 0.2,\ x_0 = 0.1 \end{aligned} \tag{4.1}$$

then

$$D_1 = \begin{bmatrix} 0 & 1 \\ -39.508 & -2.513 \end{bmatrix}, D_2 = \begin{bmatrix} 0 & 1 \\ -39.49 & -2.513 \end{bmatrix}, D_3 = \begin{bmatrix} 0 & 1 \\ -39.44 & -2.513 \end{bmatrix}, \tag{4.2}$$
$$D_4 = \begin{bmatrix} 0 & 1 \\ -39.44 & -2.513 \end{bmatrix}, D_4 = \begin{bmatrix} 0 & 1 \\ -39.476 & -2.513 \end{bmatrix}, D_6 = \begin{bmatrix} 0 & 1 \\ -39.532 & -2.513 \end{bmatrix}$$

are Hurwitz and choose a positive definite matrix P_0 and define $P_i > 0,\ i = 1..6$ recursively by

$$P_i D_i + D_i^t P_i = -P_{i-1},\ i = 1..6 \tag{4.3}$$

has an analytic expression as

$$P_6 = \int_0^\infty e^{D_6^t t_6} \int_0^\infty e^{D_5^t t_5} ... \int_0^\infty e^{D_1^t t_1} P_0 e^{D_1 t_1} dt_1 e^{D_2 t_2} dt_2 ... e^{D_6 t_6} dt_6 \tag{4.4}$$

and is called the N-B type of CQLF with initial matrix P_0.

Setting $P_0 = I_2$ and using N-B structure, we can calculate positive definite matrices P_i, sequentially and define

$$P_{i,j} = P_j D_i + D_i^t P_j,\ [D_6, D_i] = C_{6,i},\ i = 1..6,\ j = 5,6 \tag{4.5}$$

It is easy to check that

$$P_{i,5} + P_6 C_{6,i} + C_{6,i}^t P_6 < 0,\ i = 1..5 \tag{4.6}$$

Thus P_6 is a CQLF of D_i, $i = 1..6$.

5. Concluding remarks

The paper has analyzed the asymptotic behavior of a one degree of freedom structural system with a hysteretic restoring force represented by the generalized Bouc-Wen model for highly asymmetric hysteresis.

The theoretical result of the extension of the Lyapunov–Malkin theorem on the (simple) stability of the zero solution in the critical case of a simple zero eigenvalue for the Jacobian matrix calculated in zero to a switched system in the same critical situation is used.

The CQLF constructed in (4.6) is sufficient to ensure local uniform stability of the zero solution of the switched non-linear system (2.2) and a regular asymptotic behavior.

References

[1] Popescu, I., Halanay, A.: *Stability analysis for a switched system issued from a hysteretic Bouc-Wen model*, The International Conference of Differential Geometry and Dynamical Systems, 2010.

[2] Halanay, A., Ursu, I: *Stability of equilibria of some switched non-linear systems with applications to control synthesis for electrohydraulicservomechanisms*, IMA Jurnal of Applied Mathematics, 2009.

[3] Ikhouane, F., Manosa, V., Rodellar, J. : *Dynamic properties of the hysteretic Bouc-Wen model*, Systems & Control Letters 56, 197-205, 2007.

[4] Song, J., Der Kiureghian, A.: G*eneralized Bouc-Wen Model for Highly Asymmetric Hysteresis*, Journal of Engineering Mechanics, Vol. 132, No. 6, June 1, 2006.

[5] Zhu, Y.H., Cheng, D.Z., Qin, H.S.: *Constructing Common Quadratic Lyapunov Functions for a Class of Stable Matrices*, Acta Automatica Sinica, vol. 33, No.2, 2007

LAGRANGE-JACOBI AND SUNDMAN RELATIONS IN THE N-BODY PROBLEM ATTACHED TO QUASI-HOMOGENEOUS POTENTIALS

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Abstract: We prove relations analogous to the Lagrange–Jacobi equality and to Sundman inequality from three body problem, in the framework of the *n*-body problem attached to quasi-homogeneous potentials.
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Key words: *n*-body problem, quasi-homogeneous fields, collision

1. Introduction

Newton was the first to study a quasi-homogeneous model in classical celestial mechanics. He considered a gravitational force deriving from a potential of the form $A/r + B/r^2$. After Newton, such a potential was considered by Clairaut. In our days, Delgado et al. ([1]), Diacu et al. ([2]), Mioc and Stoica ([9]) or Mioc and Stavinschi ([7], [8]) have considered quasi-homogeneous models.

In this paper we shall tackle a much more general model of quasi-homogeneous potential which covers all the above quoted models and many others.

We will call *quasi-homogeneous* a potential having the form of a sum of homogeneous potentials:

$$U(\mathbf{r}) = \sum_{k=1}^{N} U_k(\mathbf{r}) = \sum_{k=1}^{N} \frac{A_k}{|\mathbf{r}|^{\gamma_k}}, \tag{1.1}$$

where the parameters A_k have different analytical expressions according to the field they characterize (but they depend neither on **r**, nor explicitly on time), γ_k are real numbers ($\gamma_k < \gamma_{k+1}, k = \overline{1, N-1}$), whereas **r** stands for the radius vector of one particle with respect to another in the force field generated by this potential. We observe that the potential (1.1) is much more general than the above quoted ones for a twofold reason: (i) γ_k may run all along the real axis; (ii) such a model allows the study of particle dynamics under hybrid forces of totally different nature. In many applications in astronomy, the expression (1.1) represents a truncated series. However, we also consider here the case $N = \infty$ for generality, even if in studies of concrete situations N is finite. Our results provide a unifying viewpoint (physical and mathematical) for a lot of problems of particle dynamics.

2. Basic equations

Let us consider a system of n interacting particles $m_i > 0,\ i = \overline{1,n}$; let $\mathbf{r}_i = (x_i, y_i, z_i) \in \mathbf{R}^3$ be their position vectors with respect to an arbitrary origin; let $\mathbf{r} = (\mathbf{r}_1, \mathbf{r}_2, ..., \mathbf{r}_n) \in \mathbf{R}^{3n}$ be the configuration of the system. Let the motion of the system be ruled by a quasi-homogeneous force deriving from a potential function of the form (1.1), in which

$$U_k(\mathbf{r}) = \sum_{1 \le i < j \le n} \frac{A_{k,ij}}{r_{ij}^{\gamma_k}}. \tag{2.1}$$

Here $U_k : (\mathbf{R}^{3n} \setminus \Delta) \to \mathbf{R}$ for $\gamma_k > 0$, and $U_k : \mathbf{R}^{3n} \to \mathbf{R}$ for $\gamma_k \le 0$; $r_{ij} = |\mathbf{r}_i - \mathbf{r}_j|$; $\Delta = \cup_{1 \le i < j \le n} \{\mathbf{r} \mid \mathbf{r}_i = \mathbf{r}_j\}$ is the collision set, whereas $A_{k,ij} : \mathbf{R}^2 \to \mathbf{R}$, are symmetric functions (mainly of masses, but not only, as we shall see in the last section): $A_{k,ij} = A_{k,ji}$.

The dynamics of this n-body system in such a field is described by the vectorial equation

$$m_i \ddot{\mathbf{r}}_i = \frac{\partial U(\mathbf{r})}{\partial \mathbf{r}_i} = -\sum_{1 \le i < j \le n} (\mathbf{r}_i - \mathbf{r}_j) \sum_{k=1}^{N} \gamma_k \frac{A_{k,ij}}{r_{ij}^{\gamma_k + 2}}. \tag{2.2}$$

To be able to tackle the case $N = \infty$ too, we state that the series of functions $\sum_{k=1}^{\infty} \gamma_k A_{k,ij} / r_{ij}^{\gamma_k + 2}$ converges uniformly on $\mathbf{R}^{3n} \setminus \Delta$. Because the series $\sum_{k=1}^{\infty} A_k / |\mathbf{r}|^{\gamma_k}$ is simply convergent to $U(\mathbf{r})$ and the series of derivatives $\sum_{k=1}^{\infty} \gamma_k A_{k,ij} / r_{ij}^{\gamma_k + 2}$ is uniformly convergent, then, by the Theorem of differentiation term by term of the series of functions, the series of derivatives tends to $\partial U(\mathbf{r}) / \partial \mathbf{r}_i$ and is continuous on $\mathbf{R}^{3n} \setminus \Delta$.

It is clear that, putting $\mathbf{q}_i = \mathbf{r}_i$, $\mathbf{q} = \mathbf{r}$ (the configuration vector), $\mathbf{p}_i = m_i \dot{\mathbf{r}}_i$, $\mathbf{p} = (\mathbf{p}_1, \mathbf{p}_2, ..., \mathbf{p}_n) \in \mathbf{R}^{3n}$ (the momentum vector), and defining $H(\mathbf{q}, \mathbf{p}) = T(\mathbf{p}) - U(\mathbf{q})$ as the Hamiltonian function (where T is the kinetic energy), equations (2.2) can be transposed into a canonical form.

Results of the theory of differential equations ensure, for given initial conditions $(\mathbf{r}, \dot{\mathbf{r}})(t = 0)$, the existence and uniqueness of an analytic solution of the system (2.2), defined on an interval (t^-, t^+), $t^- < 0 < t^+$. This can be analytically extended to a maximal interval $(\tilde{t}^-, \tilde{t}^+)$, $-\infty \le \tilde{t}^- \le t^- < 0 < t^+ \le \tilde{t}^+ \le +\infty$. If $\tilde{t}^{\pm} = \pm\infty$, the solution is regular; else, it encounters a singularity.

There is no difficulty to prove that there exist ten classical first integrals for the system (2.2): the integrals of momentum $\sum_{i=1}^{n} m_i \dot{\mathbf{r}}_i = \boldsymbol{\alpha}, \boldsymbol{\alpha} \in \mathbf{R}^3$; the integrals of mass centre $\sum_{i=1}^{n} m_i \mathbf{r}_i - (\sum_{i=1}^{n} m_i \dot{\mathbf{r}}_i)\, t = \boldsymbol{\beta}, \boldsymbol{\beta} \in \mathbf{R}^3$; the integrals of angular momentum

$$\sum_{i=1}^{n} (m_i \mathbf{r}_i \times \dot{\mathbf{r}}_i) = \mathbf{C}, \quad \mathbf{C} \in \mathbf{R}^3; \tag{2.3}$$

and the integral of energy

$$T(\dot{\mathbf{r}}) - U(\mathbf{r}) = h, \quad h \in \mathbf{R}, \tag{2.4}$$

where $\boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{C}$ and h are integration constants. In the last relation, the kinetic energy of the system has the expression

$$T : \mathbf{R}^{3n} \to [0, +\infty), \quad T(\dot{\mathbf{r}}) = \frac{1}{2}\sum_{i=1}^{n} m_i \, |\dot{\mathbf{r}}_i|^2 .$$

3. Lagrange–Jacobi relation and Sundman-type inequalities

Consider the n-body system of interacting particles $m_i > 0,\ i = \overline{1,n}$, and let $\mathbf{r}_i = (x_i, y_i, z_i) \in \mathbf{R}^3$ be their position vectors with respect to the origin $\mathbf{0} = (0,0,0) \in \mathbf{R}^3$. The moment of inertia $J(\mathbf{r})$ of the system is defined by

$$J(\mathbf{r}) = \frac{1}{2}\sum_{i=1}^{n} m_i \, |\mathbf{r}_i|^2 . \tag{3.1}$$

The moment of inertia represents a physical measure of the distribution (scattering) of the bodies (particles) in space.

THEOREM 3.1. *In the n-body problem associated to a quasi-homogeneous field, the following relation holds:*

$$\ddot{J}(\mathbf{r}) = \sum_{k=1}^{N} (2 - \gamma_k) U_k(\mathbf{r}) + 2h ,$$

where $\ddot{J}(\mathbf{r})$ *is the second derivative of* $[J(\mathbf{r})]$ *with respect to the time.*

Within the Newtonian model, the inequalities of Sundman connect the moment of inertia and the angular momentum (of course, under the respective potential). We shall prove that inequalities of this type hold within the quasi-homogeneous models, too.

THEOREM 3.2. *In a quasi-homogeneous field, the following inequality holds:*

$$|\mathbf{C}|^2 \le 2J(\mathbf{r})[\ddot{J}(\mathbf{r}) + \sum_{k=1}^{N} \gamma_k U_k(\mathbf{r})] .$$

THEOREM 3.3. *In a quasi-homogeneous field, an inequality stronger than previous holds:*

$$|\mathbf{C}|^2 \le 2J(\mathbf{r})[\ddot{J}(\mathbf{r}) + \sum_{k=1}^{N} \gamma_k U_k(\mathbf{r})] - [\dot{J}(\mathbf{r})]^2 .$$

4. Applications

We shall point out some concrete fields covered by the quasi-homogeneous model featured by Definition 1.1. Our previous results are obviously valid within all these fields.

4.1. Classical gravitational models:

– Manev's model, with $N = 2$, $\gamma_1 = 1$, $\gamma_2 = 2$;

– The problem of satellite dynamics, with $N = 2$, $\gamma_1 = 1$;

– The zonal satellite problem, with $N \to \infty$, $\gamma_1 = 1$, $\gamma_2 = 0$, $\gamma_k = k \ (k \ge 3)$.

4.2. Relativistic gravitational models:

– Schwarzschild model, with $N=2$, $\gamma_1=1$, $\gamma_2=3$;

– Fock's model, with $N=4$, $\gamma_k=k, k=\overline{1,4}$;

– Schwarzschild – de Sitter model, with $N=3$, $\gamma_1=-2$, $\gamma_2=1$, $\gamma_3=3$.

4.3. Classical non gravitational models:

– The Lennard-Jones model with $N=2$, $\gamma_1=6$, $\gamma_2=12$;

– A planetary magnetic field, with $N\to\infty$, γ_k integers.

4.4. Mixed classical models:

– The photo gravitational model (gravitation plus radiation) with a non-Newtonian gravitational force.

– The gravito-elastic model, with $N=2$, $\gamma_1=-2$, $\gamma_2=1$;

– Models from atomic physics: the potential energy of an outward electron in the field of the nucleus, with $N=2$, $\gamma_1=1$, $\gamma_2=2$.

References

[1] Delgado, J., Diacu, F., Lacomba, E. A., Mingarelli, A., Mioc, V., Perez, E., Stoica, C.: The global flow of the Manev problem, *J. Math. Phys.*, **34** (1996), 2748-2761.

[2] Diacu, F., Mingarelli, A., Mioc, V., Stoica, C.: The global flow of the Manev problem in R. P. Agarwal (ed.), *Dynamical Systems and Applications*, World Scientific Series in Applicable Analysis, Vol. **4**, World Scientific, Singapore, 1996.

[3] Fock, V. A.: *The Theory of Space, Time and Gravitation*, Pergamon Press, New York, London, Paris, Los Angeles, 1959.

[4] Mioc, V., Pérez-Chavela, E.: The 2-Body Problem Under Fock's Potential, *Discrete Cont. Dyn. Syst.*, ser. S, **1** (2008), 611-629.

[5] Mioc, V., Popescu, E., and Popescu, N.A.: Phase-space structure in Lennard-Jones-type problems, *Rom. Astron. J. Suppl.* **18** (2008), 129-148.

[6] Mioc, V., Popescu, E., and Popescu, N.A.: Groups of symmetries in Lennard-Jones-type problems, *Rom. Astron. J.* **18** (2008), 151-166.

[7] Mioc, V. and Stavinschi, M.: Binary collisions in quasi-homogeneous fields, *Phys. Lett. A* **279** (2001), 223-225.

[8] Mioc, V., Stavinschi, M.: On the Schwarzschild-Type Polygonal (n+1)-BODY Problem and on the Associated Restricted Problem, *Serb. Astron. J.*, **158** (1998), 637-651.

[9] Mioc, V., Stoica, C.: The Schwarzschild Problem in Astrophysics, *Astrophysics and Space Science*, **249** (1997), 161-173.

AGGREGATE LOSS FOR A SPECIAL CLASS OF CLAIMS

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Abstract: The following three values can be associated to any insurance policy: the claim the company has to pay in case that a damage occurs, the incoming, and their difference, more precisely the loss. We assume this convolution is also a mixture of the two corresponding distribution functions. We determine in this case the total loss of the company at a certain moment of time.
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Key words: insurance theory, compound sum, aggregate loss, convolution – mixture.

1. The general hypothesis

Let $(X_n)_n$, $(Y_n)_n$ be two sequences of independent identically distributed random variables. For the n^{th} insurance policy, let X_n represent the value of the claim, Y_n the company incoming, and $\xi_n \overset{not}{=} X_n + Y_n$ the loss corresponding to the n^{th} contract.

Let also N_t be the number of all losses until moment t with $N_0 \overset{def}{=} 0$. We assume that $(X_n)_n$, $(Y_n)_n$, $(N_t)_t$ are independent. We now consider the following additional hypothesis on the loss and on the counting process:

i) X_n and Y_n are discrete random variables satisfying $F_X([0,\infty)) = F_Y((-\infty,0]) = 1$, where F_X, F_Y are the distribution function of *X*, and *Y*, respectively and

$$\exists \quad c \in (0,1) \text{ such that } F_X * F_Y = cF_X + (1-c)F_Y$$

(we call this the convolution – mixture property and say that (*X*, *Y*) are *c* – conjugated).

ii) $P \circ N_t^{-1} :\overset{not}{=} n_t(\gamma)$, $P \circ N_0^{-1} :\overset{not}{=} \delta_0$ and $n_t(\gamma) * n_s(\gamma) = n_{t+s}(\gamma)$, for any natural numbers t, s.

In this context, we now intend to determine the compound sum S_t defined by

$$S_t :\overset{def}{=} \sum_{i=1}^{N_t} \xi_i, \quad \forall t > 0 , \qquad S_0 \overset{def}{=} 0 ,$$

which represents the total loss (the aggregate loss) that the company might have at time *t*.

Notation. For two real distribution functions $p = \sum_{n \geq 0} p_n \delta_n$ and *G*, let us denote

$$\sum_{n \geq 0} p_n G^{*n} \overset{not}{:=} p \vee G .$$

Remark 1. With the notations $P\circ S_t^{-1} \overset{not}{:=} s_t, \forall t>0$ and $P\circ \xi_n^{-1} \overset{not}{=} F, \ \forall n\geq 1$, it becomes obvious that $s_t = n_t \vee F, \ \forall t\geq 1$. Since also

$$F = F_X * F_Y \text{ and } p\vee (G_1 * G_2) = (p\vee G_1) * (p\vee G_2), \ \forall\ p, G_1, G_2,$$

we therefore only need to know $n_t \vee F_X$ and $n_t \vee F_Y$.

2. Examples of discrete c-conjugated random variables

1. It is known (see [1]) that the most general case of such (X, Y) pair is defined by:

$$F_X = x_0\delta_0 + x_1 \sum_{n\geq 1}\alpha^{n-1}\ \delta_n, \qquad\qquad F_Y = y_0\delta_0 + y_1 \sum_{n\geq 1}\beta^{n-1}\ \delta_{-n}$$

with

$$\frac{\overline{x_0}\ (c-y_0)}{c(1-y_0)} \overset{not}{:=} \alpha, \ \frac{\overline{y_0}\ (\bar{c}-x_0)}{(1-c)(1-x_0)} \overset{not}{=} \beta, \ \ 0<y_0<c<\overline{x_0}<1.$$

Now, for every constant $\theta\in(0,1)$, let us denote $1-\theta \overset{not}{:=} \bar{\theta}$. Then, the following three pairs (F_X, F_Y) come as particular cases of the previous one:

2. $F_X = \bar{c}\ \delta_0 + \dfrac{c\,\bar{c}}{1-y_0}\displaystyle\sum_{n\geq 0}\left(\frac{c-y_0}{1-y_0}\right)^n \delta_{n+1}, \quad F_{-Y} = B_1(\overline{y_0})$ (take $\beta = 0$ in the general case);

3. $F_Y = c\delta_0 + \dfrac{c\,\bar{c}}{1-x_0}\displaystyle\sum_{n\geq 0}\left(\frac{\overline{x_0}-c}{1-x_0}\right)^n \delta_{-n-1}, \qquad F_X = B_1(\overline{x_0})$ (take $\alpha = 0$ in the general case);

4. $F_X = B_1(c), \qquad\qquad F_{-Y} = B_1\left(\bar{c}\right)$ (take $\alpha=\beta=0$ in the general case).

Lemma 1. The variable X has the distribution function $F_X = x_0\delta_0 + x_1\sum_{n\geq 0}\alpha^n\ \delta_{n+1}, \ x_1 = \overline{x_0}\ \bar{\alpha}$, $x_0 < \bar{\alpha}$ if and only if $X = X_1 + X_2$, with X_1, X_2 independent having $F_{X_1} = B_1\left(\dfrac{\bar{\alpha}-x_0}{1-\alpha}\right)$ and $F_{X_2} = Geom\left(\ \bar{\alpha}\ \right)$.

We notice that, in order to compute $n_t \vee F_X$, we need to know the compound sum of the counter N_t with a *Bernoulli* (respectively with a *Geometric*) distributed random variable.

3. The main result

Theorem. Let $(X_n)_n$ be a sequence of discrete independent identically distributed random variables with $P(X_1\geq 1)=1$, $(N_t)_{t\geq 1}$ a counting process independent of X, having the distributions $n_t(\gamma) \overset{not}{=} P\circ N_t^{-1}, \ \ \forall t\geq 0$ which verify $n_t(\gamma) * n_r(\gamma) = n_{t+r}(\gamma), \ \ \forall r,t\geq 0$. If $S_t = \sum_{i=1}^{N_t} X_i, \ \forall t\geq 1, \ \ S_0 \overset{def}{=} 0$ with $s_t \overset{not}{:=} n_t(\gamma)\vee F, \ \ \ F = P\circ X_1^{-1}$, then

$$s_{t+1}(n) = \frac{t+1}{n}\sum_{i=1}^{n}\ i\, s_1(i)\ s_t(n-i), \quad \forall n\geq 1,$$

$$s_{t+1}(0) = [P(N_1=0)]^{t+1}, \quad \forall t\geq 0.$$

Remark 2. In other words, the previous result tells that if the distribution of the number of claims received by the insurance company until moment t=1 (precisely n_1) is known, then the distribution function of the total loss at any time $t \geq 2$ (namely s_t) can be determined using the above recurrence.

Remark 3. One can easily notice that $n_t(\gamma) * n_r(\gamma) = n_{t+r}(\gamma), \quad \forall r, t \geq 0$ implies $s_t = s_1^{*t}, \quad \forall t \geq 1$. But the theorem states that we don't need to compute this convolution.

In the next section, we consider two examples of counters verifying the conditions in the theorem and for each of them we determine the corresponding compound sum.

4. Particular cases

We now analyze the *binomial* and the *negative-binomial* distributed counters.

Lemma 2. If $F = F_X * F_Y$ is the distribution function of the loss, if the number of claims until moment *t* has $n_t(\gamma) = P \circ N_t^{-1}$ with $\gamma \in (0,1)$ and

a) if $n_t(\gamma) = B_t(\gamma)$ then $s_1(n) = \begin{cases} \bar{\gamma} & if \quad n = 0 \\ \gamma F(n) & if \quad n \geq 1 \end{cases}$ and

$$s_{t+1}(n) = \begin{cases} \bar{\gamma}^{t+1} & if \quad n = 0 \\ \dfrac{\gamma}{1-\gamma} \displaystyle\sum_{j=1}^{n} \left(\frac{(t+1)j}{n} - 1 \right) F(j) s_t(n-j) & if \quad n \geq 1 \end{cases}, \ \forall t \geq 1$$

b) if $n_t(\gamma) = Negbin(t, \gamma)$ then $s_1(n) = \begin{cases} \gamma & if \quad n = 0 \\ \gamma \displaystyle\sum_{i=1}^{n} F(i) s_1(n-i) & if \quad n \geq 1 \end{cases}$ and

$$s_{t+1}(n) = \begin{cases} \gamma^{t+1} & if \quad n = 0 \\ \dfrac{t+1}{n} \displaystyle\sum_{i=1}^{n} iF(i) s_t(n-i) & if \quad n \geq 1 \end{cases}, \ \forall t \geq 1.$$

Applying this lemma to the distribution functions from cases 1, 2, and 4 (case 3 is obviously similar to 2), we obtain the following results.

Corollary 3. Let $F_X = x_0 \delta_0 + x_1 \sum_{n \geq 1} \alpha^{n-1} \delta_n$ and $F_Y = y_0 \delta_0 + y_1 \sum_{n \geq 1} \beta^{n-1} \delta_{-n}$. With the above notations, we have $s_1 = n_1(\gamma) \vee F_{X+Y} = [n_1(\gamma) \vee F_X] * [n_1(\gamma) \vee F_Y]$. Then

$$\text{a) } n_1(\gamma) = B_1(\gamma) \Rightarrow \begin{cases} s_1(0) = \bar{\gamma}^2 + \gamma\bar{\gamma}(x_0 + y_0) + \gamma^2 \left(x_0 y_0 + \dfrac{x_1 y_1}{1 - \alpha\beta} \right) & for \quad n = 0 \\ s_1(-n) = u\beta^{n-1}, & \\ s_1(n) = v\alpha^{n-1}, & for \quad n \geq 1 \end{cases},$$

with $\gamma y_1 \left(\bar{\gamma} + \gamma x_0 + \gamma x_1 \dfrac{\beta}{1-\alpha\beta} \right) \overset{not}{=} u$ and $\gamma x_1 \left(\bar{\gamma} + \gamma y_0 + \gamma y_1 \dfrac{\alpha}{1-\alpha\beta} \right) \overset{not}{=} v$.

$$\text{b) } n_1(\gamma) = Geom(\gamma) \Rightarrow \begin{cases} n_1(\gamma) \vee F_X = Geom\left(\dfrac{\gamma\bar{\alpha}}{(1-\alpha) - (1-\gamma)x_0} \right) * f \\ n_1(\gamma) \vee F_{-Y} = Geom\left(\dfrac{\gamma\bar{\beta}}{(1-\beta) - (1-\gamma)y_0} \right) * h \end{cases},$$

with $f = f(0) \left\{ \delta_0 + \bar{\alpha}\,\bar{\gamma} \sum_{n \geq 1} r^k \delta_k \right\}$, $f(0) = \dfrac{\gamma}{1-(1-\gamma)(1-\alpha)}$, $r = \dfrac{\alpha}{1-(1-\gamma)(1-\alpha)}$ and

$$h = h(0)\left\{\delta_0 + \overline{\beta}\,\overline{\gamma}\sum_{n\geq 1} u^k \delta_k\right\},\ h(0) = \frac{\gamma}{1-(1-\gamma)(1-\beta)},\ u = \frac{\beta}{1-(1-\gamma)(1-\beta)}.$$

As we can see, even if the counter distribution function is easy to be handled, finding s_1 can be a challenge.

Corollary 4. Let's consider $F_X = x_0\delta_0 + x_1\sum_{n\geq 1}\alpha^{n-1}\delta_n$ and $F_Y = \delta_{-1}\,\overline{y_0} + \delta_0\, y_0$. Then

a) $$n_1(\gamma) = B_1(\gamma) \Rightarrow s_1(k) = \begin{cases} \gamma\overline{y_0}\left(\overline{\gamma} + \gamma x_0\right) & if \quad k = -1 \\ \left(\overline{\gamma} + \gamma x_0\right)\left(1 - \gamma\overline{y_0}\right) + \gamma^2\,\overline{y_0}\, x_1 & if \quad k = 0 \\ u\alpha^{k-1} & if \quad k \geq 1 \end{cases},$$

with $\gamma x_1\left(1 - \gamma\overline{y_0}\,\overline{\alpha}\right) \overset{not}{=} u$;

b) $$n_1(\gamma) = Geom(\gamma) \Rightarrow (n_1(\gamma)\vee F_X)(n) = \begin{cases} f(0)\, w\overline{w}^n \left(1 + \overline{\alpha}\,\overline{\gamma}\dfrac{1-\left(\dfrac{r}{1-w}\right)^{n+1}}{1-\dfrac{r}{1-w}}\right) & if \quad n \geq 1 \\ \dfrac{\gamma}{1-(1-\gamma)x_0} & if \quad n = 0 \end{cases},$$

with $f(0) = \dfrac{\gamma}{1-(1-\alpha)(1-\gamma)}$, $r = \dfrac{\alpha}{1-(1-\alpha)(1-\gamma)}$, $w = \dfrac{\gamma\overline{\alpha}}{(1-\alpha)-(1-\gamma)x_0}$, and

$$(n_1(\gamma)\vee F_Y)(-n) = l\overline{l}^n,\quad \forall n \geq 0,\quad l = \frac{\gamma}{(1-y_0)+\gamma y_0}.$$

Corollary 5. Let $F_X = B_1(c)$ and $F_Y = \delta_{-1}\,\overline{c} + \delta_0\, c$. Then

a) $$n_1(\gamma) = B_1(\gamma) \Rightarrow s_1(k) = \begin{cases} \gamma\overline{c}\,(1-\gamma c) & if \quad k = -1 \\ (1-\gamma c)\left(1-\gamma\overline{c}\right) + \gamma^2 c\overline{c} & if \quad k = 0 \\ \gamma c\left(1-\gamma\overline{c}\right) & if \quad k = 1 \end{cases};$$

b) $$n_1(\gamma) = Geom(\gamma) \Rightarrow s_1(k) = \begin{cases} \overline{q}^n\,\gamma & if \quad k = -n,\ n \geq 1 \\ \gamma & if \quad k = 0 \\ \overline{v}^n\,\gamma & if \quad k = n,\ n \geq 1 \end{cases},\ v = \frac{\gamma}{c+(1-c)\gamma},\ q = \frac{\gamma}{\gamma c + (1-c)}.$$

Remark 4. The negative values of s_1 can be interpreted as profit. Since $c = P(X = 1)$ can be considered to be given, we can maximize this profit for proper values of the constant γ.

References

[1] Răducan, A. : *The Dugue problem in the discrete case,* 10-th Workshop of Department of Mathematics and Computer Science, Technical University of Civil Engineering, Bucharest 23 May 2009, Proceedings, Ed. Matrix Rom Bucuresti, ISSN 2067-3132, pp 132-136.

[2] Vernic, R. and Sundt, B. : *Recursion for Convolutions and Compound Distributions with Insurance Applications,* Springer, 2009.

Estimation of the facies distribution of a reservoir using Ensemble Kalman Filter

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1. Introduction

A facies is a body of rock with relatively uniform characteristics such as porosity or permeability and there are significant differences in these petrophysical characteristics between facies types. Consequently, the spatial distribution and the location of the different facies types presents in the reservoir have significant impact on fluid flow. Therefore the estimation of the facies distribution into a reservoir domain has two goals; one is related with the high predictive capacity of flow measurements of a reservoirs that are geologic realistic and the other one is related with the need of accurate properties to predict the consequences of changing condition in the reservoir lifetime (e.g. the response to enhance recovery).The first step in the assessment of the facies distribution is to define a geological model to simulate plausible facies maps that are consistent with the prior knowledge about subsurface geology (numbers of facies type that occur, facies proportion, core information, type of patterns etc).The geological model can be construct based on a parameterization of the facies maps with some uncertain variables(parameters)through which we are able to adjust them such that these will honor the underlying knowledge of subsurface geology. These parameters can be calibrated (hence, as well the maps) using additional informations about flow measurements into a process named in reservoir engineering History Matching (HM) (Oliver et al. 2008). One of the most promising History Matching method is the Ensemble Kalman Filter (EnKF, Evensen 2006) which can be summary described as a Monte Carlo technique introduced in the Kalman filtering theory where the probability density of the state is represented by an ensemble of possible realizations that are simultaneously updated. In our study, for simulation of the facies maps we used a truncation plurigaussian method (Galli et al. 1994) .The method consists in a projection from a continuous space (the space determined by the Gaussian Random Fields) into a discrete space (the facies maps space) through a map designed in the GRF space and defined by intersection of some threshold (values for one GRF, curves for two GRF, surfaces for three GRF, etc.). In this paper we investigate the reservoirs having a complex geology defined by three facies type that occur each two could having direct contact.

2. The geological simulation model

In every well (production or injection) of our reservoir we are sure (perfect observations) about the type of facies present there, so we can use that information for the whole grid where the well is situated. Let be (i, j) a grid block of our reservoir domain and suppose that in this grid facies type 1 occur. Consider $A_k^{i,j}$ the event in which in the grid (i,j) the facies type k occur, then, in terms of probabilities we have $P(A_1^{i,j}) = 1, P(A_2^{i,j}) = 0, P(A_3^{i,j}) = 0$. Hence, every facies type generates a field defined on the reservoir domain whose values in grids cells are 0 or 1 depending on whether the facies type occurs or not in those locations. These fields have binary values (discrete fields) and to estimate them we will use probabilities fields which are defined as random fields with values in interval [0,1] having spatial correlations. The probabilities fields are modeled through projection in [0,1] of some Gaussian Random Fields defined on the reservoir domain with a truncation function(projection function).The truncation function used is $\varphi_m(t)=\begin{cases}-\frac{|t|}{m}+1 & if \quad |t|\le m\\ 0 & if \quad |t|>m\end{cases}$, where m represents a truncation parameter. The parameter m can be initially chosen based on geological prior knowledge about the facies proportions in the certain case and estimated in the process of history matching. We start with two Random Gaussian Fields y_1 and y_2 defined on the entire reservoir domain and α_1 and α_2 will be the projection of the Gaussian fields in [0,1]. Then $\alpha_k^{i,j}=\varphi_{m_k}(y_k^{i,j})$, $k\in\{1,2\}$, represent an estimator for the probability of occurrence of the facies k at the location (i,j), where m_k represent the truncation parameter for the random field y_k. In order to appoint the estimator for the probability of the third facies we will use the following rules: $\alpha_3^{i,j}=\begin{cases}1-(\alpha_1^{i,j}+\alpha_2^{i,j}) & if \ \alpha_1^{i,j}+\alpha_2^{i,j}<1\\ 0 & otherwise\end{cases}$. At a certain location (i,j) we assign facies of type $k\in\{1,2,3\}$ if $\alpha_k^{i,j}=\max\{\alpha_r^{i,j}, r=1,2,3\}$ with the convention that if $\alpha_1^{i,j}=\alpha_2^{i,j}\ge\alpha_3^{i,j}$ we assign facies type 1 and if $\alpha_2^{i,j}=\alpha_3^{i,j}>\alpha_1^{i,j}$ we assign facies type 2(maximization criterion). Also, if we represent the point $(y_1^{i,j},y_2^{i,j})$ in the Cartesian plain $(y_1 O y_2)$ we can assign the facies type to the grid (i,j) based on the region where the point $(y_1^{i,j},y_2^{i,j})$ falls in that two dimensional space(see Figure 1).

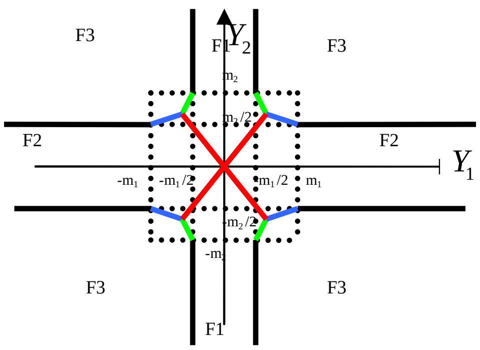

Figure 1: Truncation map for Gaussian Random Fields

3. Parameters estimation with Ensemble Kalman Filter (EnKF)

Let consider a dynamical model described by a nonlinear system of partial differential equations and assume that these equations with boundary conditions has been discretized in a space and the errors in the errors in the boundary conditions are zero. We denote with $u=u(t)\in\mathbb{R}^{N_u}$ the discretized approximation of the solution at time t. We assume that the model depends on some poorly known parameters $m\in\mathbb{R}^{N_m}$ to be estimated. Then the time equation of the state became $\frac{du(t)}{dt}=F(u,m)+w_m(t), u(t_0)=u_0$ being the initial state and $w_m(t)$ is the model error. The initial state u_0 usually is a random variable representing the uncertainty in the initial condition. In reservoirs engineering characterization u contain the dynamical variables pressure and saturation. We further assume that the model is constrained by some noisy measurements collected at times $t_1, t_2, \dots t_n$ and the measurements are related to the state through the general nonlinear relation $d_i = g_i(u(t_i), m) + v_i^d \in \mathbb{R}^{N_d}$, where $v_i^d \sim N(0, C_{d_n})$ represents the measurements errors. In the parameter estimation problems with EnKF we define an augmented state vector that contains parameters, dynamical variables and the simulated data $x_k = \begin{bmatrix} m & u_k & g(m, u_k) \end{bmatrix}^T \in \mathbb{R}^{N_m + N_u + N_d}$.

The ensemble Kalman filter is a Monte Carlo method where multiple plausible models are simultaneously updated instead of a single model as in the traditional history matching. In the first step, an ensemble of N_e states $\{x_1, x_2, \dots x_{N_e}\}$ is generated to represent the uncertainty in the initial state $x_0^t = x(t_0)$. In the second step, named the forecast step, the stochastic model propagates the distribution of the true state through the model equations according to $x_i^f(t_k) = M(x_i^a(t_{k-1})) + w_i(t_k)$ where $w_i(t_k)$ represents a realization of the noise process of the model error. For the forecasted state we calculate the mean $\overline{x}^f(t_k) = \frac{1}{N_e} \sum_{k=1}^{N_e} x_k^f(t_k)$ and based on the mean the covariance $C^f(t_k) = \frac{1}{N_e - 1} E^f(t_k) E^f(t_k)^T$, where $E^f(t_k) = \begin{bmatrix} x_1^f(t_k) - \overline{x}^f(t_k) & x_2^f(t_k) - \overline{x}^f(t_k) & \dots & x_{N_e}^f(t_k) - \overline{x}^f(t_k) \end{bmatrix}^T$ .When the measurements become available values of each ensemble member are adjusted based on the Kalman equation $x_i^a(t_k) = x_i^f(t_k) + K(t_k)[d_{obs}(t_k) - H(t_k) x_i^f(t_k) + v_i(t_k)]$, where $K(t_k) = C^f(t_k) H(t_k)^T [H(t_k) C^f(t_k) H(t_k)^T + R(t_k)]^{-1}$ is the Kalman gain, $H(t_k)$ is the observation operator, $R(t_k)$ is the covariance matrix of the measurements error and $v_i(t_k)$ is the realization of the noise added to observed measurements. At the end of the assimilation period we will have an estimator for each parameter, defined by the ensemble mean together with his uncertainty given by the forecasted covariance matrix.

4. Ensemble Kalman Filter implementation for facies update

The state vector for the j^th ensemble member at the k^th assimilation step is: $x_j^k = \begin{bmatrix} y_1 & y_2 & m_1 & m_2 & p & s & BHP & q_w & q_o & \alpha_1^w & \alpha_2^w \end{bmatrix}_j^T$,where y_1 and y_2 represents the two Random Gaussian Fields, m_1, m_2 are the truncation parameters of the random fields , p is the pressure, s is the saturation, BHP is the pressure measured at the injector, q_w, q_o are the water and oil rates measured at the producers and α_1^w, α_2^w represents the simulated facies measurements at the wells locations. The facies measurements at the wells location are written in probability terms. If facies type 1 occur then $\alpha_1^w = 1, \alpha_2^w = 0$, if facies type 2 occur then $\alpha_1^w = 0, \alpha_2^w = 1$ and if facies type 3 occur we have $\alpha_1^w = 0, \alpha_2^w = 0$.We generate an ensemble of 120 replicates where the values for pressure and saturation are kept constant for each member of the ensemble. The uncertainty in the initial ensemble is given by the choice of the two Gaussian Random Fields and the choice of the truncation parameters. The Random Gaussian Fields y_1 and y_2 are generated with sequential Gaussian simulation method specifying the Geostatistical properties (isotropy or anisotropy, principal directions and the range correlation) and with constraint given by the type of facies find in the grids where the wells are situated. If in a grid with a well located we have observation about the existence of facies type 1 then the value in this grid for y_1 is 0 and of course if we have observation about the existence of facies type 2 then we generate y_2 with value 0 in this grid.

4. Synthetic example

The simulation model is a 5-spot water flooding 2D-reservoir, black oil model with $50\times50\times1$active grid blocks. The dimension of each grid block was set at $30\times30\times1$ ft and there are one injector situated at the center of the reservoir domain and 4 producers situated in the corners. The values of the permeability (k) and porosity (ϕ), corresponding to each facies type, are: for facies type 1 k=174 md, $\phi=0.18$ for facies type 2 k=372 md, $\phi=0.25$ and for facies type 3 k=80 md, $\phi=0.14$ .In the next figures we present the reference field(the “truth”) ,

the initial fields and the estimated fields. The blue color represent facies type 1, the green color the facies type 2 and the red color the facies type 3. In the Figure 2 the light blue dots represents the wells positions. For the generation of the Gaussian fields we used isotropic geostatistics characteristics with length correlation of 17 grid blocks and the truncation parameters are generated with mean $\sqrt{2}$ and standard deviation 0.2.

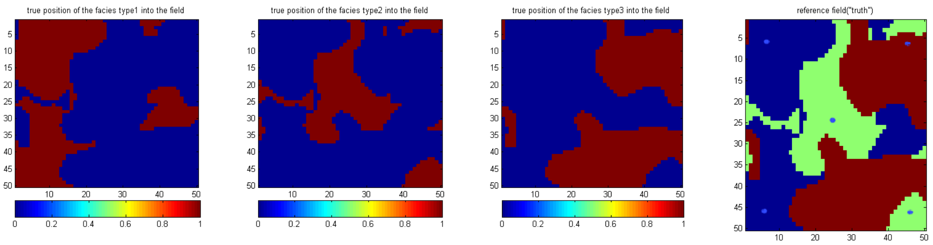

Figure 2: The binary fields defined by each facies type and the reference field

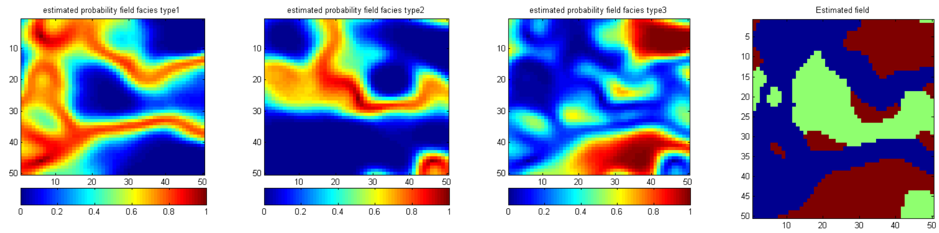

Figure 3: The estimated probability fields of each facies type and the estimated field

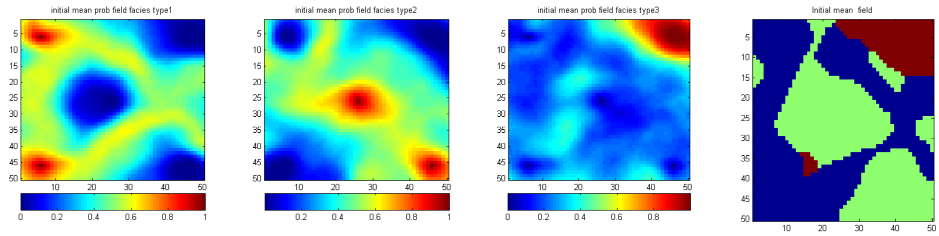

Figure 3: The initial mean probability fields of each facies type and the initial mean field

References

[1] Evensen, G., Data Assimilation: The Ensemble Kalman Filter, Springer, 2006

[2]Galli A, Beucher H, Le Loc’h G, Doligez B, Heresim Group (1994),The pros and cons of the truncated Gaussian method. In: Geostatistical simulations. Kluwer Academic, Dordrecht, pp 217–233

[3]Jazwinski AH (1970), Stochastic processes and filtering theory. Academic, San Diego, California

[4]Oliver D, Reynolds A, Liu N (2008), Inverse theory for petroleum reservoir characterization and history matching. Cambridge University Press, Cambridge

STATISTICAL BIOLOGICAL DATA IN PROGNOSTIC CANCER

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Abstract: In this paper we describe the evaluation of various biological and clinical parameters in establishing their prognostic value in prostate cancer patients. Our application was validated using clinical data from the "Prof. Dr. Th. Burghele"Clinical Hospital and "St. Ioan" Hospital from January 2008 to December 2009.
Mathematics Subject Classification (2000): 62J02
Key words: logistic regression, prostate cancer.

1. Introduction

Physiological mechanisms in human biology, the progress of disease in individual patients, hospital work-flow management: these are just a few of the many complicated processes studied by researchers in biomedicine and health-care.

For controlling the ever increasing complexity of these fields, a proper understanding of their processes is important as is the ability to reason about them.

We need statistics because we want to draw more valid conclusions from limited amounts of data and significant differences are often masked by biological variability or experimental imprecision. On the other hand, the human mind excels at finding patterns and relationships and tends to generalize too much.

Logistic regression allows one to predict a discrete outcome, such as group membership, from a set of variables that may be continuous, discrete, dichotomous, or a mix of any of these. Generally, the dependent or response variable is dichotomous, such as presence/absence or success/failure. Discriminant analysis is also used to predict group membership with only two groups. However, discriminant analysis can only be used with continuous independent variables. Thus, in instances where the independent variables are a categorical, or a mix of continuous and categorical, logistic regression is preferred.

2. Results and discussion

The work presented is to the study of antibodies with a special significance in prostate cancer. They are: p63, AMACR, 34beta, p53. P63 designates an antibody that is found, in general, in healthy gland. AMACR antibody is found only in prostate cancer. 34beta designates an antibody that is found in healthy gland. P53 antibody is present only in undifferentiated cancer, aggressive.

The table below shows the values for the logistic regression equation for predicting the dependent variable from the independent variable. Coefficients (B) having p-values less than alpha are statistically significant. Because we chose alpha to be 0.05, coefficients having a p-value of 0.05 or less would be statistically significant (i.e., we can reject the null hypothesis and say that the coefficient is significantly different from 0). The standard error (S.E.) of an estimated coefficient should be familiar from previous work on linear regression and even t-tests and z-tests. Essentially, the standard error is a measure of how stable our estimate is. A large standard error means the estimated coefficient isn't that well estimated, and a low standard error means we have a fairly precise estimate.

The values for the logistic regression equation

		B	S.E.	Wald	df	Sig.	Exp(B)
Step 1[a]	p63	-1.39	.144	.958	1	.032	1.225
	AMACR	10.93	.177	.464	1	.049	3.483
	34beta	0.45	.121	1.264	1	.026	2.625
	p53	0.73	0.81	0.023	1	.081	2.07
	Constant	-44.77	13.927	2.007	1	.015	.000

Hosmer and Lemeshow goodness-of-fit test: P=0,675

Expressed in terms of the variables used in this example, the logistic regression equation is

$$\ln(ODDS\_cancer) = -44.77 - 1.39*(p63) + 10.93*(AMACR) + 0.45*(34beta) + 0.73*(p53)$$

3. Conclusion

Prostate cancer is the most commonly diagnosed cancer in men and the second leading cause of cancer deaths in men after lung cancer. It is largely unknown what causes prostate cancer. It is thought, as with other malignancies, to be a combination of environmental risk factors in conjunction with a genetic predisposition. It is important to understand that risk factors are not "causes", but are factors that make you statistically more likely than others in the general population to have a certain condition.

References

[1] Chan TY, Partin AW, Walsh PC, Epstein JI, 2000. Prognostic significance of Gleason score 3+4 versus Gleason score 4+3 tumor at radical prostatectomy, *Urology*, vol. 56, pp. 823-827.

[2] P. A. Humphrey, 2004. Gleason grading and prognostic factors in carcinoma of the prostate, *Modern Pathology*, vol. 17, pp. 292- 306.

[3] Marbele S. Guimaraes, Maisa M. Quintal, Luciana R. Meirelles, Luis A. Magna, Ubirajara Ferreira, Athanase Billis, 2008. Gleason Score as Predictor of Clinicopathologic Findings and Biochemical (PSA) Progression Following Radical Prostatectomy, *Clinical Urology*, vol. 34(1), pp. 23-29.

[4] T. Le Chap, 2003. *Introductory Biostatistics*, New York: Wiley.

[5] N. Teodorescu, 2006. *Probabilități şi statistică matematică cu Maple*, Editura Bren, Bucureşti.

A HYPERELASTIC CYLINDER TREATED BY LEM

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Abstract. The interest in hyperelastic materials is continuosly growing because of their multiple applications in technics. In the present paper, it is considered an Ogden type model for a cylindrical rubbery shell, previously established by Dana Petroşanu, who also proved its consistency and the uniqueness of the solution. The model consists of a nonlinear two-point problem that could been tackled only numerically. Here, the model is treated by using the linear equivalence method - LEM - introduced and extended by the author, with numerous applications in nonlinear dynamics. By LEM, analytical approaches of the solution and of its derivative were obtained, yielding a qualitative study and a comparison with the linearized model.
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Key words: hyperelastic materials, linear equivalence method, nonlinear two-point problems

1. Introduction

In this paper it is considered a model of deformation of a rubbery cylindrical shell of Ogden type [3], established by Dana Petroşanu in [4][5]; the model results in solving a two-point boundary value problem for a second order non-linear ODE. Most frequently, such problems cannot be solved analitically and, consequently, they are treated numerically. Here, this problem is treated by using the *linear equivalence method* (LEM). LEM was previously introduced by the author to study both numerically and qualitatively the solutions of nonlinear dynamical systems in a classical linear frame [12]-[21].
We firstly present the hyperelastic model, as it was established in [5]. After an outline of LEM, limited to the case studied here, we obtain the LEM solution of the considered two-point nonlinear problem and also the parametric algebraic equation giving the value of its derivative at the left end of the considered interval. Finally, this solution is tested numerically and pointwisely computed.

2. The deformation of a cylindric rubbery shell

The behaviour of the isotropic elastic materials is basically described by non-linear laws, most often extensions of Hooke's law. The Ogden model assumes that the material behaviour can be described by means of a strain energy density function, from which the stress-strain relationships can be derived. Conveniently fitting the material parameters, the Ogden model accurately describes the rubber-like material behaviour.
Consider a rubbery thin shell in form of a circular cylinder, of height L and of basis radii A and B.
Dana Petroşanu found the following equation for the unknown function $f = f(R)$

$$f''(R) = -\frac{f'}{R}\cdot\frac{7(2k+1)R^2 f'^2 + (29k+1)Rff' + 3(5k+4)f^2 - 3(k+2)}{3(2k+1)R^2 f'^2 + 6(2k+1)Rff' + (5k+4)f^2 - (k+2)}, \tag{1}$$

where $k = a_1 / a_2$ is a non-dimensional material constant. She also proved that the denominator of (1) does not vanish within the considered domain.
The boundary conditions were chosen as follows

$$f(A)=\frac{a}{A}, \quad f(B)=\frac{b}{B}. \tag{2}$$

In [5] it was proved that this problem allows a unique solution.

3. LEM solutions for two-point problems

The main idea of LEM consists of an exponential mapping depending on parameters that associates to a nolinear ODS two linear equivalents: a linear PDE, always of first order with respect to the independent variable of the system and alinear, while infinite, ODS, whose associated matrix is at least column-finite and, if the ODS is polynomial, it is also row-finite; this enables the calculus by block partitioning.
In this section we show how to apply LEM to non-linear two-point problems. The obtained results concern only polynomial ODEs, but they can be extended to more general classes of non-linear operators.
Consider the second order polynomial ODE

$$y'' = \sum_{j+k=0}^{p} a_{jk}(x) y^j (y')^k, \quad a_{jk} \in C^\infty(I), I \equiv [x_0, x_1], \tag{3}$$

to which we associate the two-point (Picard) conditions

$$y(x_0) = y_0, y(x_1) = y_1. \tag{4}$$

Let us apply LEM to this equation, taking for y the initial conditions

$$y(x_0) = y_0, y'(x_0) = \beta, \tag{5}$$

β being a parameter. Denoting by $\mathbf{\Pi}$ the LEM inverse of the polynomial operator defined by (3), it follows that the solution of the polynomial initial problem (3), (5) is given by the first component $v_{10}(x,\beta)$ of the vector

$$\mathbf{\Pi}(x-x_0)\mathbf{V}_\beta(x_0) + \mathbf{\Theta}(x-x_0), \quad \mathbf{V}_\beta(x_0) = \left[y_0^j \beta^k \right]_{j+k \in \mathrm{N}}. \tag{6}$$

Denote by $f(\beta)$ the value of this component at the point x_1, if it is consistent

$$f(\beta) = v_{10}(x_1, \beta). \tag{7}$$

The following theorem is a direct application of the above considerations.

Theorem 1. [13],[19]. *Let* a_{jk} *be infinitely differentiable on* $I = [x_0, x_1]$ *and suppose that* $\left\| a_{jk} \right\|_m \le C, m \in \mathrm{N}^*$. *Then, if* $f(\beta)$ *is consistent,*

i) the set of the analytic on I *solutions of the two-point problem* (3), (5) *has the same cardinal as the set* Γ *of the solutions of the functional equation*

$$f(\beta) = y_1; \tag{8}$$

ii) the analytic on I *solutions of the two-point problem* (3), (5) *coincide with the first components of the vectors*

$$\mathbf{\Pi}(x_1-x_0)\mathbf{V}_{\bar\beta}(x_0) + \mathbf{\Theta}(x_1-x_0), \quad \bar\beta \in \Gamma. \tag{9}$$

Theorem 1 emphasizes some of LEM's advantages when applied to a two-point problem:

- LEM works, even if the solution is not unique;
- numerically, it leads to non-standard algorithms, completely distinct from the usual ones (e.g., collocation, shooting, etc.);
- if uniqueness does not hold, then, by these algorithms, one can separate and effectively determinate the solutions.

These advantages were put into evidence in the study by LEM of the simply supported, as well as in the case of the hyperstatic bar [16],[19].

4. LEM solutions for the hyperelastic cylinder

The model consists, as previously specified, of the nonlinear two-point problem (3), (5). The existence and uniqueness of the solution is ensured in this case too [5]. We can apply the techniques from the previous section, considering the series expansion of the ratio in the right member with respect to f, f'. The third order approximation gives

$$f'' = -\frac{3f'}{R} - k_1 ff'^2 - k_2 Rf'^3, \qquad R \in [A, B], \tag{10}$$

where we used the notations

$$k_1 = \frac{7k+17}{k+2}, \qquad k_2 = \frac{2(2k+1)}{k+2}. \tag{11}$$

This is again a second order polynomial ODE, with variable coefficients, odd with respect to the unknown function *f* and its derivative, similar to that corresponding to the hyperelastic sphere.

The LEM mapping will depend on two parameters in this case too. The second LEM equivalent reads

$$v'_{ij} = iv_{i-1,j+1} + j\left[-\frac{3}{R}v_{ij} - k_1 v_{i+1,j+1} - k_2 R v_{i,j+2}\right], \quad i,j \in \mathrm{N}. \tag{12}$$

Going as far as third order effects, we are led to the (finite) truncated ODS

$$\frac{\mathrm{d}\mathbf{V}^{(3)}}{\mathrm{d}R} = \mathbf{A}^{(3)}\mathbf{V}^{(3)}, \quad \mathbf{V}^{(3)} = \begin{bmatrix} \mathbf{V}_1^{(3)} \\ \mathbf{V}_3^{(3)} \end{bmatrix}, \quad \mathbf{V}_1^{(3)} = \begin{bmatrix} v_{10}^{(3)} \\ v_{01}^{(3)} \end{bmatrix}, \quad \mathbf{V}_3^{(3)} = \begin{bmatrix} v_{30}^{(3)} \\ v_{21}^{(3)} \\ v_{12}^{(3)} \\ v_{03}^{(3)} \end{bmatrix}. \tag{13}$$

The truncated LEM matrix has the form

$$\mathbf{A}^{(3)} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{13} \\ \mathbf{0} & \mathbf{A}_{33} \end{bmatrix}, \tag{14}$$

with

$$\mathbf{A}_{11} = \begin{bmatrix} 0 & 1 \\ 0 & -\frac{3}{R} \end{bmatrix}, \quad \mathbf{A}_{33} = \begin{bmatrix} 0 & 3 & 0 & 0 \\ -\frac{3}{R} & 0 & 2 & 0 \\ 0 & -\frac{6}{R} & 0 & 1 \\ 0 & 0 & -\frac{9}{R} & 0 \end{bmatrix}, \quad \mathbf{A}_{13} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -k_1 & -k_2 \end{bmatrix}. \tag{15}$$

By virtue of the oddness of (10), the LEM matrix will be “contracted”, containing only cells with odd indices.
Let us apply now theorem 1. We chose the following Cauchy conditions

$$f(A)=\frac{a}{A},\quad f'(A)=\alpha\,, \tag{16}$$

α being a parameter that will be determined from the boundary conditions.
Denote now by f_0 the solution of the linear part of the ODE (10), i.e.

$$f''=-\frac{3f'}{R},\qquad R\in[A,B], \tag{17}$$

subjected to the initial conditions (16). We first get

$$f_0'=\alpha\frac{A^3}{R^3}\,,\qquad f_0=\frac{\alpha A}{2}\left(1-\frac{A^2}{R^2}\right)+\frac{a}{A}\,. \tag{18}$$

The vector $\mathbf{V}_1^{(3)}$ is the solution of the non-homogeneous linear ODS

$$\frac{\mathrm{d}\mathbf{V}_1^{(3)}}{\mathrm{d}R}=\mathbf{A}_{11}\mathbf{V}_1^{(3)}+\mathbf{A}_{13}\mathbf{V}_3^{(3)}, \tag{19}$$

This leads to the following linear non-homogeneous second order ODE

$$v_{10}''+\frac{3}{R}v_{10}'=-k_1Rf_0\,f_0'^2-k_2Rf_0'^3\,, \tag{20}$$

to be solved under the initial conditions

$$v_{10}(A)=\frac{a}{A},\quad v_{10}'(A)=\alpha\,. \tag{21}$$

Replacing f_0, f_0' by their expressions (18), we obtain

$$v_{10}(\alpha,R)=c_{10}-\frac{c_{01}}{2R^2}-\frac{k_1}{8}a\alpha^2\frac{A^5}{R^4}-\alpha^3\frac{A^3}{48}\left[3k_1\frac{A^4}{R^4}+(2k_2-k_1)\frac{A^6}{R^6}\right], \tag{22}$$

where

$$c_{01}=\alpha A^3-\frac{k_1}{2}a\alpha^2A^3-\alpha^3\frac{A^5}{8}(2k_2+k_1)\,. \tag{23}$$

Applying the two-point conditions

$$v_{10}(A)=\frac{a}{A},\quad v_{10}(B)=\frac{b}{B}\,, \tag{24}$$

we get the following algebraic equation for α

$$\alpha a_1+\alpha^2a_2+\alpha^3a_3=a_0\,, \tag{25}$$

where $r=A/B$, the coefficients a_j being given by

$$a_0 = \frac{\frac{b}{B} - \frac{a}{A}}{1 - r^2}, \qquad a_1 = \frac{A}{2}, \qquad a_2 = -\frac{k_1}{8} aA\left(1 - r^2\right),$$
$$a_3 = \frac{A^3}{48}\left[3k_1\left(1 + r^2\right) + \left(2k_2 - k_1\right)\left(1 + r^2 + r^4\right) - 6k_2 - 3k_1\right]. \qquad (26)$$

Table 1 gives, for the same range of values of a, b, A, B, the corresponding values of α, in the case of the hyperelastic cylinder, compared with those for the linear case.

Table 14.3. The values of $f'(A)$ for the hyperelastic cylinder

Nr.	A	B	a	b	α	α (linear)
1.	0.1	0.200	0.15	0.250	-3.6994	-6.6666
2.	0.1	0.200	0.10	0.220	5.4733	2.6666
3.	0.1	0.150	0.15	0.200	-3.7337	-6.0000
4.	0.2	0.225	0.19	0.215	0.2706	0.2647
5.	0.2	0.250	0.30	0.350	-1.9533	-2.7777
6.	0.2	0.500	0.25	0.450	-2.2286	-4.1666
7.	0.2	0.300	0.30	0.400	-1.8668	-3.0000
8.	0.2	0.400	0.20	0.440	2.7366	1.3333
9.	1.0	1.500	1.10	1.700	1.4850	1.2000
10.	2.0	3.000	2.20	3.000	-0.1359	-0.1800
11.	2.0	4.000	2.00	4.400	0.2736	0.1333

5. Conclusions

In this paper, we set up a method based on LEM to get solutions for an Ogden type hyperelastic model for a rubbery cylindric shell involving a non-linear two-point problem. It is seen that, unlike by pure numerical methods, by LEM we get analytic approximations for the solution. It should be mentioned that the analytic approximations have the advantage of emphasizing the qualitative behaviour of the solution.

Finally, we observe that the above hyperelastic model has a special form, which induces a certain pattern in solving similar models. Together with an extended study of the LEM solutions for such models, this common pattern represents a perspective for future researches.

References

[1] Munteanu, L., Badea, T., Chiroiu, V., Linear equivalence method for the analysis of the double pendulum's motion, *Complexity International Journal*, **9**, pp. 26-43, 2002.

[2] Munteanu, L., Donescu, Ş., *Introduction to the soliton theory, Applications to mechanics*, Book Series: Fundamental Theories of Physics, Kluwer Academic publishers, 2004.

[3] Ogden Raymond, *Non- Linear Elastic Deformations*, New York, J. Wiley and Sons, 1984.

[4] Petroşanu, D., Exemples de grandes déformations pour un reservoir sphérique et un tube cylindrique, *University Politehnica of Bucharest, Sci. Bull., Series A: Appl. Math. and Physics*, **66**, *1*, pp. 37-46, 2004.

[5] Petroşanu, D., Sur la déformation d'un tube cylindrique, *Rev. Roum.Sci. Techniques, Série de Mécanique Appliquée*, **55**, *1*, pp. 39-50, 2010.

[6] Soare, M.V., Teodorescu, P.P., Toma, I., *Ordinary differential equations with applications to mechanics*, Springer, 2006.

[7] Teodorescu, P.P., *Mechanical systems. Classical models*, **1-3**, Springer, 2006-2009.

[8] Teodorescu, P.P., Toma, I., A class of elastic structures with the same mathematical core, in *Honorary volume dedicated to professor emeritus Ioannis D. Mittas, Aristotle Univ. of Thessaloniki*, Fac. of Eng., Dept. of Math, Phys.Sci., Division of Math., pp. 499-508, 2000.

[9] Teodorescu, P.P., Toma, I., Nonlinear damped pendulum treated by linear equivalence, *Mech. Res. Comm*, **27**, *3*, pp. 373-380, 2000.

[10] Teodorescu, P.P., Toma, I., New integral LEM formulae applied to the nonlinear bar, *Mech. Res. Comm*., **31**, *1*, pp. 161-168, 2004.

[11] Teodorescu, P.P., Toma, I., A neo-Hookean model treated by LEM, Special issue of *The Annals of "Dunărea de jos" University of Galați, fasc. 14, Mechanical Engineering*, 12th international symposium of experimental stress analysis and testing materials under the patronage of ARTENS and Romanian Technical Science Academy, 24-25.10.2008, Galați University Press, pp.99-104, 2008.

[12] Toma, I., On polynomial differential equations, *Bull. Math. Soc. Sci. Math. de la Roumanie*, **24**(**72**), *4*, pp. 417-424, 1980.

[13] Toma, I., Solutions of bilocal polynomial problems by linearization, *Analele Univ. Bucureşti, Seria Matematică*, **30**, pp. 71-80, 1981.

[14] Toma, I., Normal LEM representations for the non-linear forced pendulum, II NNMAE, in *Proc. of the International Conference in Memoriam of Professor P.D. Panagiotopoulos*, C.C. Baniotopoulos (ed), Thessaloniki, Greece, 7-8.07.2006, pp.329-332, 2006.

[15] Toma, I., Specific LEM techniques for some polynomial dynamical systems, in *Topics in Applied Mechanics*, Ed. Academiei Române, V. Chiroiu, T. Sireteanu (eds.), vol.III, pp. 427-459, 2006.

[16] Toma, I., LEM solutions in mechanics and engineering, in *Proc. of ICTCAM 2007*, 20-23.06.2007, G. Păltineanu, E. Popescu, I. Toma (eds.), pp. 123-128, 2007.

[17] Toma, I., The nonlinear pendulum from a LEM perspective, in *Research Trends in Mechanics*, Ed. Academiei Române, Bucharest, D.Popa, V. Chiroiu, I.Toma (eds.), vol.I, pp.395-423, 2007.

[18] Toma, I., *Extensions of LEM to non-autonomous systems*, Research Trends in Mechanics, Ed. Academiei Române, Bucharest, eds. D.Popa, V. Chiroiu, I.Toma, vol.II pp. 361-378, 2008.

[19] Toma, I., *The linear equivalence method and its applications in mechanics*, Ed. Tehnică, Bucharest, 2008 (in Romanian).

[20] Toma, I., LEM solutions for two neo-Hookean models, in *Research Trends in Mechanics*, Ed. Academiei Române, D.Popa, V. Chiroiu, L. Munteanu (eds.), vol.III, pp.444-468, 2009.

[21] Toma, I., An abstract pattern for some dynamical models, *Revue Roumaine des Sciences Techniques, série de Mécanique Appliquée*, **55**, *3*, 2010.

TESTING SOME HYPOTHESES FOR THE DISCHARGES OF THE DANUBE RIVER

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Abstract

In this paper we will test the mutual independence of the discharges using the Wald-Wolfowitz test and the Kendall's turning points test, their homogeneity using the Mann-Whitney-Wilcoxon test, and the lack of the trend using the Mann-Kendall test. The application is in the case of Orşova, for which we have yearly data from 1900 to 2008.

1. Introduction

The Wald-Wolfowitz test, known also as the sequences test (see [5,1]) checks with a given error ε if two given samples $X_1,...,X_m$ and $Y_1,...,Y_n$ are independent.

We order first the common sample (built from the above two samples) in increasing order. Next we count the number of sequences containing values only from $X_1,...,X_m$, or only from $Y_1,...,Y_n$. We denote this number by R.

The expectation and the variance of the above random variable are (see [5,1])

$$\begin{cases} E(R) = \frac{2 \cdot m \cdot n}{m+n} + 1 \\ Var(R) = \frac{(E(R)-1)(E(R)-2)}{m+n-1} \end{cases}. \tag{1}$$

If $m,n > 20$ we can approximate the distribution of the random variable R with the normal distribution, hence we can consider

$$Z = \frac{R - E(R)}{\sqrt{Var(R)}} \sim N(0,1). \tag{2}$$

If $m \leq 20$ or $n \leq 20$ then the cumulative distribution function (cdf) can not be approximate with the normal distribution. The discrete distribution of R if $m \leq n$ is given by (see [5,1])

$$\begin{cases} P(R=2\cdot s)=\frac{2}{C_{m+n}^{n}}\left(C_{m-1}^{s-1}\right)^{2} \; for\; 1\le s\le m \\ P(R=2\cdot s+1)=\frac{C_{m-1}^{s}\cdot C_{m-1}^{s-1}+C_{n-1}^{s}\cdot C_{n-1}^{s-1}}{C_{m+n}^{n}} \; for\; 1\le s< m \\ P(R=2\cdot m+1)=\frac{C_{n-1}^{m}}{C_{m+n}^{m}} \; if \; m<n \\ P(R=r)=0 \; otherwise \end{cases}, \qquad (2')$$

and we switch m and n in the case $m>n$.

We accept the hypothesis of mutual independence if and only if $|Z|<Z_{1-\frac{\varepsilon}{2}}$, where $Z_{1-\frac{\varepsilon}{2}}$ is the centil of the standard normal distribution for the level $1-\frac{\varepsilon}{2}$.

Noticing that $|Z|<Z_{1-\frac{\varepsilon}{2}}$ is equivalent to $\frac{\varepsilon}{2}<\Phi(Z)<1-\frac{\varepsilon}{2}$, where Φ is the cdf of the standard normal variable, we can write the condition for the independence in the case $m\le 20$ or $n\le 20$: $\frac{\varepsilon}{2}<F(R)<1-\frac{\varepsilon}{2}$, where $F(r)$ is computed using (2').

For the mutual independence of only one sample we compute the median of the sample, μ, and we delete from the sample the values equal to μ. Considering m and n the number of values less than, respectively greater than μ, we take as order before counting the number of sequences the order of appearing in the sample. Therefore a sequence contains only values less than μ, or only values greater than μ.

Another test for mutual independence is the turning points test (Kendall: see [2]). This is a non-parametric statistical test and it can be used to test the null hypothesis that the elements of the sequence are mutually independent and identically distributed (iid).

Let $X_1, X_2, ..., X_n$ be a sample of size *n* and one says that there is a turning point at time *i*, $1<i<n$, if $X_{i-1}<X_i>X_{i+1}$ or if $X_{i-1}>X_i<X_{i+1}$. In these conditions, the probability to have a turning point at time *i* is $\frac{2}{3}$.

If *T* is the number of turning points of the above sample, the expectation and the variance of the random variable *T* are

$$\begin{cases} E[T]=\frac{2\cdot(n-2)}{3} \\ Var[T]=\frac{16n-29}{90} \end{cases}. \qquad (3)$$

It is well known that the random variable $Zc=\frac{T-M[T]}{\sqrt{Var[T]}}$ tends to the normal standard random variable.

We accept the mutual independence if and only if $|Z|<Z_{1-\frac{\varepsilon}{2}}$, where ε is the first degree error of the test (analogous to the Wald-Wolfowitz test).

The Mann-Whitney-Wilcoxon test tests with a given error ε if the values from the sample $X_1, ..., X_n$ are homogenous (see [5,1]). First we compute the sample median μ.

We denote by R_1 the sum of the ranges (of the positions) for the values less than μ (m values), respectively by R_2 the sum of the ranges for the values greater than μ (*n* values). Computing

$$\begin{cases} W_1=m\cdot n+\frac{m(m+1)}{2}-R_1 \\ W_2=m\cdot n+\frac{n(n+1)}{2}-R_2 \end{cases}, \qquad (4)$$

we take $W=\max(W_1, W_2)$. The expectation and the variance of W are (see [5,1])

$$\begin{cases} E(W)=\frac{m\cdot n}{2} \\ Var(W)=\frac{m\cdot n\cdot(m+n+1)}{12} \end{cases}. \qquad (4')$$

If $m,n \geq 8$ then the distribution of W can be approximated with the normal distribution. Therefore

$$Z=\frac{W-E(W)}{\sqrt{Var(W)}} \sim N(0,1). \qquad (5)$$

We accept the hypothesis of homogeneity if and only if $Z < Z_{1-\varepsilon}$.

The Mann-Kendall test is used to verify the null hypothesis of the lack of the trend with a given error ε for a sample $X_1,...,X_n$. We compute first (see [1,6])

$$T=\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\operatorname{sgn}(x_j-x_i). \qquad (6)$$

The variance of T is

$$Var(T)=\frac{n(n-1)(2\cdot n+5)-\sum_{p=1}^{g}t_p(t_p-1)(2\cdot t_p+5)}{18}, \qquad (7)$$

where g is the number of tied groups (in a tied group we have the same value of the sample), and t_p is the number of points in the tied group p. We compute (see [1,6])

$$Z_C=\begin{cases} \frac{T-1}{\sqrt{Var(T)}},\ if\ T>0 \\ 0,\ if\ T=0 \\ \frac{T+1}{\sqrt{Var(T)}},\ if\ T<0 \end{cases}. \qquad (8)$$

We accept the lack of trend if and only if $|Z_C| < Z_{1-\frac{\varepsilon}{2}}$. If we reject the lack of the trend and $Z_C > 0$ we have a positive trend. If we reject the lack of the trend and $Z_C < 0$ we have a negative trend.

2. The method to apply tests

Each test from the above section consists in comparing a statistics to a given centil of the standard normal distribution. We can compute the centil Z_α by a numeric method, or by the Monte Carlo method.

By a numeric method we have to compute $y(\alpha)$ by solving the Cauchy problem

$$\begin{cases} y'(x)=\frac{1}{\varphi(y)} \\ y(0.5)=0 \end{cases}, \qquad (9)$$

where φ is the probability density function of the standard normal distribution: $\varphi(x)=\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}$. The methods to solve the above Cauchy problem can be in our $C++$ program the Euler method, the modified Euler method or the Runge-Kutta method (see [4]). In this paper we refer only to the Runge-Kutta method, because it is the most accurate.

To compute the centil by the Monte Carlo method, we generate first 10000 standard normal variables, and next we order them in increasing order. The centil Z_α is now the value on the position $10000\cdot\alpha$. The method to generate the normal variables can be the central limit method, the Box-Muler method, or the Butcher 1 method (see [7]). We refer only to the Box-Muler method because it is the most rapid. In the following table we write the values of some centils obtained by the Runge-Kutta method.

α	0.9	0.95	0.975	0.99	0.995

z_α	1.28155	1.64485	1.95996	2.32635	2.57583

To compute $\Phi(Z)$ by a numeric we can use the rectangles method, the trapezoids method or the Simpson method (see [4]). To compute this value by the Monte Carlo method we generate 10000 standard normal variables, and $\Phi(Z)$ is the ratio of the generated values less than or equal to Z.

This methods to compute the centils of the normal random variable and its cdf were used in [3] to make normal residues in linear regression.

3. Application

Consider the 109 yearly data (from 1900 to 2008) on the discharges of the Danube at Orşova. The maximum annual discharges were selected for statistical analysis. According to the above mentioned tests the selected values are mutual independent only for $\varepsilon = 0.01$ but the null hypothesis is rejected for $\varepsilon > 0.01$ (Wald Wolfowitz test. In the case of Kendall test, the null hypothesis is accepted for $\varepsilon < 0.05$. On the contrary, the other two tests put into evidence the null hypothesis of homogeneity and the lack of trend.

4. Conclusions

The lack of independence of the maximum annual discharges for some values of ε can be explained by the inertia of the solar cycles, which influence the hydrologic reaction of the Danube river basin system. Despite the lack of a full independence, taking into account that other two basic assumptions of the statistic computations (the homogeneity and the lack of trend) are fulfilled, the registered maximum annual discharges were used for statistical processing of the main characteristics of the flood waves: discharges and volumes corresponding to different probabilities of exceedance.

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References

[1] Armeanu, I. and Petrehuş, V.: *Probabilități si Statistică Aplicate în Biologie*, Matrix Rom, Bucureşti, 2006.

[2] Brockwell, P.J. and Davis, R.A.: *Springer Texts in Statistics, Introduction to Time Series and Forecasting*, Springer-Verlag, 2002.

[3] Ciuiu, D.: "Numerical and Monte Carlo Methods to Make Normal Residues in Regression", *Romanian Journal of Economic Forecasting*, **4** (2009), 119-131.

[4] Păltineanu, G., Matei, P. and Mateescu, G.D.: *Analiză numerică*, Conspress, Bucureşti, 2010.

[5] Petrehuş, V and Popescu, S.A.: *Probabilități şi Statistică*, Ed. UTCB, Bucureşti, 1997.

[6] Prashanth Khambhammettu: "Mann-Kendall Analysis for the Fort Ord Site", *HydroGeoLogic, Inc.-OU-1 2004 Annual Groundwater Monitoring Report-Former Fort Ord, California*, 2005.

[7] I. Văduva: *Modele de simulare*, Ed. Universității Bucureşti, 2004.

APPLICATIONS OF FULMAN-MUHLY-WILLIAMS THEOREM ABOUT CONTINUOUS TRACE GROUPOID CROSSED PRODUCTS

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Abstract: In the context of G being a second countable, locally compact groupoid with Haar system and $\mathcal{A}$ a bundle of C^*-algebras defined over the unit space of G on which G acts continously, Fulman-Muhly-Williams theorem establishes conditions under the asociated crossed product $C^*(G, \mathcal{A})$ is a continous trace C^*-algebra. In this paper are analysed some particular cases of the concrete groupoids and bundles of C^*-algebras: the case when the groupoid G is a group, the case when G is a group transformation and the case when the bundle of C^*-algebras $\mathcal{A}$ is a bundle with constant fiber, the C^*-algebra A.
Mathematics Subject Classification (2000): 46L05
Key words: General theory of C^*-algebras

1. Introduction

Throughout $C^*(G, \mathcal{A})$ will be denoted C^*-algebra crossed product asociated to a second countable grupoid G with Haar system of measures $\{\lambda^u\}_{u\in G^{(0)}}$,where $G^{(0)}$ is the unit space of G, $\mathcal{A}$ a bundle of C^*-algebras over $G^{(0)}$, and from G to $Iso(\mathcal{A}) = \{(u,V,v) / V : A(v) \to A(u)$ a C^*-algebras isomorphism, $v,u \in G^{(0)}\}$ it exists a continuos homomorphism σ, such that $\sigma_g : A(s(g)) \to A(r(g))$ is a C^*-algebras isomorphism for any $g \in G$. In the following sentences it will be denoted by $C_0(G^{(0)}, \mathcal{A})$, C^*-algebra of continuous sections of fiber $\mathcal{A}$ who vanish at infinity over $G^{(0)}$, and by X the spectrum of $C_0(G^{(0)}, \mathcal{A})$. If we consider $\hat{p}$ the canonical surjection from X to $G^{(0)}$, the groupoid G will act on X, using $\hat{p}$, in this way: if $x\in X$, π_x represents an ireductibil representation of $A(u)$ with the equivalence class, $[\pi_x]$. If $X * G$ is the space $\{(x,g)\in X\times G / \hat{p}(x) = r(g)\}$, for every pair $(x,g)\in X*G$, $x\cdot g := [\pi_x \circ \sigma_{g^{-1}}]$ will define a continuous action of G on X. In which conditions $C^*(G, \mathcal{A})$ is a continuous trace C^*-algebra is given by the Fulman-Muhly-Williams theorem (theorem 1 from [1]):

Theorem 1.1.(*Fulman-Muhly-Williams*) If C^*-algebra $C_0(G^{(0)}, \mathcal{A})$ is a continuous trace C^*-algebra and if the action of G on X is free, then the crossed product $C^*(G, \mathcal{A})$ is continuous trace C^*-algebra if and only if the action of G on X is proper.

The purpose of this paper is to analyse the special cases when the groupoid G is a group and a transformation group, and the case when $\mathcal{A}$ is a bundle with constant fiber, the C^*-algebra A.

2. The case when *G* is a group

In the case when the groupoid *G* is a topological, locally compact group, the unit space will contain only one element, the neutral element *e*, and the bundle of C^*-algebras, $\mathcal{A}$, will be formed from only one C^*-algebra, denoted *A*. In this case, the action σ will be a continuous homomorphism from *G* to the automorphismes group of *A*, and the triplet (A, G, σ) will be, in the clasical terminology, a C^*- dynamical system. The theorem Fulman-Muhly-Williams will be trasformed in this way:

Theorem 2.1. If (A, G, α) is a C^*- dynamical system like above, if C^*-algebra A is a continuous trace C^*-algebra and if the action of *G* on the spectrum of *A* is free, than the crossedproduct $C^*(G, A)$ is a continuous trace C^*-algebra if and only if the action of *G* on the spectrum of *A* is proper.

The result from theorem 2.1. is almost the same like the result of Raeburn and Rosenberg obtained in [2] (theorem 1.1) in a different manner, with the specification that in the above result is eliminated the condition that the spectrum of C^*-algebra *A* to be Hausdorff.

3. The case when *G* is a transformation group

In this case we have to remind that if *H* is a group who act (to the right) on a set *T*, the set denoted $G = T \times H$ will become a groupoid if we consider the composable elements to be $\{(t_1, h_1), (t_2, h_2) / t_2 = t_1 h_1, t_1, t_2 \in T, h_1, h_2 \in H\}$ and the natural operations of a groupoid $(t_1, h_1)(t_1 h_1, h_2) = (t_1, h_1 h_2)$ and $(t, h)^{-1} = (th, h^{-1})$. The unit space, in this case, will be identified with the set *T*. Moreover, if *H* is a locally compact group and *T* a locally compact, topological space, the groupoid *G* is a topological, locally compact groupoid with the product topology induced by the topologies of *T* and *H* and the system of measures given by $\{\lambda^t\}_{t \in T} = \{\delta_t \times \lambda\}_{t \in T}$ where λ is Haar meaasure on *H* and δ_t Dirac measure will be a Haar system. With these observations, theorem 1.1. will become:

Theorem 3.1 If $G = T \times H$ is a locally compact transformation group, $\mathcal{A}$ a bundle of C^*-algebras over *T*, *X* the spectrum of C^*-algebra $C_0(T, \mathcal{A})$ and if we assume that $C_0(T, \mathcal{A})$ is a continuous trace C^*-algebra and the action of *G* on *X* is free, than the crossed product $C^*(G, \mathcal{A})$ is a continuous trace C^*-algebra if and only if the action of *G* on *X* is proper.

4. The case when the bundle $\mathcal{A}$ is a bundle with constant fiber, the C^*-algebra *A*

In this case, the space $Iso(\mathcal{A})$ will become the group of automorhismes of C^*-algebra *A*, usually noted $Aut(A)$, and the triplet (A, G, σ) will be a C^*-groupoid dynamical system ($\sigma : G \to Aut(A)$ a continuous homomorphism). Theorem 1.1. will become:

Theorem 4.1. We consider (A,G,σ) a C^*-groupoid dynamical system. If C^*-algebra $C_0(G^{(0)},\ A)$ is a continuous trace C^*-algebra and if the action of G on spectrum of A is free, than the following sentences are equivalent:

i) the crossed product $C^*(G,A)$ is a continuous trace C^*-algebra
ii) the action of G on $\hat{A}$ (the spectrum of A) is proper.

Moreover, because we can identify $C_0(G^{(0)},A)$ with the tensor product $C_0(G^{(0)})\otimes A$ and because, $G^{(0)}$ is a Hausdorff space , $C_0(G^{(0)})$ will be a continuous trace C^*-algebra we can adapt theorem 4.1. in this way:

Theorem 4.2. We consider (A,G,σ) a C^*-groupoid dynamical system. If C^*-algebra A is a continuous trace C^*-algebra and if the action of G on spectrum of A is free, than the following sentences are equivalent:

i) the crossed product $C^*(G,A)$ is a continuous trace C^*-algebra
ii) the action of G on $\hat{A}$ (the spectrum of A) is proper.

<u>Observation</u> The result from theorem 4.2. extends the result from theorem 2.1. at the case in which the group G is a groupoid.

References

[1] Igor Fulman, Paul Muhly, Dana P. Williams, *Continuos-Trace groupoid crossed products,* Proceedings of the American Mathematical Society, vol 132, nr. 3, (2004), 707-717

[2] Raeburn, J., Rosenberg J., *Crossed products of continuos-trace $C^* -$algebras by smooth actions,* Transactions of the American Mathematical Society, volume 305, (1988), 1-45

[3] Raeburn, J. and Williams, D.: *Morita equivalence and continuous trace C^*-algebras*, Mathematical Surveys and Monographs, vol. 134, American Mathematical Society, Providence, 2007

ULAM-HYERS STABILITY OF THE OPERATORIAL EQUATIONS ON COMPLETE VECTOR LATTICES

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Abstract: In this paper we present Ulam-Hyers stability using the weakly Picard operators, Ulam stability of the coincidence equations and of the operatorial inclusions, generalized Ulam-Hyers stability.

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Key words: weakly Picard operators, fixed point equation, coincidence point equation, operatorial inclusions, Ulam-Hyers stability, generalized Ulam-Hyers stability.

1. Ulam-Hyers stability of the weakly Picard operators

Let X be a complete vector lattice and $T: X \to X$ an operator. We denote by $\text{Fix}(T) = \{x \in X \mid T(x) = x\}$ the fixed point set of the operator T.

Definition 1.1 The operator $T: X \to X$ is weakly Picard operator if the sequence of successive approximations $\{T^n(x)\}_{n\in\mathbb{N}}$ (o)-converges for all $x \in X$ and the limit is a fixed point of T.

If T is a weakly Picard operator and if the operator $T^\infty: X \to X$ is defined by $T^\infty(x) = (o) - \lim\limits_{n\to\infty} T^n(x)$, then $T^\infty(x) = \text{Fix}(T)$.

If T is a weakly Picard operator and if $\text{Fix}(T) = \{x^*\}$, then by definition T is a Picard operator and $T^\infty(x) = x^*$, $\forall x \in X$.

Definition 1.2 Let $T: X \to X$ be a weakly Picard operator and $c > 0$, $c \in \mathbb{R}$. By definition the operator T is a c-weakly Picard operator if

$$\left|x - T^\infty(x)\right| \le c\left|x - T(x)\right|, \ \forall x \in X.$$

Example 1.1 Let X be a complete vector lattice, $\varphi: X_+ \to X_+$ and $T: X \to X$ an operator with closed graphic. We suppose that:

(i) T is φ-Caristi operator, i.e., $\left|x - T(x)\right| \le \varphi(x) - \varphi(T(x))$, $\forall x \in X$;

(ii) there exists $c > 0$, such that $\varphi(x) \le c\left|x - T(x)\right|$, $\forall x \in X$.

Then T is a c-weakly Picard operator.

Definition 1.3 Let X be a complete vector lattice and $T: X \to X$ an operator. The fixed point equation

$$x = T(x) \tag{1.1}$$

is Ulam-Hyers stable if there exists $c_T > 0$, $c_T \in \mathbb{R}$ such that: for each $u > \mathbf{0}$ and each solution y^* of the inequation

$$\left|y - T(y)\right| \le u \tag{1.2}$$

there exists a solution x^* of the equation (1.1) such that:

$$\left|y^* - x^*\right| \le c_T \cdot u$$

Remark 1.1 If T is a c-weakly Picard operator then the fixed point equation (1.1) is Ulam-Hyers stable.

2. Generalized Ulam-Hyers stability of a fixed point equation

Let X be a complete vector lattice, $Y \subset X$, $Y \neq \emptyset$ and $T : Y \to X$ an operator.

Notations

$I(T) = \{Z \subset Y \mid T(Z) \subset Z,\ Z \neq \emptyset\}$ the set of all invariant subset of T.

$(MI)_T = UI(T)$ -the maximal invariant subset of T

$(AB)_T(x^*) = \left\{x \in Y \mid T^n(x) \text{ is defined for all } n \in \mathbb{N} \text{ and } T^n(x) \overset{o}{\to} x^* \in \text{Fix}(T)\right\}$ - the attraction basin of the fixed point x^* with respect to T.

$(AB)_T = \bigcup_{x^* \in \text{Fix}(T)} (AB)_T(x^*)$ -the attraction basin of T.

Definition 2.1 By definition an operator $T : Y \to X$ is weakly Picard operator if $\text{Fix}(T) \neq \emptyset$ and $(MI)_T = (AB)_T$. If $\text{Fix}(T) = \{x^*\}$ then a weakly Picard operator is said to be Picard operator.

Definition 2.2 For each weakly Picard operator $T : Y \to X$ we define the operator $T^\infty : (AB)_T \to (AB)_T$ by $T^\infty(x) = \lim_{n\to\infty} T^n(x)$.

Definition 2.3 Let $\psi : X_+ \to X_+$ be an increasing function which is (o)-continuous in $\mathbf{0}$ and $\psi(\mathbf{0}) = \mathbf{0}$. An operator $T : Y \to X$ is said to be a ψ- weakly Picard operator if it is weakly Picard operator and

$$\left|x - T^\infty(x)\right| \le \psi\left(\left|x - \psi(x)\right|\right),\ \forall x \in (MI)_T.$$

If $\psi(u) = c \cdot u$ with $c > 0$, $c \in \mathbb{R}$ we say that T is c- weakly Picard operator.

Let us consider the fixed point equation

$$x = f(x) \tag{2.1}$$

and the inequation:

$$\left|y - f(y)\right| \le u \tag{2.2}$$

Definition 2.4 The equation (2.1) is generalized Ulam-Hyers stable if there exists $\psi : Y_+ \to Y_+$ increasing and (o)-continuous in $\mathbf{0}$ with $\psi(\mathbf{0}) = \mathbf{0}$ such that for each $u > \mathbf{0}$ and for each solution $y^* \in (AB)_T$ of (2.2) there exists a solution x^* of (2.1) such that

$$\left|y^* - x^*\right| \le \psi(u)$$

If $\psi(t) = c \cdot t$ with $c > 0$, the equation (2.1) is said to be Ulam-Hyers stable.

Remark 2.1 If an operator $T : Y \to X$ is ψ- weakly Picard operator, then the fixed point equation (2.1) is generalized Ulam-Hyers stable. If T is c- weakly Picard operator, then the equation (2.1) is Ulam-Hyers stable.

Example 2.1 Let X be a complete vector lattice, $Y \subset X$, and $T : Y \to X$ a strict φ contraction with $\text{Fix}(T) \neq \emptyset$. Then the equation (2.1) is generalized Ulam-Hyers stable.

3. Ulam stability of the coincidence equations

Let X and Y be two complete vector lattices. If $T,U: X \to Y$ are two operators then, we denote by:

$$C(T,U) = \{x \in X \mid f(x) = g(x)\}$$

the coincidence point set of the pair T, U.

Definition 3.1 Let $c > 0$, $c \in \mathbb{R}$. By definition the pair $T, U: X \to Y$ is c- weakly Picard operator pair if there exists an operator $V: X \to X$ such that:

(i) V is weakly Picard operator;

(ii) $\text{Fix}V = C(T,U)$;

(iii) $|x - V^\infty(x)| \le c|T(x) - U(x)|$, $\forall x \in X$.

We remark $V^\infty(x) = C(T,U)$.

Definition 3.2 The equation

$$T(x) = U(x) \tag{3.1}$$

is Ulam-Hyers stable if there exists $c > 0$, $c \in \mathbb{R}$ such that for each $u > \mathbf{0}$ and for each solution y^* of the inequation

$$|T(y) - U(y)| \le u \tag{3.2}$$

there exists a solution x^* of (3.1) such that

$$|y^* - x^*| \le c \cdot u$$

In a similar way we can define the generalized Ulam-Hyers stability of equation (3.1).

Remark 3.1 If a pair $T, U: X \to Y$ is a c- weakly Picard pair, then the equation (3.1) is Ulam-Hyers stable.

4. Ulam stability of the operatorial inclusions

Let X be a complete vector lattice, $A, B \in P(X)$ and $T: X \to P(X)$ a multivalued operator. We denote:

$P_{cp}(x) = \{Y \in P(X) \mid Y \text{ a compact subset of } X\}$

$D(A,B) = \inf\{|a - b| \mid a \in A, b \in B\}$

$G(T) = \{(x,y) \mid x \in X, y \in T(x)\}$ - the graphic of T

Definition 4.1 An operator $T: X \to P(X)$ is a multivalued weakly Picard operator iff for each $x \in X$ and each $y \in T(x)$ there exists a sequence of successive approximations $\{x_n\}_{n \in \mathbb{N}}$, $x_{n+1} \in T(x_n)$, $n \in \mathbb{N}$, such that:

(i) $x_0 = x$, $x_1 = y$;

(ii) $x_n \xrightarrow{o} x^* \in \text{Fix}(T)$

Definition 4.2 For a multivalued weakly Picard operator T we define the multivalued operator

$$T^\infty : G(T) \to P(\text{Fix}(T))$$

by $T^\infty(x,y) = \{z \in \text{Fix}(T) \mid$ there exists a sequence of successive approximations of T starting from (x,y) that $(o)-$converge to $z\}$.

Definition 4.3 Let $\psi : X_+ \to X_+$ be an increasing function which is (o)-continuous in $\mathbf{0}$ and $\psi(\mathbf{0}) = \mathbf{0}$. An operator $T : X \to P(X)$ is ψ-multivalued weakly Picard operator if there exists a selection t^∞ of T^∞ such that

$$\left|x - t^\infty(x, y)\right| \le \psi(|x - y|),\ \forall (x, y) \in G(T).$$

If $\psi(t) = c \cdot t$, $c > 0$, $c \in \mathbb{R}$ then T is called a c-multivalued weakly Picard operator.

Definition 4.4 The equation:

$$x \in T(x) \qquad (4.1)$$

is Ulam-Hyers stable if there exists $c > 0$ such that: for each $v > \mathbf{0}$ and for each solution u^* of the inequation:

$$D(u, T(u)) \le v \qquad (4.2)$$

there exists a solution x^* of (4.1) such that

$$\left|u^* - x^*\right| \le c \cdot v$$

Definition 4.5 The equation (4.1) is generalized Ulam-Hyers stable if there exists a function $\psi : X_+ \to X_+$ (o)-continuous in $\mathbf{0}$ with $\psi(\mathbf{0}) = \mathbf{0}$ such that: for each $v > \mathbf{0}$ and for each solution u^* of the inequation (4.2) there exists a solution x^* of (4.1) such that

$$\left|u^* - x^*\right| \le \psi(v)$$

Remark 4.1 If the operator $T : X \to P_{cp}(X)$ is a c-multivalued weakly Picard operator then the equation (4.1) is Ulam-Hyers stable. If $T : X \to P_{cp}(X)$ is a ψ-multivalued weakly Picard operator then the equation (4.1) is generalized Ulam-Hyers stable.

References

[1] A. Chis-Novac, R. Precup, I. A. Rus, *Data dependence of fixed point for non-self generalized contraction*, Fixed Point Theory, **10**(2009), No.1, 73-87.

[2] A. Petrusel, *Multivalued weakly Picard operators and applications*, Scientiae Mathematica Japonicae, **59**(2004), 167-202.

[3] D.Popa, *Hyers-Ulam stability of the linear recurence with constant coefficients*, Adv. In Difference Equations, **2**(2005), 101-107.

[4] I. A. Rus, *Picard operators and applications*, Sciantiae Math. Japonicae, **58**(2003), No.1, 191-219..

[5] I. A. Rus, A. Petrusel, A. Sîntămărian, *Data dependence of the fixed point set of some multivalued weakly Picard operators*, Nonlinear Anal., **52**(2003), 1947-1959.

ON FULL HILBERT C*-MODULES AND THEIR REPRESENTATIONS

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Abstract: This paper is dedicated to the full Hilbert C*-modules and their faithful representations.
In Section 2, the first result about full Hilbert C*-modules involves a construction that is relevant to the notion of Morita equivalence ([5]). We present a theorem which characterizes stable isomorphism for σ-unital C*-algebras in terms of Morita equivalence ([3], [5]).
In Section 3 we discuss about sets of generators $\{x_i ; i \in I\}$ for a full Hilbert A-module X for which the operators $b \mapsto \langle x_i, x_i \rangle b \langle x_i, x_i \rangle$, $i \in I$, on A are compact, or equivalently, the operators $x \mapsto x_i \langle x, x_i \rangle$, $i \in I$, on X are compact and we show that the existence of such set of generators characterizes Hilbert C*-modules over compact operators ([2]). In particular, examples of such sets of generators are orthonormal bases.
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Key words: (full) Hilbert C*-module, adjointable operator, Morita equivalence, faithful representation.

1. Introduction

Hilbert C*-modules were first introduced by Kaplansky in 1953 and his idea was to generalise Hilbert space by allowing the inner product to take values in an unital C*-algebra instead of the field **C** of complex numbers. In some ways, Hilbert C*-modules behave like Hilbert spaces, but there are some fundamental Hilbert space properties which are not established.
In Section 2 we present a theorem of Brown, Green and Rieffel which characterizes stable isomorphism for σ-unital C*-algebras in terms of relation called Morita equivalence (Theorem 2.12).
It is well known that every Hilbert C*-module can be isometrically embedded into a Banach space $L(H_1, H_2)$, for some Hilbert spaces H_1 and H_2, so that it is allow to extend the notion of representation from C*-algebras to Hilbert C*-modules.
In [**2**], Arambašić works with a certain class of sets of generators for a full Hilbert C*-module in order to characterize Hilbert C*-modules over the C*-algebra of compact operators (Theorem 3.7). Moreover, in is obviously that an arbitrary Hilbert C*-module does not have an orthornormal basis and then, Corollary 3.8 establishes the existence of orthonormal bases in full Hilbert C*-modules over a C*-algebra of compact operators.

2. Full Hilbert C*-modules and Morita equivalence

Definition 2.1. A *Hilbert C*-module over a C*-algebra* A (or a *Hilbert* A*-module*) is a linear space X that is a right A-module equipped with an A-valued inner-product $\langle \cdot, \cdot \rangle$ on $X \times X$ which is A-linear in the second and conjugate linear in the first variable, such that X is a Banach space with the norm given by $\|x\| = \left(\|\langle x, x \rangle\|\right)^{\frac{1}{2}}$ $(x \in X)$.

Definition 2.2. If X, Y are two Hilbert A-modules, we denote by $L(X,Y)$ or $L_A(X,Y)$ the space of all *adjointable operators* acting between X and Y, that is, the set of all bounded A-linear maps $T: X \to Y$ for which there is a map $T^*: Y \to X$ (called the *adjoint* of T) such that $\langle Tx, y\rangle = \langle x, T^* y\rangle$, $x \in X, y \in Y$. Furthermore, by $K(X,Y)$ or $K_A(X,Y)$ we denote the closed linear subspace of $L(X,Y)$ spanned by $\{\theta_{x,y};\, x \in Y, y \in X\}$, where $\theta_{x,y} \in L(X,Y)$ is defined by $\theta_{x,y}(z) = x\langle y, z\rangle$, $z \in X$. The elements of $K(X,Y)$ are often called "compact" operators. We abbreviate the C*-algebras $L(X) = L(X,X)$ and $K(X) = K(X,X)$.

Definition 2.3. A Hilbert A-module X is said to be *full* if the ideal $\langle X, X\rangle \subseteq A$ coincides with A, where $\langle X, X\rangle$ is the closed linear subspace of A generated by $\{\langle x, y\rangle;\, x, y \in X\}$ (an ideal in a C*-algebra always means a closed two-sided ideal).

Observation 2.4. If A is a C*-algebra, X, Y are two Hilbert A-modules and $B = K_A(X)$, then $K_A(X,Y)$ is a Hilbert B-module, with the B-valued inner-product defined by

$$\langle T, S\rangle_B = T^* S, \text{ for } T, S \in K_A(X,Y).$$

Theorem 2.5. ([5]) Let A be a C*-algebra, let X, Y be Hilbert A-modules, let $B = K_A(X)$ and let Z be the Hilbert B-module $K_A(X,Y)$. If X is full, then

$$K_B(Z) \cong K_A(Y) \text{ and } L_B(Z) \cong L_A(Y).$$

Definition 2.6. If A is a C*-algebra (regarded as a full Hilbert A-module with the inner-product given by $\langle a, b\rangle = a^* b$, $a, b \in A$) and H is a Hilbert space (regarded as a Hilbert C*-module over algebra **C**), then $H \otimes A$ is called *the exterior tensor product of* H *and* A and is defined as a Hilbert $\mathbf{C} \otimes A \cong A$-module, where the $\mathbf{C} \otimes A \cong A$-valued inner-product is given by

$$\langle h_1 \otimes a_1, h_2 \otimes a_2\rangle = \langle h_1, h_2\rangle \otimes \langle a_1, a_2\rangle = \langle h_1, h_2\rangle\, a_1^* a_2,\ h_1, h_2 \in H,\ a_1, a_2 \in A.$$

In the case where H is a separable infinite-dimensional Hilbert space, the Hilbert A-module $H \otimes A$ plays a special role in the theory of Hilbert C*-modules and we denote it by H_A.

The C*-algebra $K(H \otimes A) = K(H_A)$ is called the *stable algebra of* A.

Obviously, we have $K(H_A) \cong K(H) \otimes K(A) \cong K(H) \otimes A$, where $K(H)$ is the C*-algebra of all compact operators on H.

Definition 2.7. A C*-algebra A is called σ*-unital* if it has a countable approximate unit or it has a strictly positive element.

Definition 2.8. If X is a Hilbert A-module, then $\{x_i;\, i \in I\} \subseteq X$ is a *generating set for* X if the closed submodule of X of finite A-linear combinations $\left\{\sum_{j \in J} x_j a_j;\ J \subset I \text{ finite}, a_j \in A\right\}$ is the whole of X. We say that X is *countably generated* if it has a countable generating set.

Definition 2.10. Two C*-algebras A and B are said to be:

1) *stably isomorphic* if their stable algebras $K(H_A)$ and $K(H_B)$ are isomorphic.

2) *Morita equivalent* if there is a full Hilbert A-module X such that $B \cong K_A(X)$.

Morita equivalence is an equivalence relation.

The next result can be expressed as follows: if A is a σ-unital C*-algebra and X is a Hilbert A-module which is "large enough" (full) but "not too large" (countably generated), then X is stably unitarily equivalent to A, where "stable" means here "when tensored with H".

Proposition 2.11. ([5]) Let A be a σ-unital C*-algebra, let X be a full Hilbert A-module and let H be a separable Hilbert space. Then:

(i) there is a Hilbert A-module Y such that $H \otimes X \cong A \oplus Y$

(ii) if X is countably generated, then $H \otimes X \cong H_A$.

Theorem 2.12. ([3], [5]) Two σ-unital C*-algebras are stably isomorphic if and only if they are Morita equivalence.

3. Faithful representations of Hilbert C*-modules

Definition 3.1. A *representation* of a Hilbert A-module X is a map $\Phi : X \to L(H_1, H_2)$, where H_1, H_2 are Hilbert spaces, for which there is a representation $\varphi : A \to L(H_1)$ of C*-algebra A such that $\langle \Phi(x), \Phi(x) \rangle = \varphi(\langle x, y \rangle)$, for all $x, y \in X$ (where $L(H_1, H_2)$ is the space of all linear bounded operators acting between H_1 and H_2). The representation Φ is called *faithful* if it is injective.

Definition 3.2. For a Hilbert C*-module X over a C*-algebra A, *the linking algebra* $\mathcal{L}(X)$ may by written as the matrix algebra of the form

$$\mathcal{L}(X) = \begin{bmatrix} K(A) & K(X, A) \\ K(A, X) & K(X) \end{bmatrix}.$$

We observe that $\mathcal{L}(X)$ is in fact the C*-algebra of all "compact" operators on Hilbert A-module $A \oplus X$ and it can be rewritten as

$$\mathcal{L}(X) = \begin{bmatrix} K(A) & K(X, A) \\ K(A, X) & K(X) \end{bmatrix} = \left\{ \begin{bmatrix} T_a & l_y \\ r_x & T \end{bmatrix}; a \in A, x, y \in X, T \in K(X) \right\}$$

where every $x \in X$ and $a \in A$ induce the maps $r_x \in L(A, X)$, $l_y \in L(X, A)$ and $T_a \in K(A)$ given by $r_x(b) = xb, b \in A$, $l_x(z) = \langle x, z \rangle, z \in X$ and $T_a(b) = ab, b \in A$ such that $l_x^* = r_x$.

Observation 3.3. 1) For any C*-algebra A, there is π a faithful $*$-representation of A on some Hilbert space H such that for any $a \in A$ it holds: the operator $\pi(a)$ on H is compact if and only if the operator $b \mapsto aba$ on A is compact.

2) For any Hilbert C*-module, there is a faithful representation to $L(H_1, H_2)$, for some Hilbert spaces H_1 and H_2.

Remark 3.4. ([1]) If $\Phi : X \to L(H_1, H_2)$ is a representation of Hilbert A-module X, then the map $\pi_{\varphi,\Phi} : \mathcal{L}(X) \to L(H_1 \oplus H_2)$ is a representation of linking algebra $\mathcal{L}(X)$ given by

$$\pi_{\varphi,\Phi}\left(\begin{bmatrix} T_a & l_y \\ r_x & T \end{bmatrix}\right)(\xi_1 \oplus \xi_2) = (\varphi(a)\xi_1 + \Phi(y)^*\xi_2) \oplus \left(\Phi(x)\xi_1 + \Phi^+(T)\xi_2\right)$$

where $\varphi : A \to L(H_1)$ and $\Phi^+ : K(X) \to L(H_2)$, $\Phi^+(\theta_{x,y}) = \theta_{\Phi(x),\Phi(y)} = \Phi(x)\Phi(y)^*$ are $*$-representations of C*-algebras, $a \in A$, $x, y \in X$, $T \in K(X)$, $\xi_1 \in H_1$, $\xi_2 \in H_2$.

The following result generalizes Observation 3.3, 1), in the context of Hilbert C*-modules:

Proposition 3.5. ([2]) Let X be a full Hilbert A-module. Then there are two Hilbert spaces H_1 and H_2 and two faithful representations $\varphi : A \to L(H_1)$ and $\Phi : X \to L(H_1, H_2)$ such that the following statements are equivalent, for an element $x \in X$:

1) $\varphi(\langle x, x \rangle) \in K(H_1)$

2) $\Phi(x) \in K(H_1, H_2)$ (where $K(H_1, H_2)$ is the space of all compact operators)

3) The operator $b \mapsto \langle x, x \rangle b \langle x, x \rangle$ on A is compact

4) The operator $y \mapsto x \langle y, x \rangle$ on X is compact.

Moreover, $\varphi(\langle x, y \rangle) = \langle \Phi(x), \Phi(x) \rangle = \Phi(x)^* \Phi(x)$, for all $x, y \in X$.

Remark 3.6. In the proof of the previous proposition, we used the fact that the map $\Phi \to \pi_{\varphi,\Phi}$ is a natural bijection between the set of all faithful representations of full Hilbert A-module X and the set of all faithful representations of its linking algebra $\mathcal{L}(X)$.

Theorem 3.7. ([2]) If X is a full Hilbert A-module, then the following conditions are equivalent:

1) A is $*$-isomorphic to a C*-algebra of compact operators

2) There is a set of generators $\{x_i; i \in I\}$ for X such that the operators $b \mapsto \langle x_i, x_i \rangle b \langle x_i, x_i \rangle$, $i \in I$, on A are compact

3) There is a set of generators $\{x_i; i \in I\}$ for X such that the operators $x \mapsto x_i \langle x, x_i \rangle$, $i \in I$, on X are compact

(where by a C*-algebra of compact operators we understand a C*-subalgebra of the C*-algebra of all compact operators on some Hilbert space).

Moreover, A is $*$-isomorphic to a C*-algebra of compact operators, then for every set of generators $\{x_i; i \in I\}$ for X, the operators $b \mapsto \langle x_i, x_i \rangle b \langle x_i, x_i \rangle$ on A and $x \mapsto x_i \langle x, x_i \rangle$ on X are compact, for every $i \in I$.

Corollary 3.8. ([2]) A full Hilbert A-module X has an orthonormal basis if and only if A is $*$-isomorphic to a C*-algebra of compact operators.

References

[1] Arambašić, L.: *Irreducible representations of Hilbert C*-modules*, Math. Proc. R. Ir. Acad. 105 (2), 11-24, 2005.

[2] Arambašić, L.: *Another characterization of Hilbert C*-modules over compact operators*, J. Math. Anal. Appl. 344, 735-740, 2008.

[3] Brown, L.G., Green, P., Rieffel, M.A: *Stable isomorphism and strong Morita equivalence of C*-algebras*, Pacific J. Math. 71, 349-363, 1977.

[4] Brückler, F.M.: *Tensor products of C*-algebras, operator spaces and Hilbert C*-modules*, Mathematical Communications 4, 257-268, 1999.

[5] Lance, C.: *Hilbert C*-modules*, London Math. Soc. Lecture Note Series 210, Cambridge Univ. Press, Cambridge, 1995.

[7] Raeburn, I., Williams, D.P.: *Morita Equivalence and Continuous Trace C*-algebras*, Math. Surveys Monogr., Volume 60, Amer. Math. Soc., 1998.

[9] Rieffel, M.A.: *Morita equivalence for operator algebras*, Proc. Sympos. Pure Math., Volume 38 (Amer. Math. Soc.), 285-298, 1982.

[10] Zamfir, M., Costache, T.L.: *About orthogonality in Hilbert C*-modules*, Proc. 10-th Workshop of Depart. of Math. and Computer Science, TUCE, Bucharest, 167-171, 2009.

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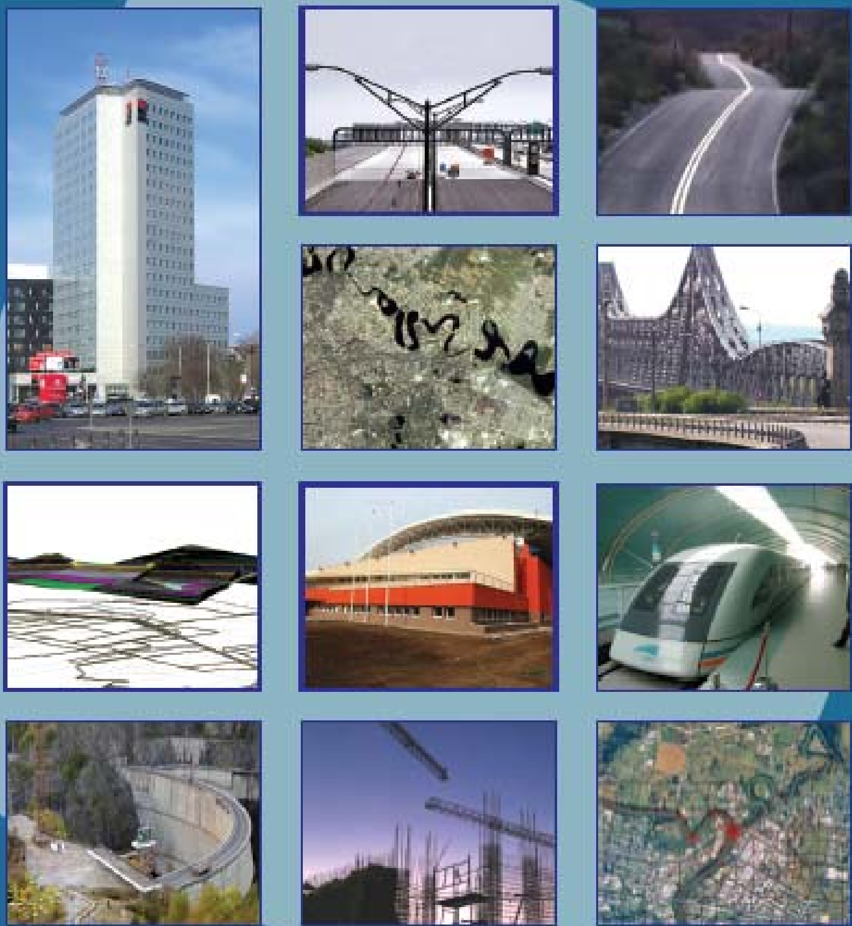

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