TECHNICAL UNIVERSITY OF CIVIL ENGINEERING BUCHAREST
DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE

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PREFACE

The 12-th Workshop of Department of Mathematics and Computer Science, Technical University of Civil Engineering was held in Bucharest, Romania, on May 25, 2013.

The aim of the workshop was the exchange of ideas, methods and problems between researchers, professors, practitioners in Mathematics and related fields. The program included 37 lectures organized in five sections:

1. Mathematical Analysis, Functional Analysis, Numerical Analysis
2. Algebra, Geometry
3. Differential Equations, Partial Differential Equations, Mechanics
4. Computer Science, Mathematical Applications in Engineering Sciences
5. Using some computing programs in research and teaching Mathematics

This volume contains 33 papers corresponding to the research talks that cover a wide variety of topics in Mathematics.

We thank the authors of the contributed papers for timely submission and participation in the workshop.

The Editors

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SOLVING LINEAR ALGEBRA AND DIFFERENTIAL GEOMETRY WITH SAGE

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Abstract: We apply extensively the software SAGE [1] to aid the understanding of Linear Algebra, Analytic Geometry, Differential Geometry, that is useful to all researchers and students in Mathematics, Physics, Engineering and other applied sciences. We shall give a plenty of SAGE applications at each step of our exposition. The mathematical software SAGE combines the power of many existing source packages into a common Python-based interface, therefore SAGE uses the Python programming language.
Mathematics Subject Classification (2010): 68N15
Key words: Linear Algebra, Analytic Geometry, Differential Geometry, Sage, Python

1. Vector spaces

Linear algebra can be regarded as a theory of the vector spaces, because a vector space is a set of objects or elements that can be added together and multiplied by numbers (the result remaining an element of the set), so that the ordinary rules of calculation are valid. An example of a vector space is the geometric vector space (the free vector space), which plays a central role in physics and technology and illustrates the importance of the vector spaces and linear algebra for all practical applications.

Besides the notions which operates mathematics, created by abstraction from environmental observation (for example the geometric concepts) or quantitative and qualitative research of the natural phenomena (for example the notion of number) in mathematics there are elements from other sciences.

The notion of vector from physics has been studied and developed creating vector calculus, which became a useful tool for both mathematics and physics. All physical quantities are represented by vectors (for example the force and velocity). A vector indicates a translation in the three-dimensional space; therefore we study the basics of the three-dimensional Euclidean geometry: the points, the straight lines and the planes.

Example 1 Find for the space of polynomials by at most four degree, the transition matrix from the basis $B_1 = \left\{1, X, X^2, X^3, X^4\right\}$ to the basis $B_2 = \left\{1, X+1, (X+1)^2, (X+1)^3, (X+1)^4\right\}$.

We shall give a solution in Sage:

```
sage: R.<x> = RR['x'];K=R^5
sage: M=K.span([[1,x,x^2,x^3,x^4]])
sage: U=M([1,x+1,(x+1)^2,(x+1)^3,(x+1)^4]);
sage: v0=vector([U[0][0],U[0][1],U[0][2],U[0][3],U[0][4]]);
sage: v1=vector([U[1][0],U[1][1],U[1][2],U[1][3],U[1][4]]);
sage: v2=vector([U[2][0],U[2][1],U[2][2],U[2][3],U[2][4]]);
sage: v3=vector([U[3][0],U[3][1],U[3][2],U[3][3],U[3][4]]);
sage: v4=vector([U[4][0],U[4][1],U[4][2],U[4][3],U[4][4]]);
sage: A = column_matrix([v0,v1, v2, v3,v4]).n(digits=3)
sage: A
[ 1.00  1.00  1.00  1.00  1.00]
[0.000  1.00  2.00  3.00  4.00]
[0.000 0.000  1.00  3.00  6.00]
[0.000 0.000 0.000  1.00  4.00]
[0.000 0.000 0.000 0.000  1.00]
```

Example 2 In the *arithmetic vector space* $\Re^3$ the following vectors are considered: $\bar{a}_1 = (2,-1,2)$, $\bar{a}_2 = (1,-1,2)$, $\bar{a}_3 = (0,3,2)$, $\bar{b}_1 = (0,1,-1)$, $\bar{b}_2 = (2,1,1)$, $\bar{b}_3 = (-1,2,1)$, $\bar{x} = (-1,2,3)$. Prove that $\mathrm{B}_1 = \{\bar{a}_1, \bar{a}_2, \bar{a}_3\}$ is a basis of $\Re^3$. Determine the coordinates of $\bar{x}$ relative to the basis B_1. Prove that $\mathrm{B}_2 = \{\bar{b}_1, \bar{b}_2, \bar{b}_3\}$ is a new basis of $\Re^3$ and write the transition matrix from the basis B_1 to the basis B_2. Write the formulas of changing a vector coordinates when one passes from the basis B_1 to the basis B_2.

```
sage: V=VectorSpace(RR,3)
sage: a1=V([2,-1,2]);a2=V([1,-1,2]);a3=V([0,3,2]);x=V([-1,2,3]);
sage: V.linear_dependence([a1, a2, a3])==[]
True
sage: W = V.subspace_with_basis([a1, a2,a3])
sage: W.coordinate_vector(x)
(-1.62500000000000, 2.25000000000000, 0.875000000000000)
sage: b1=V([0,1,-1]);b2=V([2,1,1]);b3=V([-1,2,1])
sage: V.linear_dependence([b1, b2, b3])==[]
True
sage: v1=W.coordinate_vector(b1);v2=W.coordinate_vector(b2);v3=W.coordinate_vector(b3);
sage: A = column_matrix([v1, v2, v3]).n(digits=3)
sage: A
[ 0.625   1.88 -0.875]
[ -1.25  -1.75  0.750]
[ 0.125  0.375  0.625]
sage: x1, x2, x3,y1,y2,y3 = var('x1, x2, x3,y1,y2,y3')
sage: yy = column_matrix([y1, y2, y3]);
sage: xx=A*yy
sage: xx
[ 0.625*y1 + 1.88*y2 - 0.875*y3]
[ -1.25*y1 - 1.75*y2 + 0.750*y3]
[0.125*y1 + 0.375*y2 + 0.625*y3]
```

2. Linear transformations

The linear transformations need to be studied, because they are compatible with the operations defined in a vector space and allow us to transfer algebraic situations and related problems in three dimensional space. Matrix operations clearly reflect their similarity to the operations with linear transformations; so the matrices can be used for the numerical representation of the linear transformations. The matrix representation of linear transformations is analogous to the representation of the vectors through *n* coordinates relative to a basis.

Example 3. Let $T_1, T_2 \in \mathrm{End}(\Re^3)$, be defined as

$$T_1(\bar{x}) = \left(5x^{(1)} - x^{(2)} - 5x^{(3)},\ 20x^{(1)} - 15x^{(2)} + 8x^{(3)},\ 3x^{(1)} - 2x^{(2)} + x^{(3)}\right)$$

$$T_2(\bar{x}) = \left(10x^{(1)} - 10x^{(2)} + 10x^{(3)},\ 0,\ 5x^{(1)} - 5x^{(2)} + 5x^{(3)}\right), (\forall)\bar{x} = \left(x^{(1)}, x^{(2)}, x^{(3)}\right) \in \Re^3.$$

Find the sum of the two endomorphism matrix $T = T_1 + T_2$ relative to the basis

$$B' = \left\{\bar{v}_1 = (2,3,1), \bar{v}_2 = (3,4,1), \bar{v}_3 = (1,2,2)\right\} \subset \Re^3.$$

Solving this problem in Sage, we shall have:

```
sage: V=RR^3
sage: v1=vector(RR,[2,3,1]);v2=vector(RR,[3,4,1]);v3=vector(RR,[1,2,2]);
sage: W=V.subspace_with_basis([v1,v2,v3])
sage: var("x1 x2 x3")
(x1, x2, x3)
sage: T_symbolic(x1,x2,x3) = [5*x1-x2-5*x3,20*x1-15*x2+8*x3,3*x1-2*x2+x3]
sage: T1=linear_transformation(W, W, T_symbolic)
sage: Tt_symbolic(x1,x2,x3) = [10*x1-10*x2+10*x3,0,5*x1-5*x2+5*x3]
sage: T2=linear_transformation(W, W, Tt_symbolic)
sage: print "(T1+T2).matrix(side='right')\n",(T1+T2).matrix(side='right').n(digits=3)
(T1+T2).matrix(side='right')
[ 1.00 0.000 0.000]
[0.000  2.00 0.000]
[0.000 0.000  3.00]
```

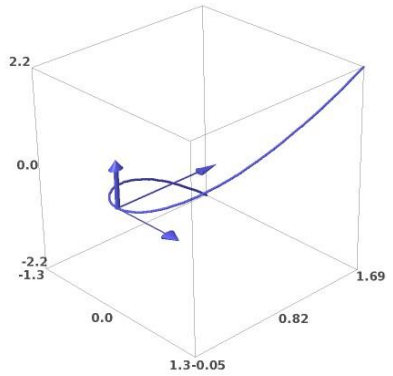
3. Differential geometry

In the Differential geometry, in the study of the geometric figures, we use the concepts and the methods of the mathematical analysis, especially the differential calculus and the theory of differential equations, presented in the sixth chapter. The physical problems lead to inhomogeneous linear differential equations of order *n* with constant coefficients.

Example 4. Find the versors of the Frenet trihedron in the origin for the curve:

$$\Gamma: \ \bar{r} = t\bar{i} + t^2\bar{j} + t^3\bar{k}.$$

```
sage: t=var('t')
sage: r=vector([t,t^2,t^3])
sage: rp=diff(r,t);rs=diff(r,t,2)
sage: n=rp.norm().simplify_exp()
sage: tau=(rp/n).subs(t=0);tau
(1, 0, 0)
sage: v=rp.cross_product(rs)
sage: vn=v.norm().simplify_exp()
sage: be=(v/vn).subs(t=0);be
(0, 0, 1)
sage: nu=be.cross_product(tau).subs(t=0);nu
(0, 1, 0)
sage: c=parametric_plot3d((r[0],r[1],r[2]),(t,-1.3,1.3),thickness=5)
sage: a1=arrow3d((0,0,0),(tau[0],tau[1],tau[2]))
sage: a2=arrow3d((0,0,0),(be[0],be[1],be[2]))
sage: a3=arrow3d((0,0,0),(nu[0],nu[1],nu[2]))
sage: c+a1+a2+a3
```



4. Conics and Quadrics

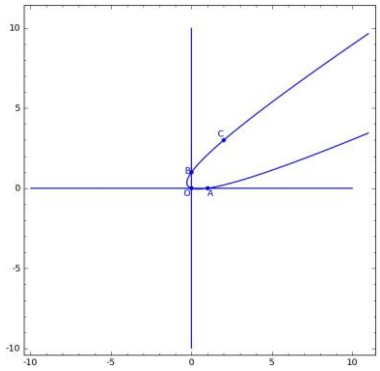
In Analytic Geometry we replace the definitions and the geometrical study of the curves and the surfaces, by the algebraic correspondence: a curve and a surface are defined by algebraic equations, and the study of the curve and the surface is reduced to the study of the equation corresponding to the each one.

Example 5. Write the equation of the parabola which passes through the points:

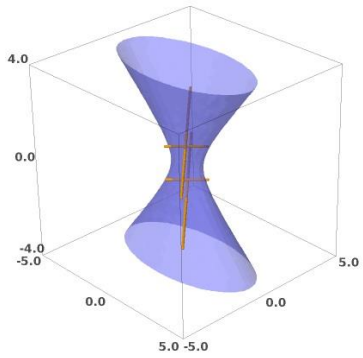
$$O(0,0), A(1,0), B(0,1), C(2,3).$$

We shall represent the achieved parabola in Sage, too:

```
sage: R.<x,y>=PolynomialRing(RR,2)
sage: I1=R.ideal([ff])
sage: I=I1.plot()
sage: A1=point2d((A[0],A[1]),size=22);t1=text("A",(A[0]+0.2,A[1]-0.3))
sage: B1=point2d((B[0],B[1]),size=22);t2=text("B",(B[0]-0.2,B[1]+0.1))
sage: O1=point2d((O[0],O[1]),size=22);t3=text("O",(O[0]-0.3,O[1]-0.3))
sage: C1=point2d((C[0],C[1]),size=22);t4=text("C",(C[0]-0.2,C[1]+0.4))
sage: I1=implicit_plot(y,(x,-10,10),(y,-10,10),linewidth=1)
sage: I2=implicit_plot(x,(x,-10,10),(y,-10,10),linewidth=1)
sage: I+A1+B1+t1+t2+O1+t3+I1+I2+C1+t4
```



Example 6. Find the rectilinear generators of the quadric: $\Sigma: x^2+3y^2+4yz-6x+8y+8=0.$



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DARBOUX PROPERTY FOR THE DERIVATE OF A MEASURE

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Abstract: In [1] it was established that if a measure λ on σ- algebra B of borel subsets of G, where G is an open connected set of X is derivable with respect to the Lebesgue measure μ, then the set $\left\{\frac{d\lambda}{d\mu}x / x \in G\right\}$ is an interval. In this paper we generalize the above result to the case where G is an arc-connected topological space which is metrizable and μ is a Vitali measure.
Mathematics Subject Classification (2010): 28A05
Key words: Darboux property, Vitali measure

Introduction

In this paper it is proved that if λ is a positive measure on an arc-connected open subset G of a metric space (X,d) and μ is a Vitali measure on (X,B), where B is the σ-algebra of borel subsets of X, rhen the set $\{\lambda'(x)/x \in G\}$ is an interval of real line whenever λ is derivable with respect to μ on G. We generalize a similar assertion [1], Theorem 2.3. where λ was the Lebesgue measure on $\mathbb{R}^n$.

1.Preliminaries and first results

Let (X,d) be a metric space and $B = B(X)$ be the set of all borel subsets of X. For any point $a \in X$ and any real number $r > 0$ we denote:

$$B(a,r) = \{ x \in X / d(x,a)=r \}.$$

A positive measure $\mu : B \to \bar{\mathfrak{R}}_+$ is termed a Vitali measure if the following "doubling condition" holds: there exist $\theta \in \mathbb{R}$, $1 < \theta < \infty$ such that for any $a \in X$ and any $r > 0$ we have:

$$\mu (B(a,2r)) \leq \theta \cdot \mu (B(a,r)) .$$

The following assertions are almost obvious:

1) If μ a Vitali measure and $\mu (B(a_0,r_0))=0$ for some point $a_o \in X$ and some positive number $r_o > 0$, then we have $\mu (X) = 0$.
2) If μ a Vitali measure and $\mu (B(a_0,r_0))=\infty$ for some point $a_o \in X$ and some positive real number $r_o > 0$ then we have $\mu (B(a,r))=\infty$ for any point $a \in X$ and any $r \in \mathbb{R}_+$, $r > 0$.

Throughout the paper we suppose that μ is a Vitali measure and there exist $r_0 > 0$, $a_o \in X$ such that : $0 < \mu (B(a_0,r_0)) < \infty$. In this case μ charges any nonempty open subset of X because $\mu (B(a,r)) > 0$ for all $a \in X$ and all $r > 0$.

The measure will be σ-finite since $X = \bigcup_{n=1}^{\infty} B(a_0, nr_0)$. Also we assume that $\mu(S(a,r)) = 0$ for all $a \in X$ and all $r \geq 0$, where $S(a,r) := \{ x \in X / d(a,x) = r \}$.

Remark: The proof off all the statements in thispoint will be omitted.

Theorem 1.1. In the above mentioned conditions on μ , the map $(x,r) \to \mu(B(a,r))$ from $X \times \mathbb{R}_+$ into $\mathbb{R}_+$ is continuous.

Definition : For any borel subset A of X we denote by $\alpha(A)$ the element of $[0, \infty)$ given by $\alpha(A) = \sup\left\{ \frac{\mu(A)}{\mu(B(x,r))} / A \subset B(x,r) \right\}$.

A sequence $(A_n)_n$ of borel subsets of X will be called ρ – regular ($\rho \in (0,\infty)$) if $\rho \leq \alpha(A_n)$ for all $n \in \mathbb{N}$. The sequence $(A_n)_n$ is called regular if it is ρ – regular for some $\rho > 0$.

We remember (see [2]) that a measure λ is derivable at a point $x_0 \in X$ if for any regular sequence $(F_n)_n$ of closed subsets of X such that $x_0 \in F_n$ for any $n \in \mathbb{N}$ and such that $\lim_{n\to\infty} \delta(F_n) = 0$ ($\delta(M)$ means the diameter of M) the sequence $\left(\frac{\lambda(F_n)}{\mu(F_n)} \right)_n$ is convergent.

The limite of the sequence $\left(\frac{\lambda(F_n)}{\mu(F_n)} \right)_n$ does not depend on the sequence $(F_n)_n$ and will be noted by $\lambda'(x_0)$.

For the proof of our main result we need the following auxiliary assertions.

Proposition 1.2 : If a topological space (X, τ) is locally arc-connected, then any connected and open subset G of X is arc-connected.

Proposition 1.3 : If the metric space (X,d) possesses a Vitali measure μ such that :

$0 < \mu(B(x,r)) < \infty$, $\forall r > 0$, $\forall x \in X$ that any bounded subset M of X is totally bounded.

Corollary 1.4 The metric space (X, d) is separable whenever there exists a Vitali measure μ on X .

2. Derivability and absolut continuity

As in the previous paragraph we suppose that μ is a Vitali measure an the metric space (X,d) such that for any $x \in X$, and any $r > 0$ we have $0 < \mu(B(x,r)) < \infty$ and $\mu(S(x,r)) = 0$.

Proposition 2.1 If G is an open subset of X, $\mathcal{B}(G)$ is the family of all borel subset of G and λ is a positive measure on $\mathcal{B}(G)$ which is derivable with respect to μ and absolutely continuous w.r. to μ, then for any $\beta \geq 0$, we have $\lambda^*(A) \leq \beta \mu^*(A)$ (respective $\beta \mu^*(A) \leq \lambda^*(A)$) for any subset $A \subset G$ with the property $\lambda' \leq \beta$ on A (respective $\lambda' \geq \beta$ on A). (Here λ^* and μ^* denote the outer measures associated with λ and μ).

Proof We suppose that at any point $a \in A$ we have $\lambda'(a) \leq \beta$. For any $\varepsilon > 0$, we consider an open set G_ε such that $A \subset G_\varepsilon \subset G$ with $\mu(G_\varepsilon) \leq \mu^*(A) + \varepsilon$ and for any $a \in A$ we consider a regular sequence $(F_n^a)_n$ of closed subsets of X, $a \in F_n^a \subset G_\varepsilon$ such that $\lim_{n\to\infty} \delta(F_n^a) = 0$ and such that $\frac{\lambda(F_n^a)}{\mu(F_n^a)} < \beta + \varepsilon$ for all n. This is possible since $\lambda'(a) < \beta + \varepsilon$.

The family $\mathcal{F} = \{ F_n^a / a \in A , n \in \mathbb{N} \}$ is a Vitaly covering [2] of A and therefore we may apply the Vitaly covering lemma:

There exist $M \subset A$ with $\mu^*(M)=0$ and a sequence $(F_k)_k$ in $\mathcal{F}$ such that $F_p \cap F_q = \emptyset$ if $p \neq q$ and $A \setminus M \subset \bigcup_k F_k$.

We shall have $\lambda^*(M) = 0$ and $\lambda^*(A \setminus M) \leq \lambda^*(\bigcup_k F_k) = \lambda(\bigcup_k F_k) = \sum_k \lambda(F_k) \leq$

$\leq (\beta+\varepsilon) \sum_k \mu(F_k) = (\beta+\varepsilon)\mu(\bigcup_k F_k) \leq (\beta+\varepsilon)\mu(G_\varepsilon) \leq (\beta+\varepsilon)(\mu^*(A)+\varepsilon)$.

Hence $\lambda^*(A) \leq \lambda^*(A \setminus M) + \lambda^*(M) = \lambda^*(A \setminus M) \leq (\beta+\varepsilon)(\mu^*(A)+\varepsilon)$.

Since $\varepsilon > 0$ is arbitrary we get $\lambda^*(A) \leq \beta \mu^*(A)$.

The other part of the proposition may be similarly proved.

Corollary 2.2 Let G be an open subset of X, let λ be a positive measure on $\mathcal{B}(G)$ which is derivable and absolutely continuous with respect to μ. If A is an element of $\mathcal{B}(G)$ with $0 < \mu(A) < \infty$ and $\lambda'(a) > \beta$ ($\lambda'(a) < \beta$) for any point $a \in A$ then we have $\lambda(A) > \beta \mu(A)$ ($\lambda(A) < \beta \mu(A)$).

Proof : We suppose $\lambda'(a) < \beta$ for any $a \in A$ and for any $n \in \mathbb{N}$, $n \neq 0$ with $\frac{1}{n} < \beta$ we denote $A_n = \{ x \in A / \lambda'(x) \leq \beta - \frac{1}{n} \}$.

Obviously $(A_n)_n$ is a sequence increasing to A and therefore the sequences $(\lambda^*(A_n))_n$, respectively $(\mu^*(A_n))_n$ are increasing to $\lambda(A)$, respectively $\mu(A)$. Since $\mu(A) > 0$ we have $\mu^*(A_n) > 0$ for sufficiently large number n.

From proposition 2.1. we have : $\lambda^*(A_n) \leq (\beta - \frac{1}{n}) \mu^*(A_n)$ and $\lambda^*(A \setminus A_n) \leq \beta \mu^*(A \setminus A_n)$.

It is known [2] that the real function $x \longrightarrow \lambda'(x)$ is μ – measurable. Hence the sets A_n are μ – measurable and therefore, using the preceding inequalities we get:

$\lambda(A) = \lambda^*(A) \leq \lambda^*(A_n) + \lambda^*(A \setminus A_n) \leq (\beta - \frac{1}{n}) \mu^*(A_n) + \beta \mu^*(A \setminus A_n) < \beta \mu^*(A_n) + \beta \mu^*(A \setminus A_n) =$

$= (\mu(A_n) + \mu(A \setminus A_n)) = \beta \mu(A)$.

The case $\lambda'(a) > \beta$ for all $a \in A$ may be treated in a similar way.

Using the same arguments as in the proof of Proposition 2.1. we get:

Proposition 2.3 If G is an open subset of X and λ is a real measure on $\mathcal{B}(G)$ which is derivable and absolutely continnous with respect to μ then for any borel subset A of G we have $\lambda(A) \leq \beta \mu(A)$ (respectively $\lambda(A) \geq \beta \mu(A)$) whenever $\lambda'(a) \leq \beta$ (respectively $\lambda'(a) \geq \beta$) for all points $a \in A$.

Corollary 2.4 Let G be an open subset of X and λ be a real measure on $\mathcal{B}(G)$ which is derivable and absolutely continnous with respect to μ. If $A \in \mathcal{B}(X)$ is such that $0 < \mu(A) < \infty$ and β is a real number for which $\lambda'(a) < \beta$ (resp. $\lambda'(a) > \beta$) for any points $a \in A$ then we have $\lambda(A) < \beta \mu(A)$ (respectively $\lambda(A) > \beta \mu(A)$).

The proof of this assertion is similar with the proof of Corollary 2.2, using Proposition 2.3.

Theorem 2.5 Let be a Vitali measure on the complet metric space (X,d) such that $\mu(S(x,r)) = 0$, $0 < \mu(B(x,r)) < \infty$ for all $x \in X$ and all $r \in \mathbb{R}$, $r > 0$. If we suppose that any ball $B(x,r)$ is arc-connected, then for any open an connected set G and for any real measure λ on B(G) which is derivable and absolutely continnous with respect to μ, the set $\{ \lambda'(x) / x \in G \}$ is an interval of the real line.

Proof : Let $\alpha, \beta \in \mathbb{R}$, $\alpha < \beta$ and let $\gamma \in \mathbb{R}$ be such that $\alpha < \gamma < \beta$. We suppose that there exist $x, y \in G$ such that $\lambda'(x) = \alpha$, $\lambda'(y) = \beta$ and we want to show that there exists $z \in G$ such that $\lambda'(z) = \gamma$.

We suppose the contrary i.e. $G = A \cup B$ where $x \in A$, $y \in B$, $A \cap B = \emptyset$ and for any element $a \in A$ and any element $b \in B$ we have $\lambda'(a) < \gamma < \lambda'(b)$.

Since G is connected we can not have simultaneously $A \cap \bar{B} = \emptyset$ and $B \cap \bar{A} = \emptyset$. So there exists $c \in G$ such that for any neighbourhood V of c we have $V \cap A \neq \emptyset \neq V \cap B$. Let us suppose that $c \in A$ and let $r \in \mathbb{R}$, $r>0$ be such that $\bar{B}(c,r) \subset G$. We choose an element $b \in B(c,r)$ such that $d(b,c) < \frac{r}{2}$ and such that $b \in B$. Since the ball $B(c,r)$ is arc-connected we may consider a continuous function $\varphi:[0,1] \to B(c,r)$ such that $\varphi(0)=c$, $\varphi(1)=b$. Since the set $K=\{\varphi(t) / t \in [0,1]\}$ is a compact subset of $B(c,r)$ we deduce that the distance $\delta(K, X \setminus B(c,r))$ between K and $X \setminus B(c,r)$ given by: $\delta(K, X \setminus B(c,r)) = \inf\{d(k,x) / k \in K, x \in X \setminus B(c,r)\}$ is a strictely positive number. If we consider now $\rho_1 > 0$, $\rho_1 < \delta(K, X \setminus B(c,r))$ we have $B(k,\rho_1) \subset B(c,r)$ for any $k \in K$ and therefore $B(\varphi(t), \rho_1) \subset B(c,r)$ for all $t \in [0,1]$. Since $\lambda'(c) < \gamma$ and $\gamma < \lambda'(b)$ we may suppose that for $\rho_1 > 0$ sufficiently small we have $\frac{\lambda(B(c,\rho_1))}{\mu(B(c,\rho_1))} < \gamma < \frac{\lambda(B(b,\rho_1))}{\mu(B(b,\rho_1))}$. By theorem 1.1, the function $\psi:[0,1] \to \mathbb{R}$, given by $\psi(t) = \frac{\lambda(B(\varphi(t),\rho_1))}{\mu(B(\varphi(t),\rho_1))}$ is continnous and $\psi(0) = \frac{\lambda(B(c,\rho_1))}{\mu(B(c,\rho_1))} < \gamma < \frac{\lambda(B(b,\rho_1))}{\mu(B(b,\rho_1))} = \psi(1)$.

Hence there exists t_0, $0<t_0<1$ such that $\psi(t_0) = \gamma$, $\frac{\lambda(B(c_1,\rho_1))}{\mu(B(c_1,\rho_1))} = \gamma$, where $c_1 = \psi(t_0)$.

$\bar{B}(c_1,\rho_1) \subset \bar{B}(c,r) \subset G$. The ball $B(c_1,\rho_1)$ and the set G have the same properties i.e. they are connected open sets and there exist a_1, b_1 in $B(c_1,\rho_1)$ such that $\lambda'(a_1) < \gamma < \lambda'(b_1)$. Indeed, we can not have $\lambda'(a) < \gamma$ for all points $a \in B(c_1,\rho_1)$ because, in the contrary case, applying corollary 2.4 we arrive at the contradictory inequality $\lambda(B(c_1,\rho_1)) < \gamma \mu(B(c_1,\rho_1))$.
The same argument show that we can not have $\lambda'(a) >$ for all points $a \in B(c_1,\rho_1)$.
Following the construction of $B(c_1,\rho_1)$ starting from the set G we may construct a ball $B(c_2,\rho_2)$ such that : $\rho_2 < \frac{\rho_1}{2}$, $\bar{B}(c_2,\rho_2) \subset B(c_1,\rho_1)$, $\frac{\lambda(B(c_2,\rho_2))}{\mu(B(c_2,\rho_2))} = \gamma$.

So, we may contruct by induction a sequence $(B(c_n,\rho_n))_n$ of balls such that $\bar{B}(c_1,\rho_1) \subset G$, $\rho_{n+1} < \frac{\rho_n}{2}$, $\bar{B}(c_{n+1},\rho_{n+1}) \subset B(c_n,\rho_n)$, $\frac{\lambda(B(c_n,\rho_n))}{\mu(B(c_n,\rho_n))} = \gamma$, for all natural number n, $n \geq 1$.
Obviously the intersection of the balls $B(c_n,\rho_n)$ reduces at a point z of G and we have
$\lambda'(z) = \lim_{n\to\infty} \frac{\lambda(B(c_n,\rho_n))}{\mu(B(c_n,\rho_n))} = \gamma$.
Hence the function $\lambda' : G \to \mathbb{R}$ has the started assertion.

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APPLICATIONS OF WAVELETS FOR COMPRESSING SIGNALS

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Abstract: We introduce the Haar and Daubechies wavelets end we show how they can be used to compress audio signals, digital image and video signal.
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Key words: Wavelets, Harmonic analysis, Numerical methods

1. Introduction

The wavelets are a rather new field of mathematics, which is connected to the harmonic analysis and the time-frequency Fourier analysis.

The 1D wavelets are $L^2(\mathbb{R})$ functions $\Psi: \mathbb{R} \to \mathbb{C}$ with 0 average and norm 1, $\hat{\Psi}(0) \equiv \int_{-\infty}^{+\infty} \Psi(t)dt = 0$ and $\|\Psi\| = 1$.

If Ψ is a wavelet then we can obtain new wavelets by translation and dilatation. Namely, if a$\neq$0, $b \in \mathbb{R}$ we define $\Psi_{a,b}(t) = \frac{1}{\sqrt{|a|}}$. By giving appropiate values to a and b, we obtain convenient orthonormal bases for the Hilbert spaces $L^2(\mathbb{R})$ with good localization in time and frequency.

Given a signal $f \in L^2(\mathbb{R})$ instead of the family $\{\hat{f}(\omega) | \omega \in \mathbb{R}\}$ depending on the parameter ω, we define the numbers $\{c_f(b,a) | a, b \in \mathbb{R}, a \neq 0\}$, where $c_f(b,a) = \langle f, \Psi_{a,b} \rangle$ the original signal can be recovered from $c_f(b,a)$ through Calderón's formula. The advantage of such decomposition is that the functions $\psi_{a,b}(t)$ are localized in time and frequency.

In practical applications we want to consider parameters with discrete values, namely of the type $a = 2^{-j}, b = k \cdot 2^{-j}$ with $j, k \in \mathbb{Z}$. An important problem was finding wavelets Ψ such that $\Psi_{a,b}(t) \equiv \Psi_{j,k}(t) = 2^{\frac{j}{2}} \cdot \Psi(2^{j \cdot t} - k)$ are an orthonormal bases for $L^2(\mathbb{R})$. In this case every signal $f \in L^2(\mathbb{R})$ can be written as. $f = \sum_{jk} \langle f, \Psi_{jk} \rangle \Psi_{jk}$. This way the analogue signal f is identified with the sequence of numbers $\langle f, \Psi_{jk} \rangle$ and so we have an analogue/digital conversion. Also f writes as $f = \sum_j \left(\sum_k \langle f, \Psi_{jk} \rangle \Psi_{jk} \right) = \sum_j f_j(t)$, which is a new type of decomposition, a sum of the "voice" f_i rather than harmonics like in the Fourier case. It is important that Ψ has a compact support, which limits considerably the number of non zero scalar products $\langle f, \Psi_{jk} \rangle$. This was achieved by Y. Meyer in 1986 and Ingrid Daubechies in 1988.

2. Signal compression

An important role in processing and transmission of signals is played by elimination of redundancies and shortening of the information we want to send. The main idea is to reduce the number of samples without losing much information. This is called compression.

Haar and Daubechies wavelets

Definition 1

The Haar wavelets is $\Psi_H : \mathbb{R} \to \mathbb{R}$, given by $\Psi_H(t) = 1$ if $t \in \left[0, \frac{1}{2}\right)$, $\Psi_H(t) = -1$ if $t \in \left[\frac{1}{2}, 1\right)$ and $\Psi_H(t) = 0$ everywhere else.

Definition 2

We fix an integer $m \geq 1$. Let $n > m$ be an integer. Let $P(z) = \frac{1}{2}\sum_{n=0}^{N} p_n \cdot z^n = \left(\frac{1+z}{2}\right)^m \cdot S(z)$, where $S(z)$ is a polynomial of degree at most $N - m$, such that $S(1) = 1,\ S(-1) \neq 0$. The Daubechies scaling function $\varphi_{D;m}(t)$ of order m is defined by the property $\hat{\varphi}_{D;m}(\omega) = \prod_{k=1}^{\infty} P\left(e^{-\frac{j\omega}{2^k}}\right)$ and the Daubechies wavelet is $\Psi_{D;m}(t) = \sum_{k=-N+1}^{j} (-1)^k p_{1-k} \varphi_{D;m}(2t-k)$, where the coefficients p_k are given by the functional equation $\varphi_{D;m}(t) = \sum_{k=0}^{n} p_k \varphi_{D;m}(2t-k)$.

When we use a wavelet Ψ Haar or Daubechies and take samples $-15 \leq m, n \leq 16$ we have $32^2 = 1024$ pairs (m, n) we get a good approximation g_1 for signal g. When we restrict ("compress") to values $-3 < m,\ n \leq 4$, so we take only $8^2 = 64$ pairs, we have another approximation

$$g_2(t) = \sum_{m=-3}^{4} \sum_{n=-3}^{4} \alpha_{mn} \cdot \Psi_{mn}(t)$$

(we rely on the fact that most coefficients α_{mn} are either 0 or very small). This way we obtained a compression ratio of 1024 : 64 = 16 : 1 .

3. Conclusions

The every ratio in the case of Haar wavelets is $E(g_2)/ E(g_1) \approx 0.80$. When using Daubechies wavelets this ratio grows to ≈ 0.95.

The compression ratio achieved for audio signals is 12:1.

For digital image with resolutions 1024x512 in three colours each with 256 shades about 1.3 MB of memory are required by using Daubechies wavelets the it is possible to obtain a 32:1 compression.

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COHOMOLOGY WITH COEFFICIENTS IN A CELLULAR CONSTANT SHEAF

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Abstract: The notion cellular constant sheaf was introduced by the author in his PhD thesis. In this note we present the cohomology of a standard n-simplex (for arbitrary $n \geq 1$) with values in a cellular constant sheaf, giving explicit formulas.
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Key words: cohomology of sheaves, cellular constant sheaf.

1. Introduction

The cellular constant sheaves appeared in the equivariant K- theory – e.g. the computation of K_G - groups for some G-spaces, where G is a compact Lie group. This notion was introduced in [2] for a standard simplex (as base), but it was extended in [4] to any CW-complex.

Let $X = [x_0, \dots, x_n]$ be a standard n-simplex - i.e. generating by exactly the vertices $x_0, \dots, x_n$ (this section contains definitions from [2] and [3]).

Definition. A sheaf $\mathcal{F}$ (of abelian groups) on X is called *cellular constant* if for any open face Y of X the restriction of $\mathcal{F}$ to Y is constant (i.e. . $\mathcal{F} \mid Y = A_Y \in \mathrm{Ab}$).

Proposition 1. (see [3]) For the cellular constant sheaf $\mathcal{F}$ we have :

a. If Y, Z are two faces of the simplex X such that $Y \cap \overline{Z} \neq \varnothing$ (this condition is equivalent with: Y is a face of Z) then there exists the morphisms $f_Z^Y : A_Y \to A_Z$.

b. If Y, Z , W are faces of the simplex X such that $Z \cap \overline{W} \neq \varnothing$ and $Y \cap \overline{Z} \cap \overline{W} \neq \varnothing$ then f_W^Y is the composition $f_W^Y = f_W^Z \circ f_Z^Y$.

Notations 1. In this paper we will denote by $G_{i_0 \dots i_q}$ the abelian group which is the value of the cellular constant sheaf $\mathcal{F}$ on the open face $X_{i_0 \dots i_q}$ of the n - standard simplex X, where $0 \leq i_0 < i_1 < \dots < i_q < \dots \leq n$. We will also assume that for $q < r$ the morphisms from the proposition 1:

$$G_{i_0 \dots i_q} \to G_{i_0 \dots i_r}$$

($0 \leq i_0 < i_1 < \dots < i_q < \dots < i_r < \dots \leq n$) are all inclusions – it's exactly the situation appeared in the computation of the equivariant K- theory for some topological spaces (see [2]).

2. By Ad X^q we will denote the set of all q – closed faces of the n - standard simplex X and by Int X^q the union of all q – open faces of X .

It's obvious that Ad $X^0 = \{x_0, x_1, \dots, x_n\}$ and that the set Ad X^{n-1} is exactly the boundary of X denoted by BX ; we also have that Ad X^q − Ad X^{q-1} is the set Int X^q for every $q = 1, 2, \dots, n-1$.

2. Cohomology of a n-standard simplex

Let X be the standard n- simplex generated by the points $x_0, \dots, x_n$ and we consider that the stalk of the cellular constant sheaf $\mathcal{F}$ at the point x_0 is the group $G_0, \dots$, at the point x_n is G_n ; using the formula

$$H^n(\mathbb{R}^m; G) = \begin{cases} 0 & if \quad n \neq m \\ G & if \quad n = m \end{cases} \qquad (1) \quad \text{(see [1] and [6])}$$

and the exact Mayer-Vietoris sequence (in cohomology) associated to the closed sets F_0 and F_1 (see [5]):

$0 \to H^0(F; \mathcal{F}) \to H^0(F_0; \mathcal{F}) \oplus H^0(F_1; \mathcal{F}) \to H^0(F_0 \cap F_1; \mathcal{F}) \xrightarrow{\delta} H^1(F; \mathcal{F}) \to \dots$
(where $F = F_0 \cup F_1$) one obtains, by induction, the folowing formulas :

$$H^0(\text{Ad } X^0; \mathcal{F}) \cong G_0 \oplus G_1 \oplus \dots \oplus G_n ;$$

$$H^q(\text{Int } X^q; \mathcal{F}) \cong \oplus G_{i_0 \dots i_q} \text{ where } 0 \leq i_0 < i_1 < \dots < i_q \leq n \text{ and } q = 1,2, \dots, n-1$$

and

$$H^n(X - B X; \mathcal{F}) \cong G_{01 \dots n} \qquad (2)$$

Given the close set F = BX now one considers the long exact sequence (in cohomology) associated to F (see [1] and [5]:

$$0 \to H^0(X - F; \mathcal{F}) \to H^0(X; \mathcal{F}) \to H^0(F; \mathcal{F}) \xrightarrow{\delta} H^1(X - F; \mathcal{F}) \to \dots$$
$$\to H^{n-1}(X - F; \mathcal{F}) \to H^{n-1}(X; \mathcal{F}) \to H^{n-1}(F; \mathcal{F}) \xrightarrow{\delta} H^n(X - F; \mathcal{F}) \to H^n(X; \mathcal{F}) \to 0$$

using also the formula (1) and all the above results (2) we obtain the isomorphisms :

$$H^q(X; \mathcal{F}) \cong H^q(BX; \mathcal{F}) \quad \text{for any } q \neq n-1, n$$

and the following exact sequence:

$$0 \to H^{n-1}(X; \mathcal{F}) \to H^{n-1}(BX; \mathcal{F}) \xrightarrow{\delta} G_{01 \dots n} \to H^n(X; \mathcal{F}) \to 0 \qquad (3)$$

But the sets Ad X^{q-1} are close in Ad X^q (q = 1, 2, …, n-1) and using again the above long exact sequence in cohomology (associated to a close subspace) one obtains the isomorphisms of groups:

$$H^k(\text{Ad } X^q; \mathcal{F}) \cong H^k(\text{Ad } X^{q-1}; \mathcal{F}) \quad \text{for any } k \neq q-1, q$$

(q = 1, 2, …, n-1) and moreover the exact sequences of groups:

$$0 \to H^q(\text{Ad } X^{q+1}; \mathcal{F}) \to H^q(\text{Ad } X^q; \mathcal{F}) \xrightarrow{\delta} \oplus G_{i_0 \dots i_q} \to H^{q+1}(\text{Ad } X^{q+1}; \mathcal{F}) \to 0 \quad (4)$$

where q = 0, 1, …, n-2.

Using all the above obtained results, the exact sequence (4) turns for q=0 in:

$$0 \to H^0(\text{Ad } X^1; \mathcal{F}) \to G_0 \oplus \ldots \oplus G_n \xrightarrow{\delta} \oplus G_{i_0 i_1} \to H^1(\text{Ad } X^1; \mathcal{F}) \to 0$$

We will use the differentials from the Čech cohomology – because the family of supports in cohomology is exactly the family of compact sets of X one results that

$$H^n(X; \mathcal{F}) \cong \widetilde{H^n}(X; \mathcal{F}) \qquad \text{(see [5], p. 234) ;}$$

then the above differential is defined by the formula:

$$\delta((g_i)_i) = (g_{i_1} - g_{i_0})$$

where $0 \leq i_0 < i_1 \leq n$.

Then one obtains finally the formulas :

$H^0(X; \mathcal{F}) \cong H^0(B X; \mathcal{F}) \cong \cdots \cong H^0(\text{Ad } X^1; \mathcal{F}) \cong \text{Ker } \delta = G_0 \cap \ldots \cap G_n$.

$$H^1(\text{Ad } X^1; \mathcal{F}) \cong \oplus G_{i_0 i_1} / (G_0^1 + G_1^1) \qquad (5)$$

(where $0 \leq i_0 < i_1 \leq n$ and we noted by G_p^q the group associated to the following sequence $i_0 \ldots i_{p-1}\, i_{p+1} \ldots i_q$)

$H^q(\text{Ad } X^1; \mathcal{F}) = 0$ for every $q > 1$.

Using now all the formulas (2) and (5), the sequence (4) turns for $q = 1$ in the following exact sequence:

$$0 \to H^1(\text{Ad } X^2; \mathcal{F}) \to \oplus G_{i_0 i_1} / (G_0^1 + G_1^1) \xrightarrow{\delta} \oplus G_{i_0 i_1 i_2} \to H^2(\text{Ad } X^2; \mathcal{F}) \to 0$$

and the 2 - differential is given by the formula:

$$\delta((\tilde{g}_{i_0 i_1})) = (g_{i_1 i_2} - g_{i_0 i_2} + g_{i_0 i_1})$$

where $0 \leq i_0 < i_1 < i_2 \leq n$ and one denotes by $(\tilde{g}_{i_0 i_1})$ the image of $(g_{i_0 i_1})$ in the quotient group $\oplus G_{i_0 i_1} / (G_0^1 + G_1^1)$.

Then it follows the 1 – cohomology of the simplex X :

$$H^1(X; \mathcal{F}) \cong H^1(\text{Fr } X; \mathcal{F}) \cong \cdots \cong H^1(\text{Ad } X^2; \mathcal{F}) \cong \text{Ker } \delta =$$
$$= \{ (\tilde{g}_{i_0 i_1}) \in \oplus G_{i_0 i_1} / (G_0^1 + G_1^1) \mid g_{i_1 i_2} - g_{i_0 i_2} + g_{i_0 i_1} = 0 \}$$

and the formulas :

$$H^2(\text{Ad } X^2; \mathcal{F}) \cong \oplus G_{i_0 i_1 i_2} / (G_0^2 + G_1^2 + G_2^2)$$

$H^q(\text{Ad } X^2; \mathcal{F}) = 0$ for any $q > 2$.

Similary one obtains the following formulas for $q = 2, 3, \ldots, n-2$:

$H^q(X;\mathcal{F}) \cong \{ (\tilde{g}_{i_0\dots i_q}) \in \bigoplus G_{i_0\dots i_q} / (G_{i_1\dots i_q} + \dots + G_{i_0\dots i_{q-1}}) \mid \sum_{p=0}^{q+1} (-1)^p \, g_{i_0\dots i_{p-1}\, i_{p+1}\dots i_q} = 0 \}$

and $H^{q+1}(\mathrm{Ad}\, X^{q+1};\mathcal{F}) \cong \bigoplus G_{i_0\dots i_{q+1}} / (G_{i_1\dots i_{q+1}} + \dots + G_{i_0\dots i_q})$.

Using the last formula and the definition of the (n-1) – differential, from the exact sequence (4) we conclude the computation of the cohomology of the n –simplex X with values in a cellular constant sheaf :

$H^{n-1}(X;\mathcal{F}) \cong \{ (\tilde{g}_{i_0\dots i_{n-1}}) \mid \sum_{p=0}^{n} (-1)^p \, g_{0\dots p-1\, p+1\dots n} = 0 \}$;

$H^n(X;\mathcal{F}) \cong G_{01\dots n} / (G_{12\dots n} + \dots + G_{01\dots n-1})$

(for the definition of the differentials and for more details see [2]).

Remarks 1. One finds again the formulas obtained for 1 and 2 – standard simplexes in the papers [2] and [3].

2. In the groups $H^q(X;\mathcal{F})$ (q = 1, 2, … , n -1) among the above $\binom{n+1}{q+2}$ relations only $\binom{n}{q+1}$ relations are independent ; then one considers only $\binom{n}{q}$ factors in the cohomology group $H^q(X;\mathcal{F})$.

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ON SOME VECTOR LATTICE CONCEPTS

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Abstract: Continuing some of my work published earlier last decade, this paper gives *algebraic descriptions* and *geometric interpretations* of certain concepts from the theory of vector lattices. So is given a new approach to investigate, besides classical concepts of *vector sublattice* and *Riesz homomorphism*, the *lattice-subspace* notion and the *restricted-lattice operator* notion. The *lattice-subspaces* were introduced in 1983, by I.A. Polyrakis and, independently by S. Miyajima, as ordered vector subspaces of a vector lattice with a lattice structure, but not the induced one. In 1992, C.D. Aliprantis and D. Brown began to apply lattice-subspaces in economics, and posed the problem of the study of finite dimensional lattice-subspaces. Such subspaces are studied by I.A. Polyrakis since 1996. In this paper we introduce *restricted-lattice operators* as a kind of Riesz homomorphisms defined not on vector sublattices, but on lattice-subspaces. Some *properties* and *typical problems* on restricted-lattice operators will be studied.
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Key words: lattice, sublattice, lattice-subspace, restricted-lattice operator, Riesz homomorphism.

1. Preliminaries

In [2], [3] and [4] we gave an algebraic description and a geometric interpretation for what it means $x \geq y$ in an ordered vector space (in short o.v.s.) E and for the lattice operations in a vector lattice (in short v.l.) E. (For terminology, see [1].)
So denoting E_+ the positive cone in the o.v.s. E we remark that, for $x, y \in E$ we have:

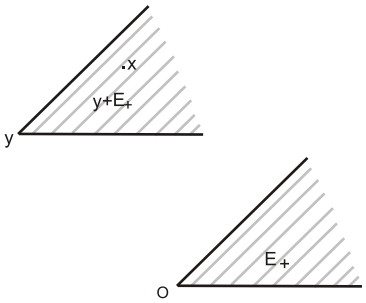
$$x \geq y \Leftrightarrow x \in y + E_+ \Leftrightarrow y \in x - E_+ \Leftrightarrow -x \in -y - E_+ \Leftrightarrow -y \in -x + E_+ \tag{1}$$

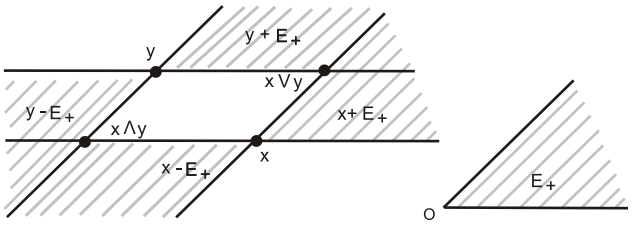
Also, if E is a v.l. and $x, y \in E$, we have for $u, v \in E$:

$$u = x \vee y \Leftrightarrow u + E_+ = \left(x + E_+\right) \cap \left(y + E_+\right) \tag{2}$$

$$v = x \wedge y \Leftrightarrow v - E_+ = \left(x - E_+\right) \cap \left(y - E_+\right) \tag{3}$$

The corresponding pictures for (1), (2) and (3) are the following:

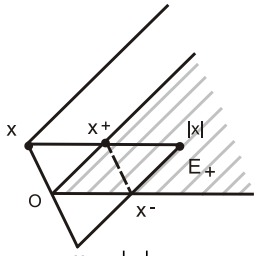




For the *positive part*, the *negative part* and the *absolute value* of an element x in a v.l. E, classical defined by $x^+ = x \vee 0$, $x^- = (-x) \vee 0$, and $|x| = x \vee (-x)$, we have:

$$x^+ + E_+ = (x + E_+) \cap E_+,\ x^- + E_+ = (-x + E_+) \cap E_+,\ \text{and}\ |x| + E_+ = (x + E_+) \cap (-x + E_+) \qquad (4)$$

The *geometric descriptions* for x^+, x^- and $|x|$ appear in the next picture:



The basic properties of x^+, x^- and $|x|$ can be intuited from this picture. So we have:

$$x^+ \wedge x^- = 0,\ |x| = x^+ + x^-,\ |x| = x \vee (-x),\ x = x^+ - x^-.$$

2. Sublattices and lattice-subspaces

In this section, E will be a v.l. and $G \subseteq E$ a vector subspace endowed with the induced ordering (that is by the cone $G_+ = G \cap E$). We say that G is an *ordered vector subspace* of E. We recall the following:

Classical definition. We say that G is a *sublattice* of E, if for all $u, v \in G$ the supremum $u \vee v$ (that exists in E) is in G (or, equivalently, $u \wedge v \in G$).

The following result gives an *algebraic description* of the concept of *sublattice*.

Proposition 1. *Let E be a vector lattice and $G \subseteq E$ an ordered vector subspace. Then, G is a sublattice if and only if for all $u, v \in G$, there exists $z \in G$ such that $z + E_+ = (u + E_+) \cap (v + E_+)$ and, in this case, $z = u \vee v$.*

As consequences of this result we have:

Proposition 2. (see [4] and [6]) *Let G be an ordered vector subspace of a vector lattice E. Then G is a sublattice of E if and only if for all $u, v \in G$, the following equality holds:*

$$u \vee v + G_+ = (u + E_+) \cap (v + E_+) \cap G.$$

Proposition 3. *If E is a vector lattice and $G_1, G_2 \subseteq E$ are two sublattices, then $G_1 \cap G_2$ is a sublattice, too.*

Proof. Let $u, v \in G_1 \cap G_2$. According to Proposition 1, there exists $z_1 \in G_1$ and $z_2 \in G_2$ such that

$$\begin{aligned} z_1 + E_+ &= (u + E_+) \cap (v + E_+), \\ z_2 + E_+ &= (u + E_+) \cap (v + E_+). \end{aligned} \qquad (5)$$

It follows that $z_1 + E_+ = z_2 + E_+$. So we obtain $z_1 \in z_2 + E_+$ and hence (by using (1)) it follows $z_1 \geq z_2$. The converse inequality ($z_2 \geq z_1$) is similar. Therefore $z_1 = z_2 \in G_1 \cap G_2$. By using (5) it follows that there exists $z \in G_1 \cap G_2$ such that

$$z + E_+ = (u + E_+) \cap (v + E_+).$$

Then by applying again Proposition 1, it follows that $G_1 \cap G_2$ is a sublattice of E. □

The following concept generalizes in a some sense the concept of *sublattice*. It was introduced

in 1983, by I.A. Polyrakis and, independently, by S. Miyajima, and has been extensively used in economics, since 1992.

Classical definition. Let E be a vector lattice and $G \subseteq E$ an ordered vector subspace. We say that G is a *lattice-subspace* of E, if G is a vector lattice, that is, for each $u, v \in G$, the supremum $u \vee_G v$ of $\{u, v\}$ exists in G. (Mention that $u \vee_G v$ is also denoted by $\sup_G \{u, v\}$ or $u \triangledown v$.)

What does mean that $u \vee_G v = z$? We have:

$$u \vee_G v = z \Leftrightarrow 1)\ z \in G \text{ and } u \le z,\ v \le z, \text{ and}$$
$$2) \text{ for each } t \in G, \text{ with } u \le t,\ v \le t, \text{ it follows } z \le t.$$

It is clear that $u \vee v \le u \vee_G v$.

Similar we will denote by $u \wedge_G v$ the infimum of $\{u, v\}$ in G, if this element exists in G. Hence, we have:

$$u \wedge_G v = w \Leftrightarrow 1')\ w \in G \text{ and } w \le u,\ w \le v, \text{ and}$$
$$2') \text{ for each } s \in G, \text{ with } s \le u,\ s \le v, \text{ it follows } s \le w.$$

Obviously the ordered vector subspace G of the vector lattice E is a *lattice-subspace* of E if and only if, for each $u, v \in G$, the infimum $u \wedge_G v$ exists in G. Obviously $u \wedge_G v \le u \wedge v$.

In [4] we gave an *algebraic description* of the notion of *lattice-subspace*. So, if $u, v \in G$, then:

$$u \vee_G v + G_+ = (u + G_+) \cap (v + G_+), \text{ and} \tag{6}$$
$$u \wedge_G v - G_+ = (u - G_+) \cap (v - G_+) \tag{7}$$

Comparing the identity (2) with (6) and (3) with (7), we better understand the difference between $u \vee v$ and $u \vee_G v$, and $u \wedge v$ and $u \wedge_G v$, respectively.

Note that if E is a v.l. and $u \vee_G v = u \vee v$ for any $u, v \in G$, then G is a sublattice of E (see [6]). Remark that $u \vee_G v$ depends on the subspace $G \subseteq X$. In other words, in this kind of subspaces we have the induced ordering and a lattice structure but not the induced one.

Remark. The class of all lattice-subspaces in a v.l. E is larger then that of all sublattices of E, because any sublattice is a lattice-subspace but the converse is not true in general.

3. Various types of linear operators commuting with lattice operations

It is well known the notion of Riesz homomorphism acting between two vector lattices E, F. So, the linear operator $T: E \to F$ is called *Riesz homomorphism* if $T(x \vee y) = T(x) \vee T(y)$ for all $x, y \in E$. In the sequel we reconsider this notion and we propose two new notions.

Classical definitions. Let E, F be two vector lattices and $T: E \to F$ a linear operator and $G \subseteq E$ an ordered vector subspace.

I) if $G \subseteq E$ is a sublattice, we say that T is a *G-Riesz homomorphism* (in short R.h.) if

$$T(u \vee v) = T(u) \vee T(v), \text{ for all } u, v \in G.$$

II) if $G \subseteq E$ is a lattice-subspace, we say that T is a *G-lattice operator* or, more general, a *restricted-lattice operator* (in short l.o.) if $T(u \vee_G v) = T(u) \vee T(v)$, for all $u, v \in G$.

III) if $G \subseteq E$ is a lattice-subspace, we say that T is a *G-quasi lattice operator* (in short q.l.o.) if $T(u \vee_G v) = T(u) \vee_{T(G)} T(v)$, for all $u, v \in G$.

Obviously, if G is a sublattice, any R.h. $T: G \to F$ is a l.o. and a q.l.o., too. Note also that in

[5] we studied some problems related to Riesz homomorphisms.
Corresponding *algebraic descriptions* for these definitions from above can be given. For example:

I') if G is a sublattice, then T is a G *-Riesz homomorphism* if

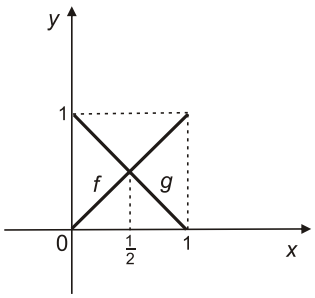
$$T(u \vee v + E_+) = T(u) \vee T(v) + T(E_+) \text{ for all } u, v \in G.$$

II') if G is a lattice-subspace of E, then T is a G *-lattice operator* if and only if

$$T(u \vee v + G_+) = T(u) \vee T(v) + T(G_+) \text{ for all } u, v \in G.$$

4. An example

It is known that if $E = C[0,1]$ is endowed with the pointwise algebraic and order structures, then E is a v.l. and its ordered vector subspace G consisting of all affine functions on $[0,1]$ is a lattice-subspace, but not a sublattice.



Take $f, g : [0,1] \to \Re$ defined by $f(t) = t$ and $g(t) = 1 - t$.
Then we have:

$$(f \vee g)(t) = \begin{cases} 1 - t, & 0 \le t \le \dfrac{1}{2} \\ t, & \dfrac{1}{2} < t \le 1 \end{cases} \text{ and } f \vee_G g = 1.$$

So, we remark that $f \vee g \le f \vee_G g$.

Now, define $T : E \to E$ by $T(f) = f$. We remark that:

1) T is a E-Riesz homomorphism.
2) T is a G-quasi lattice operator.
3) T is not a G-lattice operator.

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INTERSECTIONS THEOREMS FOR SETS WITH NONEMPTY SECTIONS IN THE FINITE DIMENSIONAL TOPOLOGICAL VECTOR SPACES SETTING

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Abstract: In this paper we will apply some fixed-point theorems for families of multimaps defined on a product of (finite dimensional) topological vector spaces, to obtain intersection theorems for sets with nonempty sections. We will use some compactness assumptions. Finally, a short economic interpretation will be given.
Mathematics Subject Classification (2010): 54H25, 91B54.
Key words: multimap, topological vector space, intersection, nonempty sections.

1. Introduction

In this paper the fixed-point theorems for multimaps families are included in the category of topological vector spaces because it handles concepts of convexity (specific to vector space structure) and the compactness contained in category of topological spaces.

The need for compatibility between the topological and algebraic structures lead to the category of topological vector spaces. Analyzing finite dimensional case was imposed by the economic interpretation of the results obtained, the economies are finite dimensional in real life. For a nonempty set Y, we denote by 2^Y its *power set* that is the collection of all subsets of Y.

Definition 1. For X and Y nonempty sets, a *multimap* is a function $T: X \to 2^Y$. This multimap is *nonempty-valued*, if for each $x \in X$, the set $T(x)$ is nonempty.

Definition 2. If X and Y are two vector spaces, a multimap $T: X \to 2^Y$ is *convex-valued*, if for each $x \in X$, the set $T(x)$ is convex.

Definition 3. If $A \subseteq X$, and $T: X \to 2^Y$ is a multimap, we denote by $T(A)$ the set $\{T(x) \in Y \mid x \in A\}$, and if $y \in Y$, we denote by $T^{-1}(y) = \{x \in X \mid y \in T(x)\}$ calling this last set the *fiber of T in y*.

Definition 4. Let X, Y be two topological vector spaces. A multimap $T: X \to 2^Y$ is

a) *compact-valued* if $T(x)$ is a compact set, for each $x \in X$.

b) *compact* if there exists a compact subset $K \subseteq Y$ such that $T(x) \subseteq K$ for each $x \in X$.

Definition 5. For a multimap $T: X \to 2^Y$ (between two nonempty sets X, Y), a point $x \in X$ is called a *fixed-point* if $x \in T(x)$.

Definition 6. Let I be an index set and for each $i \in I$ let x_i a nonempty set. Denote $X = \prod_{i \in I} X_i$ and for each $i \in I$ let $T_i : X \to 2^{X_i}$ be a multimap. A (*collectively*) *fixed-point* for the family $(T_i)_{i \in I}$ is a point $x = (x_i)_{i \in I}$ such that $x_i \in T_i(x)$, for all $i \in I$.

Remember some fixed-point results. Finally, the following two fixed-point results (see, for example, [3]) are valid in the topological vector spaces setting and use open fiber hypothesis.

Applying [1, Corollary 5.33] and Proposition 1.1. or Proposition 1.2. respectively in [4] we obtained the following two fixed-point results, having compactness assumptions of same multimaps.

Proposition 1.1.(see [4], Thm. 2.1) *Let I be an arbitrary index set, and for each $i \in I$, let X_i be a nonempty convex set in a finite dimensional topological vector space E_i. Let also $X = \prod_{i \in I} X_i$, and let $T_i : X \to 2^{X_i}$ $(i \in I)$ be a nonempty-valued and convex-valued multimap. Suppose that for each $i \in I$, the following conditions hold:*

1) $X = \bigcup_{y_i \in X_i} \operatorname{int} T_i^{-1}(y_i)$; 2) T_i is a compact multimap, (i. e. there exists a nonempty compact subset $K_i \subset X_i$ such that $T_i(x) \subseteq K_i$ for each $x \in X$).

Then, there exists $\tilde{x} = (\tilde{x}_i) \in X$ such that $\tilde{x}_i \in T_i(\tilde{x})$, for each $i \in I$.

Proposition 1.2.(see [4], Thm.2.2) *Let I be an arbitrary index set, and for each $i \in I$ let X_i be a nonempty convex set in a finite dimensional topological vector space E_i. Let also $X = \prod_{i \in I} X_i$, and let $S_i, T_i : X \to 2^{X_i}$ $(i \in I)$ be nonempty-valued multimaps. Suppose that for each $i \in I$, the following conditions hold:*

1) for each $x \in X$, $coS_i(x) \subseteq T_i(x)$; 2) $X = \bigcup_{y_i \in X_i} \operatorname{int} S_i^{-1}(y_i)$; 3) S_i is a compact multimap, that is there exists a nonempty compact subset $K_i \subseteq X_i$ such that $S_i(x) \subseteq K_i$ for each $x \in X$.

Then, there exists $\tilde{x} = (\tilde{x}_i) \in X$ such that $\tilde{x}_i \in T_i(\tilde{x})$, for each $i \in I$.

We apply these fixed-point results to obtain some intersection theorems.

2. Main Results

A problem solved by the fixed-point theory is the *problem of intersection of sets with convex sections*. Let I be an index set. In the case when I is finite it is always assumed that I contained at least two indices. Let also $(X_i)_{i \in I}$ be a family of convex sets, each in a Hausdorff topological vector space. Denote $X = \prod_{i \in I} X_i$ and $X^i = \prod_{j \in I \setminus \{i\}} X_j$ and let $(A_i)_{i \in I}$ be a family of subsets of X. Then, the problem on intersection of sets with convex sections is to find suitable conditions on $(A_i)_{i \in I}$ such that this family has a nonempty intersection.

Definition 7. For a nonempty set $A \subset X$ and $x^i \in X^i$, the set $A(x^i) = \{y_i \in X_i | [x^i, y_i] \in A\}$ is called the *set of all sections of A at x^i*, where $[x^i, y_i] \in X^i \times X_i$ is the element of X having the i *th* coordinate y_i and for $j \neq i$ having its j *th* coordinate x_j^i.

Definition 8. (see [5]) We say that $A \subseteq X$ has *nonempty sections* if for each $i \in I$ and $x \in X$, the set $A(x^i)$ is nonempty.

In 2009, R.-M. Dăneț, I.-M. Popovici and F. Voicu, see [3], obtained two intersection theorems for sets with nonempty sections and then they deduce a Nash-Ma type equilibrium theorem.

Note that in [3, def. 4.2 and 4.3] were introduced the following two notions.

Definition 9. Let I be an index set, having at least two elements and, for each $i \in I$, let X_i be a set in a Hausdorff topological vector space E_i.

Let $X = \prod_{i \in I} X_i$ and let A be a subset of X. If $i \in I$ is a fixed index, we say that *the nonempty set* $D \subseteq X$ *can be* (A, i) *sectioned with a common element of* X_i if there exists $y_i \in X_i$ such that $y_i \in \bigcap_{z \in D} A(z^i)$ (that is $[z^i, y_i] \in A$, for each $z \in D$).

Definition 10. Let I, E_i, X_i and X be as in the previous definition. Let M be an arbitrary subset of X, and for each $i \in I$, let A_i and L_i subsets of X, and respectively of X_i. We say that M *can be locally covered with a family* $(D_i)_{i \in I}$ *of open sets of* X *such that the set* D_i *can be* (A_i, i) *sectioned with a common element of* L_i, *for each* $i \in I$, if for each $x \in M$, there exists a family of open sets $(D_i)_{i \in I}$ in X, such that, for each $i \in I$, $x \in D_i$ and there exists an element $y_i \in L_i$ with $y_i \in \bigcap_{z \in D_i} A_i(z^i)$.

In the results that follows, we gave sufficient conditions in order to obtain that the set $\bigcap_{i \in I} A_i$ is nonempty. In the sequel of this section, all results are new.

Theorem 2.1. *Let* I *be an index set, and, for each* $i \in I$, *let* X_i *be a nonempty convex subset of a topological vector space* E_i, *and let* $X = \prod_{i \in I} X_i$. *Let also* C *be a nonempty compact subset of* X, *and, for each* $i \in I$, *let* A_i *be a subset of* X, *having nonempty convex section* $(A_i(x^i))_{x \in X}$. *Suppose that:*

1) X *can be locally covered with open sets which can be* (A_i, i) *sectioned with a common element of* X_i $(i \in I)$;

2) if X *is not compact, assume that* $X \setminus C$ *can be locally covered with open sets which can be* (A_i, i) *sectioned with a common element in a nonempty compact convex set* C_i *of* X_i $(i \in I)$. *Then,* $\bigcap_{i \in I} A_i \neq \varnothing$.

The proof of Theorem 2.1. uses [4, Proposition 1.1.].

Theorem 2.2. *Let* I, E_i, X_i, X *be like in Theorem 2.1, with* X *a finite dimensional topological vector space. For each* $i \in I$ *let* A_i *be a subset of* X *having nonempty convex sections. Suppose that, for each* $i \in I$,

1) X *can be locally covered with open sets which can be* (A_i, i) *sectioned with a common element of* X_i;

2) There exists a compact subset K_i *of* X_i *such that for all* $x = (x_i)_{i \in I} \in X$ *and all* $y_i \in X_i$ *with* $(x^i, y_i) \in A_i$ *it fallows that* $y_i \in K_i$.

Then, $\bigcap_{i \in I} A_i \neq \varnothing$.

Theorem 2.3. *Let I, E_i, X_i and X be like in Theorem 2.2. For each $i \in I$ let A_i and B_i be two subsets of X such that B_i has nonempty sections. Suppose that for each $i \in I$:*

1) $\text{co}B_i\left(x^i\right) \subseteq A_i\left(x^i\right)$, for each $x = \left(x_i\right)_{i\in I} \in X$,

2) X can be locally covered with open sets which can be $\left(B_i, i\right)$ sectioned with a common element of X_i;

3) There exists a compact subset L_i of X_i such that for all $x = \left(x_i\right)_{i\in I} \in X$ and all $y_i \in X_i$ with $\left(x^i, y_i\right) \in B_i$ it follows that $y_i \in L_i$.

Then, $\bigcap_{i\in I} A_i \neq \varnothing$.

The proofs of Theorem 2.2 and Theorem 2.3. uses Proposition 1.1. and Proposition 1.2. presented above.

3. An economic interpretation

Let I be an index set and, for each $i \in I$, let X_i be a nonempty convex set in a finite dimensional topological vector space. Let $X = \prod_{i\in I} X_i$, and, for each $i \in I$, consider a subset A_i of X having nonempty convex sections.

Suppose that the hypothesis of our Theorem 2.2. are valid. For each $i \in I$ define the multimap $P_i : X \to 2^{X_i}$, by $P_i\left(x\right) = A_i\left(x^i\right)$, where $x = \left(x_i\right)_{i\in I}$. It follows that P_i is a nonempty-valued and convex-valued.

Consider an economy (or a qualitative game) $\Gamma = \left(X_i, P_i\right)_{i\in I}$, where X_i is the strategy (choice) set of the agent (player) i and P_i is its preference multimap.

Then, by applying Theorem 2.2., we find a *strategies combination* (an *equilibrium point*) for Γ, namely $\tilde{x} = \left(\tilde{x}_i\right)_{i\in I} \in X$ such that $\tilde{x} \in \bigcap_{i\in I} A_i$, that is (remember that $A_i\left(x^i\right) = \left\{y_i \in X_i \,\middle|\, \left[x^i, y_i\right] \in A\right\}$, where $x^i \in X_i = \prod_{j\in I\setminus\{i\}} X_j$) $\tilde{x}_i \in A_i\left(\tilde{x}^i\right)$ (or equivalently, $\tilde{x}_i \in P_i\left(\tilde{x}^i\right)$) for all $i \in I$. In other words $\tilde{x}$ is *preferred* by all agents of the economy Γ.

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ON NORMAL SEMI-CONTINUOUS FUNCTIONS

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Abstract: The normal semi-continuous functions appear in the construction of the Dedekind completion of *C(X)*, the Riesz space (vector lattice) of all real-valued continuous functions on a topological space *X*. This paper presents some of the most important properties of this class of functions.
Mathematics Subject Classification (2010): 26A15
Key words: Semi-continuous functions, quasi-continuous functions, Dedekind order completion

1. Introduction

The normal (upper or lower) semi-continuous functions was considered for the first time by Dilworth in 1950 [3] in order to describe the Dedekind completion of $C_b(X)$, the lattice of all real-valued bounded continuous functions on a (Hausdorff) completely regular topological space *X*. More precisely, Dilworth proved the following theorem: *If X is a completely regular topological space, then the Dedekind completion of the lattice* $C_b(X)$ *is isomorphic with the lattice of all real-valued normal upper semi-continuous functions on X* ([3], Theorem 4.1). A similar result was proved three years later by Horn for $C(X)$ ([4], Theorem 11).

This paper presents the most important properties of the normal semi-continuous functions. For the terminology concerning the Riesz spaces see [7]. The symbol # marks the end of a proof or the end of an example.

2. Normal semi-continuous functions

In order to recall the definition of a normal semi-continuous function we need first to establish the notation used below. Let *X* be a Hausdorff topological space. By $B(X)$ we denote the Dedekind complete vector lattice of all bounded functions on *X*. For every *f* in $B(X)$, $I(f)$ is the lower limit function of *f* and $S(f)$ is the upper limit function of *f*, that is,

$$I(f):\ X \to \mathbb{R},\ \ I(f)(x) = \sup_{V \in N_x} \inf_{y \in V} f(y),\ \ x \in X,$$

$$S(f):\ X \to \mathbb{R},\ \ S(f)(x) = \inf_{V \in N_x} \sup_{y \in V} f(y),\ \ x \in X,$$

where N_x denotes the set of all neighborhoods of the point $x \in X$.

Obviously, $I(f)(x) \le f(x) \le S(f)(x),\ x \in X,$ and f is bounded on *X* if and only if $I(f)$ and $S(f)$ are bounded on *X*. So we obtain two nonlinear operators $I, S\ :\ B(X) \to B(X)$. I is called the *lower Baire operator* and *S* is called the *upper Baire operator* in honor of R. Baire who used these operators for the first time in his book under the name $m(f)$ for $I(f)$ and $M(f)$ for $S(f)$ [1].

The operators *I* and *S* are monotone and idempotent and their compositions $I \circ S$ and $S \circ I$ have the same properties. The sets of fixed points of these operators are denoted as follows (in brackets are written the abbreviations used below):

(a) $L_{sc}(X) = \{f \in B(X):\ I(f) = f\}$ - the lower semi-continuous (lsc) functions;

(b) $U_{sc}(X) = \{f \in B(X):\ S(f) = f\}$ - the upper semi-continuous (usc) functions;

(c) $NL_{sc}(X) = \{f \in L_{sc}(X):\ I(S(f)) = f\}$ - the normal lower semi-continuous (nlsc) functions;

(d) $NU_{sc}(X) = \{f \in U_{sc}(X):\ S(I(f)) = f\}$ - the normal upper semi-continuous (nusc) functions.

It is well known that the semi-continuous functions can be characterized in the following manner:

(a) $f \in L_{sc}(X) \Leftrightarrow \{x \in X\ :\ f(x) > \lambda\}$ is open;

(b) $f \in U_{sc}(X) \Leftrightarrow \{x \in X\ :\ f(x) < \lambda\}$ is open.

For the normal semi-continuous functions a similar characterization was given by Dilworth.

Theorem 1 ([3], Th. 3.2). *A function* $f \in U_{sc}(X)$ *is normal if and only if for each real* λ *the set* $\{x \in X\ :\ f(x) > \lambda\}$ *is the union of closures of open sets.*

A local characterization of a nusc function is given in the next theorem.

Theorem 2 ([3], Th. 3.1). *A function* $f \in U_{sc}(X)$ *is normal if and only if for each* $\varepsilon > 0$, $x \in X$, *and open set* U *containing* x, *there exists a nonempty open set* $G \subseteq U$ *such that* $f(y) > f(x) - \varepsilon$, *for all* $y \in G$.

Let us note that the set G may not contain x. A function $f : X \to \mathbb{R}$ that satisfies the second condition in the previous characterization is usually called *quasi-continuous*. The relations between quasicontinuous functions and the normal semi-continuous functions was addressed by the author in [2]. There it was shown that: *If* $f \in B(X)$ *is quasi-continuous then* $I(S(f)) = f$ *and* $S(I(f)) = S(f)$ (Prop. 3.10). Therefore, if $f \in U_{sc}(X)$ (that is $S(f) = f$) and quasicontinuous, then $S(I(f)) = f$, which means that f is nusc. Similar result holds for $f \in L_{sc}(X)$. Consequently, for semi-continuous functions normality is equivalent with quasi-continuity.

The following examples show some differences between semi-continuous functions and normal semi-continuous functions,

Example 1. Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined as follows,

$$f(x) = \begin{cases} \sin\dfrac{1}{x}, & x \neq 0, \\ a, & x = 0, \end{cases}$$

where a is a real number. Then:

(i) f is upper semi-continuous if and only if $a \geq 1$.

(ii) f is normal upper semi-continuous if and only if $a = 1$. #

The point-wise sum of two usc functions is an usc function. In contrast, the sum of two nusc functions is not, in general, a nusc function. The following example confirms this statement.

Example 2. The following two functions $f, g : \mathbb{R} \to \mathbb{R}$,

$$f(x) = \begin{cases} \sin\dfrac{1}{x}, & x \neq 0, \\ 1, & x = 0, \end{cases} \qquad g(x) = \begin{cases} \cos\dfrac{1}{x}, & x \neq 0, \\ 1, & x = 0, \end{cases}$$

are nusc, but their point-wise sum

$$(f+g)(x)=\begin{cases}\sqrt{2}\cos\left(\dfrac{1}{x}-\dfrac{\pi}{4}\right), & x\neq 0,\\ 2, & x=0,\end{cases}$$

is upper semi-continuous at $x=0,$ but it is not normal. To be normal at $x=0$ we must have $(f+g)(0)=\sqrt{2}.$ #

Let us denote by $f\vee g$ and $f\wedge g$ the point-wise supremum, respectively infimum, of two functions $f,g\in B(X).$ Using Theorem 1 it is easy to see that

$$f,g\in NU_{sc}(X)\Rightarrow f\vee g\in NU_{sc}(X). \qquad (1)$$

This means that $f\vee g$ is also the supremum of $\{f,g\}$ in the set $NU_{sc}(X)$ endowed with the point-wise order.

Not the same thing holds for infimum. If $f,g\in NU_{sc}(X),$ then $f\wedge g\in U_{sc}(X),$ but $f\wedge g$ may not be in $NU_{sc}(X).$ The following example shows such a situation.

Example 3 ([3], p.433). Let $f=\chi_{[0,\,1/2]}$ and $g=\chi_{[1/2,\,1]}$ be the characteristic functions of the closed intervals $[0,1/2]$ and $[1/2,1]$ respectively. Then

$$(f\wedge g)(x)=\begin{cases}0, & x\neq 1/2,\\ 1, & x=1/2,\end{cases}\quad x\in[0,1],$$

which shows that $f\wedge g\in U_{sc}(X),$ but $f\wedge g$ is not normal since the set

$$\{x\in[0,1]:\ (f\wedge g)(x)>0\}=\{1/2\}$$

is not a union of closures of open sets (see Theorem 1). #

These examples shows that the set of nusc functions $NU_{sc}(X).$ endowed with point-wise algebraic operations and point-wise order is not a Riesz space (vector lattice).

3. Kaplan operators and their use in the study of discontinuous functions

For every function $f\in B(X)$ we define two new functions:

$$\ell(f)=\sup\{g\ :\ g\in C_b(X),\ g\le f\},$$
$$u(f)=\inf\{g\ :\ g\in C_b(X),\ g\ge f\}.$$

Let us note that the above sets are nonempty, since $|f|\le M\chi_X$, for some positive real constant $M.$ This means that the Riesz ideal generated by $C_b(X)$ in $B(X)$ is the whole $B(X)$, that is, $I(C_b(X))=B(X).$

In this manner we have two new operators $\ell,u\ :\ B(X)\to B(X)$ and for every $f\in B(X)$ the following inequalities hold

$$\ell(f)\le I(f)\le S(f)\le u(f).$$

If X is a completely regular topological space, then ([3], Lemma 4.1),

$$\ell(f)=I(f)\ \text{ and }\ u(f)=S(f).$$

The properties of the operators ℓ and u have been studied in details by Kaplan in [5] and [6]. In Kaplan's papers these operators are defined on the second dual of $C(X)$, with X compact, but, in general, their properties depend only on the order between functions and not of the structure of the functions. So the properties of ℓ and u also hold in our settings. We will called the operators ℓ and u Kaplan operators.

Below we enumerate some useful properties of the Kaplan operators and show how these operators can be used in the study of semi-continuous functions.

Theorem 3 ([5], [6]). Let X be a completely topological space. For any functions $f, g \in B(X)$ we have:

(i) $\ell(f) \le f \le u(f)$.

(ii) $\ell(f) = f \Leftrightarrow f \in L_{sc}$, $u(f) = f \Leftrightarrow f \in U_{sc}$.

(iii) $\ell(f) + \ell(g) \le \ell(f+g) \le \ell(f) + u(g) \le u(f+g) \le u(f) + u(g)$.

(iv) $\ell(f) - u(g) \le \ell(f-g) \le \begin{matrix} u(f) - u(g) \\ \ell(f) - \ell(g) \end{matrix} \le u(f-g) \le u(f) - \ell(g)$.

(v) $\ell(f \wedge g) = \ell(f) \wedge \ell(g)$, $u(f \vee g) = u(f) \vee u(g)$.

The following theorem is well known but its proof with Kaplan operators is new.

Theorem 4. $f, g \in U_{sc} \Rightarrow f+g, f \vee g, f \wedge g \in U_{sc}$.

Proof. If $f, g \in U_{sc}$, then $u(f) = f$ and $u(g) = g$. Using Theorem 3, (i) and (iii), we have $f + g \le u(f+g) \le u(f) + u(g) = f + g$, hence $u(f+g) = f+g$, that is $f + g \in U_{sc}$. Similarly, using (v), we have for supremum, $f \vee g \le u(f \vee g) = u(f) \vee u(g) = f \vee g$, and for infimum, $f \wedge g \le u(f \wedge g) \le u(f) \wedge u(g) = f \wedge g$. Hence $f \vee g$ and $f \wedge g$ are in U_{sc}. #

Using Kaplan operators we can give the following proof of the implication (1). Indeed, let $f, g \in NU_{sc}(X)$. Then, using Theorem 4, (v), and the fact that Kaplan operators are monotone and idempotent, we have

$$f \vee g = u(f) \vee u(g) = u(f \vee g) \ge u\ell(f \vee g) \ge u[\ell(f) \vee \ell(g)] = u\ell(f) \vee u\ell(g) = f \vee g$$

This shows that $u\ell(f \vee g) = f \vee g$, therefore $f \vee g \in NU_{sc}(X)$.

The isomorphism proved by Dilworth on the Dedekind completion of $C_b(X)$, in the theorem quoted in the Introduction, refers to the lattice structure only. As we noted at the end of Section 2, the set $NU_{sc}(X)$ endowed with the point-wise algebraic operations and point-wise order is not a Riesz space.

Problem: can define algebraic and lattice operations on $NU_{sc}(X)$ (of course, not point-wise) such that this set become a Riesz space? The answer is yes, but the results will be published elsewhere.

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USING GENETIC AND GRADIENT ALGORITHMS FOR SOLVING HYDROGEOLOGICAL OPTIMIZATION PROBLEMS

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Abstract: Gradient and Genetic Algorithms (GAs) have been applied with great success in optimization problems in groundwater field. Two optimization problems are investigated: a parameter estimation problem in an aquifer using a gradient method and a pumping management problem using GAs. The aim of this paper is to investigate the performance of local and global optimization techniques, respectively, a gradient based Gauss–Marquard–Levenberg (GML) algorithm and a GA. In order to solve inverse problem in hydrogeology, GML is used to estimate the aquifer's parameters, so that the computed solution to match field-observed values for water level. The objective of the optimization model is to minimize the difference between observed and calculated heads using the Least Squares method. Regarding the transport optimization problem, a GA is coupled with an existing groundwater flow and transport model to find an optimal set of pumping/injection well rates.
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Key words: global and local optimization methods, transport optimization, Genetic algorithm, Inverse Problem, parameter estimation, groundwater remediation

1. Introduction and purpose

We compare the local gradient method based on GML algorithm and a global technique using GAs. Local methods use deterministic rules and find a single parameter set while global ones involve the evaluation of the function, usually at a random sample of points in the parameter space, followed by a subsequent manipulation of the sample using probabilistic rules. Regarding aquifer's parameter estimation by optimization algorithms first step is to solve the forward problem that finds the unknown heads assuming that the parameter values and boundary conditions are known. The second step is to solve the inverse problem that finds unknown parameters by fitting the simulated solutions to observed heads using automatic calibration procedures. To achieve this purpose we are using **PEST** (**P**arameter **EST**imation) for model calibration. [2]

The second problem is about Groundwater Remediation Problems and how to apply a global optimization algorithm. The optimization code **MGO** (A **M**odular **G**roundwater **O**ptimizer) first purpose was to reduce the costs associated with a pump-and-treat remediation system. The MGO code can be applied to a variety of groundwater resource management issues: to minimize cleanup time or contaminant mass remaining in aquifer. We are using the GAs MGO in order to find rates and/or locations at each extraction/injection well. [5]

2. Gradient algorithms used in parameter estimation in groundwater models

In the simulation problem we predict the unknown heads using the two dimensional groundwater flow equation:

$$\frac{\partial}{\partial x}\left[T_x \frac{\partial H}{\partial x}\right] + \frac{\partial}{\partial y}\left[T_y \frac{\partial H}{\partial y}\right] + R = S\frac{\partial H}{\partial t} \qquad (1)$$

where (x, y) are cartesian coordinates $[L]$, t is time $[T]$, $H = H(x, y, t)$ is the hydraulic head $[L]$, $S = S(x, y)$ is the storativity, $T = T(x, y)$ is the transmissivity $[L^2T^{-1}]$ and $R = R(x, y, t)$ is the average recharge of the aquifer $[LT^{-1}]$.

The inverse model uses known values of hydraulic head in order to estimate values of unknown parameters such as transmissivity, recharge or storativity.

To avoid the subjectivity of the trial-and-error procedure we will transform the inverse problem into an optimization problem defined by:

$$\begin{cases} \min E(p) \\ E(p) = \sum_{i=1}^{n}[(H_i(p))^{calc} - (H_i)^{obs}]^2 = \sum_{i=1}^{n} f_i^{\,2}(p) \end{cases} \qquad (2)$$

where $(H_i(p))^{calc}$ is the calculated head, $(H_i)^{obs}$ is the observed head, n is total number of measurements and $p = (p_1, p_2, ..., p_m)^T$ is the parameter vector needed to be estimated and $f = (f_1, f_2, ..., f_n)^T$ represents the difference between calculated head and observed head.

In order to find a parameter set that minimizes the objective function we are using an iterative process defined by the equation $p^{k+1} = p^k + \alpha_k d^k$ where α_k is the size step along the displacement direction d^k. The gradient methods use the negative gradient direction as the search direction in each iteration: $p^{k+1} = p^k - \alpha_k G^k$ where $G = \nabla E(p); G^k = G(p^k)$.

Another powerful tool used in optimization based on gradient methods is PEST (**P**arameter **Est**imation). PEST uses the GML method, a combination between the gradient descent method and Gauss-Newton method. [2], [3]

The iterative process from GML algorithm has the form

$$p^{k+1} = p^k - (J^{k^T} J^k + \lambda_M diag(J^{k^T} J^k))^{-1} J^{k^T} f^k \qquad (3)$$

where J is the Jacobian matrix of f, $J^k = J(p^k) = (\frac{\partial f_i}{\partial p_j})_{\substack{i=\overline{1,n} \\ j=\overline{1,m}}}$ calculated at the current point p^k; λ_M is *Marquardt parameter*. If λ_M tends to infinity we have the steepest descent method and if $\lambda_M = 0$ we have the Gauss-Newton method and its convergence is slow but safe.

3. Genetic algorithms used in groundwater remediation problems

Because gradient calculation is a major source of numerical difficulty who can lead to instability and convergence problems, a new class of optimization methods based on heuristic search techniques have emerged. Among these "gradient-free" optimization techniques that identify the global optimum we find genetic algorithms (GAs) based on biological evolution.

Genetic algorithms deal only with unconstrained optimization problems, so if we have a constrained optimization problem it's necessary to be converted into an unconstrained problem before using these techniques.

GAs use concepts of "Natural Selection" and "Genetic Inheritance" and follow the idea of survival of the fittest: better and better solutions evolve from previous generations until a near optimal solution is obtained. The first step is to generate a random initial set of decision vectors. Starting with these decision vectors after applying Selection, Crossover, Mutation,

Acceptation and Replace of the old generation with new population, the optimization model obtains an optimal set of parameters.
The optimization problem can be formulated as

$$\begin{cases} \min E = \sum_{i=1}^{N} y_i |Q_i| \\ Q_{\min} \le |Q_i| \le Q_{\max} \\ Q_{inj} = \frac{1}{M} \sum_{i=1}^{P} |Q_i| \end{cases} \quad (4)$$

where Q_i is pumping rate of each well $[m^3 d^{-1}]$, N is the number of pumping wells, y_i is a binary variable equal to either 1 if parameter i is active (i.e., the associated flow rate is not zero) or zero if parameter i is inactive, E is the absolute sum of all pumping rates, Q_{inj} is injection rates of each injection well chose so that the total pumping rate from P active wells is split between the M injection wells so extraction volume equals injection volume. We apply GAs from MGO to design a containment zone meant to prevent the plume from spreading [4], [5].

4. Numerical application

Consider an homogenous and confined aquifer which has an area of $10 \times 6\ km^2$. The north and south boundary of the aquifer are impermeable, the west boundary is a recharge boundary, i.e. the recharge rate is $0.005 m d^{-1}$ and the east boundary is restricted to sea level $0 m$. Aquifer transmissivity is $600 m^2 d^{-1}$ and there are two abstraction wells located at points $(7500, 3500)$-well 1 and $(3500, 2500)$-well 2. The total amount of pumping is $4000 m^3 d^{-1}$ and should be divided between wells 1 and 2 in such a way that the risk of well pollution by chlorinated water is eliminated. Adapted from [3].
We are using Groundwater Vistas because it can solve optimization problems with gradient and genetic algorithm.
The direct problem is solved with MODFLOW using the finite difference with block-centered grid. The aquifer is divided into 50×30 nodes, $\Delta x = \Delta y = 200 m$. Pumping from two wells was designed in such a way that there would be minimal risk for seawater intrusion to pumping wells: from well 1, respectively well 2, we extract $1500 m^3 d^{-1}$ and $2500 m^3 d^{-1}$. The hydraulic head in well 1 is $0.06 m$.
The inverse problem estimates T, Q_w, the transmissivity and unknown inflow through the west-boundary. Regarding the automatic calibration procedure using a nonlinear least-squares technique we can use GV Calibrations or PEST model-independent calibration software. GV's Calibration or Pest models for parameter estimation employs Marquardt's modification to the Gauss-Newton nonlinear least-squares. GV Calibration technique is simplified over other codes which makes it easier to use.
We have eight steady-state measurements of hydraulic heads located in:
$H = 5.998, (1500; 4500)$; $H = 4.687, (2500, 3500)$
$H = 2.804, (3500, 2500)$; $H = 2.545, (4500, 2500)$; $H = 2.054, (5500, 4500)$
$H = 0.522, (7500, 4500)$; $H = -0.142, (7500, 3500)$; $H = 1.231, (7500, 2500)$
GV Calibration estimates a multiplier of the initial parameter value: 1.36 for T and 0.967 for recharge so $T = 816.17 m^2 d^{-1}$; $Q_w = 5796 m^3 d^{-1}$.
To validate the model we can use in EXCEL successive over relaxation iterative method with a good convergence rate for the direct problem. For the inverse problem, the optimization

program called SOLVER is based on another gradient method named Generalized Reduced Gradient. SOLVER is not very well suited for solving inverse groundwater problems because the program had to be restarted more times until the target function had reached a minimum value.

Based on groundwater flow and transport model we want to optimize a set of five pumping wells, using the two existing wells (well 1 and well 2) and three new wells using optimization formulation from equations (4). The maximum for the magnitude of each pumping rate to be optimized is $4000 m^3 d^{-1}$. The injection rates at two injection wells are each required to be one half of the total pumpage from the five pumping wells. The purpose is to find optimal pumping rate in order to achieve containment of the contaminant plume. The objective function E after a total of seven generations satisfying all the constraints and converges to a value of $10839.1 m^3 d^{-1}$.

Well location	**Pumping rate**
(13 ; 38)	-129.00
(18; 18)	-387.10
(10 ; 38)	-2323.00
(10, 35)	-4000.00
(13; 35)	-4000.00
Total	**-10839.10**

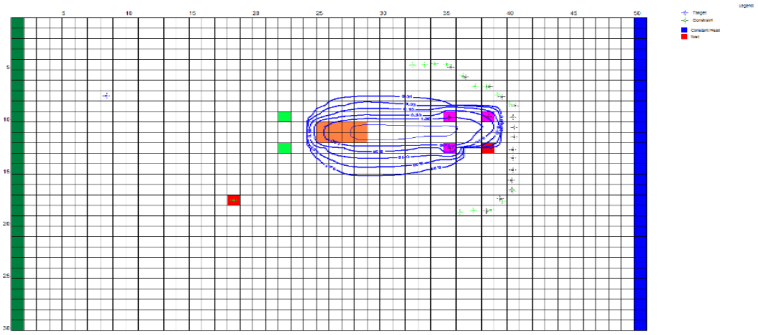

Fig1. a) Optimized pumping rates from the GA solutions;
b) Configuration of the capture zone design based on the optimized pumping rates

5. Conclusion

This paper makes a combination of a global and local techniques in a hydrogeology optimization problem. The advantages of GAs are that the objective function does not need to be continuous or differentiable because they are not using the gradient or Hessian matrix. A classical gradient optimization method is finding the solution in the neighborhood of a starting point so it is a local optimum while GAs can search very large spaces of candidate solutions for finding a global optimum. In the real world problems every methods has his advantages or drawbacks depending on the formulation and the set-up of the model.

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EXISTENCE OF THE BOUNDED SOLUTIONS OF THE NONLINEAR STRING EQUATION

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Abstract: In this paper, the existence of the bounded generalized solutions of the nonlinear equation that describes the wave propagation in the heterogeneous and anisotropic strings, is proved

$$y_{tt} = A^2(y_x, y_t)y_{xx}, \quad y(x,0) = y^{(0)}(x), \; y_t(x,0) = y^{(1)}(x), \tag{1}$$

where $y(x,t)$ is the displacement, $y^{(0)}$, $y^{(1)}$ are known functions and $A(y_x, y_t) > 0$ is a function with positive values representing the local wave propagation velocity in the string, which satisfies the conditions

$$(A^2 - 1)e_x - \eta y(1+\delta e^2)^{3/2} = 0, \; A = A(e,z), \; e(x,0) = e^{(0)}(x), \tag{2}$$

with $e = y_x$, $f = y_t$, μ, ν, γ, η, δ dimensionless material constants, and $A^{(0)}, A^{(1)}$, $e^{(0)}$ known constants. A theorem is formulated to proof the existence of at least one bounded generalized solution for equation (1) using LEM [2]. The theorem demonstrates that bounded generalized solutions are likely solitons, the localized waves which preserve their properties if interacting with other waves and having a behavior like particles [3-5].
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Key words: nonlinear waves; string; solitons.

1. Introduction

The study of remarkable equations as sine-Gordon, Burger, Korteweg-de Vries and Schrödinger equation, has emerged some localized wave solutions called *solitons*, whose mathematical properties are like the particles. In 1955, Fermi Pasta and Ulam studied in laboratories in Los Alamos the wave propagation in granular media with applications to rocks and soils [6]. They proposed some equations that admit the soliton solutions in which the amplitude is dependent to the speed. At the collision they behave like the particles, verifying the nonlinear superposition property [7]. Waves described by these solutions collide elastically, without changing their shape and transferring to each other energy, momentum and amplitude, or passing through one another, interacting in pairs and introducing some phase shifts [8]. Seymour and Varley [9] were reopen in 1982 the Fermi Pasta Ulam problem under the form of three types of equations for the velocity of propagation A depending on y_x and y_t ($A(y_x, y_t) > 0$)

$$y_{tt} = A^2(y_x, y_t)y_{xx}, \; y(x,0) = y^{(0)}(x), \; y_t(x,0) = y^{(1)}(x), \tag{1}$$

$$A_e - A^{3/2}(\mu + \nu A) = 0, \quad e = y_x, \; A = A(e), \; A(0) = A^{(0)}, \tag{2}$$

$$A_z + \gamma A^2 = 0, \quad z = y_t, \; A = A(z), \; A(0) = A^{(1)}, \tag{3}$$

$$(A^2 - 1)e_x - \eta\, y(1+\delta e^2)^{3/2} = 0, \; A = A(e,z), \; e(x,0) = e^{(0)}(x), \tag{4}$$

where $y(x,t)$ is the bounded displacement for $x \to \infty$ and $t \to \infty$, and μ, ν, γ, η, δ are dimensionless material constants, while $A^{(0)}, A^{(1)}$, $e^{(0)}$ are known quantities.

The case $y_x = y_x(x,t)$ and $A(y_x, y_t) = A(y_x(x,t)) = A(x,t)$ where (1) and (2) can be written as

$$y_{tt} = A^2(x,t)y_{xx},\ A_x = A^{3/2}(\mu + \nu A)y_{xx}. \tag{5}$$

has been analyzed in [10] by using the linear equivalence method (LEM) [2]. A bounded solution of (2) is given by [8]

$$A(y_x) = \frac{\lambda[e_3 + (e_2 - e_3)\mathrm{sn}^2(\sqrt{e_1 - e_3}\,y_x + \delta')]}{1 + \rho[e_3 + (e_2 - e_3)\mathrm{sn}^2(\sqrt{e_1 - e_3}\,y_x + \delta')]}, \tag{6}$$

where λ, ρ are material constants expressed as functions to μ, ν, and e_i, $i = 1,2,3$ are the solutions of the equation $4y^3 - g_2 y - g_3 = 0$, where g_2, g_3 depend on μ, ν.

The solution (6) depends on the amplitude and exhibits a cnoidal dependence on A and y_x.

For $m = 1$ ($m = \dfrac{e_2 - e_3}{e_1 - e_3}$), the solution (6) becomes

$$A(y_x) = \frac{\lambda[e_1 - (e_1 - e_3)\,\mathrm{sech}^2(\sqrt{e_1 - e_3}\,y_x + \delta')]}{1 + \rho[e_1 - (e_1 - e_3)\,\mathrm{sech}^2(\sqrt{e_1 - e_3}\,y_x + \delta')]}, \tag{7}$$

In this paper we study the problem (1)

$$y_{tt} = A^2(y_x, y_t)y_{xx},\ y(x,0) = y^{(0)}(x),\ y_t(x,0) = y^{(1)}(x), \tag{8}$$

where $A(y_x, y_t) > 0$ depends on y_x and y_t. In (8) $y(x,t)$ represents the displacement, and $y^{(0)}$, $y^{(1)}$ are known functions. The velocity $A(y_x, y_t) > 0$ verifies the conditions [7]

$$(A^2 - 1)e_x - \eta\, y(1 + \delta e^2)^{3/2} = 0,\ A = A(e,z),\ e(x,0) = e^{(0)}(x), \tag{9}$$

with $e = y_x$, $z = y_t$, μ, ν, γ, η, δ dimensionless material constants and $A^{(0)}, A^{(1)}$, $e^{(0)}$ known quantities.

Equation (8) describes the wave propagation in an inhomogeneous and anisotropic string. The equations (8) and (9) were studied by Synge in 1981 [1]. He highlighted the boundless nature of the solutions to this problem in the general case. He said that the problem depends on the properties of A, and for certain values of this function, the solutions become unstable. From the physical point of view, the unbounded nature of the solutions translates into the instability and tendency towards chaos.

In this paper we consider the problem of determining the nature of the bounded generalized soliton solutions for (8) and (9).

2. The existence theorem

We obtain from (8) and (9)

$$y_{tt} = A^2(e,z)y_{xx},\ (A^2 - 1)e_x - \eta\, y(1 + \delta e^2)^{3/2} = 0. \tag{10}$$

By removing the function A from both equations (10) and introducing the change of variable specific to the progressive waves, $\xi = k\,x + \omega t$, where k is the wave number and ω is the angular frequency, with $\rho = \sqrt{\delta k}\; y$, we have

$$\rho'' + c^2\rho(1+\rho'^2)^{3/2} = 0,\ c \neq 0, \tag{11}$$

where *prime* represents the differentiation with respect to ξ. The initial conditions are $\rho(0) = 0$ and $\rho'(0) = \rho^{(0)}$. This equation is known into the literature as the Bernoulli-Euler equation of the elastic beams, and it was solved by using LEM by Ileana Toma [2]. For this reason no detailed calculation is given.

The existence theorem

Equation (11) admits at least one generalized solution with the following LEM representation

$$\rho(\xi) = a_1 \sin d\xi - b_1\xi\cos d\xi + a_3 \sin 3d\xi - b_3\xi\cos 3d\xi + ... \tag{12}$$

where the points represent the higher order terms, and $\xi = k\,x + \omega t$, $c^2 = \dfrac{\eta\sqrt{\delta k}}{k^2 - \omega^2}$,

$$a_1 = \frac{\rho^{(0)}}{d\sqrt{\delta k}}\left(1 + \frac{9\rho^{(0)2}}{16(1+\rho^{(0)2})}\right),\ b_1 = \frac{9\rho^{(0)3}}{16d\sqrt{\delta k}(1+\rho^{(0)2})},\ a_3 = \frac{cb_1}{3d^3}, b_3 = \frac{cb_1^2}{4d^2},\ d = c(1+\rho^{(0)2})^{3/4}.$$

In addition, the solution (12) is bounded only if $c^2 = \dfrac{\eta\sqrt{\delta k}}{k^2 - \omega^2} \neq 0$.

Numerous studies have been conducted to analyse the equation (11) with respect to the variation of the parameter c. It has been proved that the solution is unstable for $c \to 0$. In addition, the generalized solution (12) can be rewritten under the cnoidal representation [8]

$$\begin{aligned}\rho(\xi) &= \sum_{n=0}^{\infty}\frac{q^{n+1/2}}{1-q^{2n+1}}\left(\mathrm{sn}^{2n+1}(\xi,m) + \mathrm{cn}^{2n+1}(\xi,m)\right)\mathrm{sech}(\xi,m) = \\ &= \frac{2\pi}{K\sqrt{m}}\sum_{n=0}^{\infty}\left(a_{2n+1}\sin(2n+1)\,d\xi + (-1)^{2n+1}b_{2n+1}\cos(2n+1)\,d\xi\right),\end{aligned} \tag{13}$$

where $q = \exp(-\pi cK'/K)$, $K = \int_0^{\pi/2}\dfrac{d\vartheta}{(1-m\sin^2\vartheta)^{1/2}}$ and $K(m) = K'(1-m)$.

The cnoidal representation (13) proves the solitonic character of (12) due to the cnoidal functions. For details on the cnoidal functions see [8]. The instability appears for $c \to 0$ and is linked to the properties of A. By applying again the LEM to $(10)_1$, we obtain for A the expression

$$A(e,z) = \frac{5z(q-1)}{6qe}\,\frac{ae\cosh(c\xi - 2\pi\mathrm{i}) - bz\cos(\delta\xi)\cos(d\xi)}{a\cosh(c\xi) - b\cos(\delta\xi)\cos(2d\xi)}. \tag{14}$$

with $a = \dfrac{A^{(0)}\eta}{d\sqrt{\delta k}}\left(\dfrac{3}{4} + \dfrac{A^{(0)2}}{1+A^{(0)2}}\right)$ and $b = 4 + \dfrac{A^{(0)3}\eta}{2d\sqrt{\delta k}}$.

We recognize in (14) the breather soliton wave [10]. So, the instability character of (12) is given by the behavior of the breather local wave velocity (14). The graphical representation of $A(e)$ with respect to e and c is given in Fig.1. Red indicates the function values at which the instability occurs (amplitude tends to ∞). We see from (14) that $A(e,z)$ is bounded only if $c \neq 0$. The bounded nature of the wave propagation velocity is proved physically.In conclusion, the instability of the wave propagation in the inhomogeneous and anisotropic string occurs due to the nature of the velocity of the propagation, which is, in this case, a

breather local soliton. The generalized solution is bounded for a bounded wave velocity, i.e. $c^2 = \frac{\eta\sqrt{\delta k}}{k^2 - \omega^2} \neq 0$.

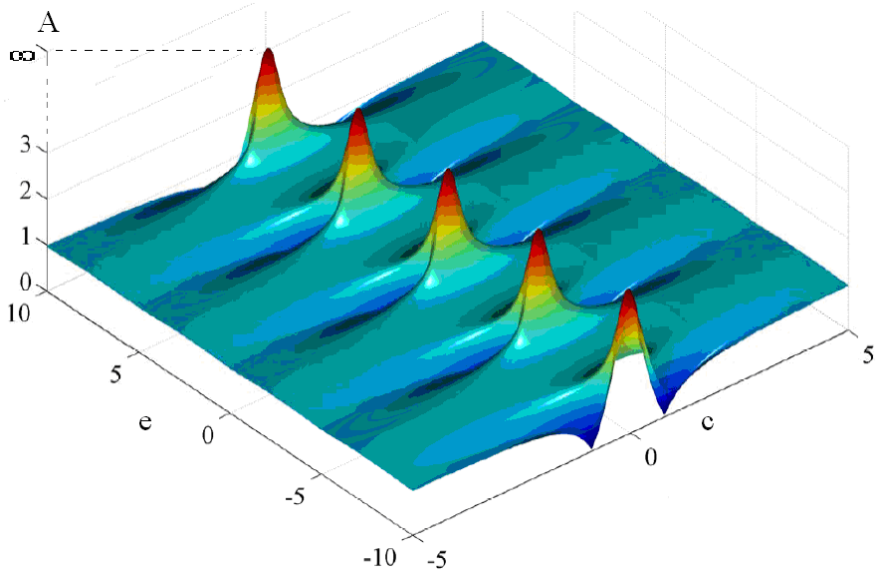

Fig.1. The plot of the function $A(e)$.

That means that the waves propagate in the inhomogeneous and anisotropic string without attenuation and dissipation. In addition, the bounded generalized solution (13) has a solitonic character. Solitons can be regarded as physical entities, quasi-particles that retain their character and interact with the environment and other waves as particles. The solitons (mathematically expressed by hyperbolic functions sech, tanh, or Jacobian elliptic functions cn, sn) are related to the interesting phenomenon in which the waves propagate without attenuation in dissipative media with attenuation.

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EXTREMAL POINTS IN BANACH SPACES

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Abstract: In this article we present some results about extremal points of the closed unit ball of a Banach space. We will use in the paper the notion of **extremal point** in order to highlight the geometry of the closed unit ball.
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Key words: Extremal Points

Notations: Let V be a linear normed space. We use the notation $B_V = \left\{ x \in V \middle| \|x\| \le 1 \right\}$ for the closed unit ball of the space V, and $S_V = \left\{ x \in V \middle| \|x\| = 1 \right\}$ for unit sphere of the space V.

Definition 1. Let A be a convex subset of the linear space V, $A \neq \varnothing$. *A* subset K, $K \subseteq A, K \neq \varnothing$, is called *extremal subset* of A if from $\alpha x_1 + (1-\alpha) x_2 \in K$, where $x_1, x_2 \in A$ and $\alpha \in (0,1)$, it results that $x_1, x_2 \in K$.

If the subset K has only one element, $K = \{x_0\}$, then x_0 is called *extremal point* of the set A.

Remark 1: From Definition 1 it can be concluded that x_0 is an extremal point of the set A if one of the following statements is true:

a) Let $x_0 = \alpha x_1 + (1-\alpha) x_2$, where $x_1, x_2 \in A$ and $\alpha \in (0,1)$, then $x_1 = x_2 = x_0$; or

b)Let $x_0 = \alpha_1 x_1 + \alpha_2 x_2$, where $x_1, x_2 \in A$ and $\alpha_1, \alpha_2 \in (0,1), \alpha_1 + \alpha_2 = 1$, then $x_1 = x_2 = x_0$.

Notation: We use the notation $Ext(A)$ for the set of the extremal points of the set A.

Proposition 1. If $(V, \| \ \|)$ is a Banach space, then any extremal point of the closed unit ball B_V has norm equal to1.

Proof. Let us suppose that $x \in B_V$, $x \neq 0$ is an extremal point and $\|x\| < 1$.

From the equality $x = \alpha x_1 + (1-\alpha) x_2$, in which we consider $\alpha = \|x\|$, $x_1 = \dfrac{x}{\|x\|}$, $x_2 = 0$, it results that $x = x_1 = x_2$ if and only if $x = 0$.

This contradicts our initial supposition $x \neq 0$.In conclusion x is not an extremal point for B_V.

Remark 2: From Proposition 1 it can be concluded that $Ext(B_V) \subseteq S_V$.

Example 1: Let $V = \Re^3$ and $\|x\|_2 = \sqrt{x_1^2 + x_2^2 + x_3^2}$, $(\forall) x = (x_1, x_2, x_3) \in V$.
We have

$$B_V = \left\{ x = (x_1, x_2, x_3) \in V \middle| x_1^2 + x_2^2 + x_3^2 \leq 1 \right\}$$

and

$$S_V = \left\{ x = (x_1, x_2, x_3) \in V \middle| x_1^2 + x_2^2 + x_3^2 = 1 \right\}$$

In this case $Ext(B_V) = S_V$.

Example 2: Let $V = \Re^3$ and $\|x\|_1 = \|(x_1, x_2, x_3)\|_1 = |x_1| + |x_2| + |x_3|$,
$(\forall) x = (x_1, x_2, x_3) \in V$.
In this case we have

$$B_V = \left\{ x = (x_1, x_2, x_3) \in V \middle| |x_1| + |x_2| + |x_3| \leq 1 \right\}$$

$$S_V = \left\{ x = (x_1, x_2, x_3) \in V \middle| |x_1| + |x_2| + |x_3| = 1 \right\}$$

and

$$Ext(B_V) = \{(1,0,0);(0,1,0);(0,0,1);(-1,0,0);(0,-1,0);(0,0,-1)\}$$

Therefore $Ext(B_V) \subset S_V$.

Example 3: Let $V = \Re^3$ and $\|x\|_\infty = \|(x_1, x_2, x_3)\|_\infty = \max\{|x_1|, |x_2|, |x_3|\}$,
$(\forall) x = (x_1, x_2, x_3) \in V$.
In this case

$$Ext(B_V) =$$

$$= \{(1,1,1);(-1,1,1);(-1,-1,1);(1,-1,1);(1,1,-1);(-1,1,-1);(-1,-1.-1);(1,-1,-1)\}$$

Therefore it is clear that $Ext(B_V) \subset S_V$.

Proposition 2. Let V be a linear space and A a non-empty convex subset of V. If $x_0 \in A$, then the following statements are equivalent:

a) x_0 is an extremal point of the set A;

b) If $x_1, x_2 \in A, x_1 \neq x_2$ and $\alpha \in [0,1]$ so that $x_0 = \alpha x_1 + (1-\alpha) x_2$ then $\alpha = 0$ or $\alpha = 1$;

c) The set $A \setminus \{x_0\}$ is convex;

d) If $x_1, x_2 \in A$ and $x_0 = \dfrac{x_1 + x_2}{2}$, then $x_1 = x_2 = x_0$.

Proof. $a) \Rightarrow b)$. We will use the "reductio ad absurdum"method (proof by contradiction). Let us suppose that $\alpha \neq 0$ and $\alpha \neq 1$.

From $x_0 = \alpha x_1 + (1-\alpha)x_2$ and the fact that x_0 is an extreme point of the set A (see a)), it results that $x_1 = x_2 = x_0$ (see Remark 1, a)), which contradicts $x_1 \neq x_2$.
This means that our supposition is false. Therefore we have $\alpha = 0$ or $\alpha = 1$.

$b) \Rightarrow a)$. Let $x_1, x_2 \in A$, $\alpha \in (0,1)$ and $x_0 = \alpha x_1 + (1-\alpha)x_2$.
If we suppose that $x_1 \neq x_2$ from b) it results that $\alpha = 0$ or $\alpha = 1$, which is in contradiction with $\alpha \in (0,1)$. Therefore we have $x_1 = x_2$.
From $x_0 = \alpha x_1 + (1-\alpha)x_2$ it results that $x_0 = x_1 = x_2$.

$a) \Rightarrow c)$. Let $x_1, x_2 \in A \setminus \{x_0\}$ and $\alpha \in [0,1]$. It is clear that $x_1 \neq x_0$ and $x_2 \neq x_0$.
It has to be proved that $\alpha x_1 + (1-\alpha)x_2 \in A \setminus \{x_0\}$, which is clear for $\alpha = 0$ and $\alpha = 1$.
If the set $A \setminus \{x_0\}$ would not be convex, it would exist $\alpha \in (0,1)$ so that $\alpha x_1 + (1-\alpha)x_2 \notin A \setminus \{x_0\}$. But A is convex, so this means that $\alpha x_1 + (1-\alpha)x_2 \in A$.
Therefore we have $\alpha x_1 + (1-\alpha)x_2 = x_0$. From a) it results that $x_1 = x_2 = x_0$, which contradicts $x_1 \neq x_0$ and $x_2 \neq x_0$. This means that $A \setminus \{x_0\}$ is convex.

$c) \Rightarrow d)$. Let $x_1, x_2 \in A$ and $x_0 = \dfrac{x_1 + x_2}{2}$.
It can be noticed that if $x_1 = x_0$, then also $x_2 = x_0$.
The other way a round, if $x_2 = x_0$, then also $x_1 = x_0$.
If we suppose that $x_1 \neq x_0$ and $x_2 \neq x_0$, then $x_1, x_2 \in A \setminus \{x_0\}$. As $A \setminus \{x_0\}$ is a convex set, it results that $x_0 = \dfrac{x_1 + x_2}{2} \in A \setminus \{x_0\}$. But $x_0 = \dfrac{x_1 + x_2}{2}$. This means that $x_0 \in A \setminus \{x_0\}$, which is a contradictory result. This way we proved that $x_1 = x_2 = x_0$.

$d) \Rightarrow a)$. Let $x_1, x_2 \in A$ and $\alpha \in (0,1)$ so that $x_0 = \alpha x_1 + (1-\alpha)x_2$.
Considering $\alpha = \dfrac{1}{2}$, we obtain $x_0 = \dfrac{x_1 + x_2}{2}$ and from d) we have $x_1 = x_2 = x_0$.
If $\alpha \in \left(0, \dfrac{1}{2}\right)$, considering $x_3 = 2x_0 - x_2$ we obtain $x_3 = \lambda x_1 + (1-\lambda)x_2$, where $\lambda = 2\alpha \in (0,1)$; so we have $x_3 \in A$.
From $x_3 = 2x_0 - x_2$ we have $x_0 = \dfrac{x_2 + x_3}{2}$ and using the hypothesis from d) we obtain $x_2 = x_3 = x_0$.
From $x_0 = \alpha x_1 + (1-\alpha)x_2$ it results that $\alpha x_1 = \alpha x_2$. This is possible if and only if $x_1 = x_2$.

This way we proved that if $\alpha \in \left(0, \frac{1}{2}\right)$ then $x_1 = x_2 = x_0$

For $\alpha \in \left(0, \frac{1}{2}\right)$ the proof is similar. We consider $\lambda = 2\alpha - 1 \in (0,1)$, $x_3 = 2x_0 - x_1$ and we obtain $x_3 = \lambda x_1 + (1-\lambda) x_2$ so we have $x_3 \in A$. As in the case $\alpha \in \left(0, \frac{1}{2}\right)$ we can proved that $x_1 = x_2 = x_0$.

In conclusion, x_0 is an extremal point for the set A.

An important and well-known result about the set of extremal points of a set is the *Krein-Milman Theorem*:

Any non-empty, compact and convex subset K of a locally convex Hausdorff space V is the closed convex hull of its extremal points.

Therefore we have the equality: $K = \overline{\boldsymbol{co}(ExtK)}$.

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STRONGLY EXPOSED POINTS IN NORMED SPACES

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Abstract: In this article we present some results about strongly exposed points a non-empty, convex and bounded subset of a Banach space. Since a strongly exposed point is a special type of extrem point, it can be concluded that the Radon-Nikodym property is stronger than the Krein-Milman property.
Mathematical Subject Classification (2010): 46A55
Key words: Strongly Exposed Points, Extremal Points

Definition 1. Let V be a normed space and A a non-empty, convex and bounded subset of V. A point $x \in A, x \neq 0$ is called a **strongly exposed point in** A, if there exists $f \in V'$ (V' is the dual of V) so that:

(i) $f(x) = \sup f(A)$, and

(ii) for any sequence $(x_n)_{n \in N}$ from A with $\lim_{n \to \infty} f(x_n) = f(x)$, we have $\lim_{n \to \infty} \|x_n - x\| = 0$.

We could also say that x **is strongly exposed in** A **by** f or f **strongly exposes** x **in** A.
An element $x \in V, x \neq 0$ is called a **strongly exposed point in** V (or even simpler a **strongly exposed point**), if it is a strongly exposed point in $B(0, \|x\|) := \{ y \in V \mid \|y\| \leq \|x\| \}$.

Remark 1: In the definition of the strongly exposed points from above (Definition 1), the restriction (i) ca be replaced by
(i') $f(x) = \|f\| \cdot \|x\| \neq 0$ or $f(x) = 1$.

Proposition 1. If $x \in V, x \neq 0$, is strongly exposed through $f \in V'$ and a, b are real numbers so that $a \cdot b > 0$, then $a \cdot x$ is strongly exposed by $b \cdot f$ or $b \cdot x$ is strongly exposed by $a \cdot f$.

Proposition 2. If $f \in V'$ strongly exposes $x_1 \in V$ and $x_2 \in V$ and $\|x_1\| = \|x_2\| \neq 0$, then $x_1 = x_2$.

Proof. We have $f(x_1) = f(x_2) = \sup f(A)$.

Let $(y_n)_{n\in N}$ be a sequence of $B\left(0,\|x\|\right)$, so that $\lim_{n\to\infty} f(x_n) = f(x_1) = f(x_2)$. Then we have $\lim_{n\to\infty}\|y_n - x_1\| = \lim_{n\to\infty}\|y_n - x_2\| = 0$. This means that $x_1 = x_2$.

Definition 2. An element $x\in A, x\neq 0$ is called a **tooth element** of the subset $A\subset V, A\neq\varnothing$, if for every $\varepsilon>0$ there is $x\notin\overline{CO}\left(A\setminus B(x,\varepsilon)\right)$, where $B(x,\varepsilon)$ is the closed unit ball of center x and radius ε. The subset A is called tooth-type set if it contains at least one tooth.

Definition 3. If we have $A\subset V, A\neq\varnothing$, $f\in V'$ and $\alpha\in N$ so that $0\le\alpha\le\sup\{f(x): x\in A\}$, then the set $S(f,\alpha,A) := \{x\in A: f(x)\ge\alpha\}$ is called **slice** of set A.

Proposition 3. An element $x\in A, x\neq 0$ is a strongly exposed point of the set A if there exists a functional $f\in V'$ so that:

(i'') $f(x) > f(a)$ for any $a\in A\setminus\{x\}$, and

(ii'') $\lim_{n\to\infty}\delta\left(S\left(f, f(x)-\frac{1}{n}, A\right)\right) = 0$, where $\delta(M)$ is the diameter of set M .

Proposition 4. If x is a strongly exposed point through f, then the restriction (ii) from the definition of a strongly exposed point is equivalent with any of the following assertions:

(ii')for any sequence $(x_n)_{n\in N}$ from $S\left(0,\|x\|\right) := \{y\in V\,|\,\|y\| = \|x\|\}$ with $\lim_{n\to\infty} f(x_n) = f(x)$ we have $\lim_{n\to\infty}\|x_n - x\| = 0$;

(ii")for any sequence $(x_n)_{n\in N}$ from $B\left(0,\|x\|\right) := \{y\in V\,|\,\|y\| \le \|x\|\}$ with $\lim_{n\to\infty} f(x_n) = f(x)$ there exists a subsequence $(x_{n_k})_{k\in N}$ of $(x_n)_{n\in N}$ so that $\lim_{k\to\infty}\|x_{n_k} - x\| = 0$;

(ii''') for any sequence $(x_n)_{n\in N}$ from $S\left(0,\|x\|\right)$ with $\lim_{n\to\infty} f(x_n) = f(x)$, there exists a subsequence $(x_{n_k})_{k\in N}$ of $(x_n)_{n\in N}$ so that $\lim_{k\to\infty}\|x_{n_k} - x\| = 0$.

Proposition 5. Let V be a Banach space. The following statements are equivalent:

(a) Any bounded subset $A\in V, A\neq\varnothing$ is tooth-type set:

(b) Any bounded closed and convex subset $A\in V, A\neq\varnothing$, has at least one strongly exposed point.

(c) Any bounded closed and convex subset $A \in V, A \neq \varnothing$ is the closure in norm of the convex hull of the set made of its strongly exposed points.

(d) The space V has the Radon-Nikodym property.

Remark 2: Since a strongly exposed point is a special type of extremal point, it can be concluded that the Radon-Nikodym property is stronger than the Krein-Milman property.

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SOME RESULTS CONNECTED TO FRACTIONAL LAGUERRE FUNCTIONS

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Abstract: The present paper presents a new approach connected to the fractional Laguerre functions $L_{\beta}^{(\alpha)}(x)$ introduced in [El-Sayed, 1999], [Ishteva et al., 2005]. The properties of these functions are derived by using the Riemann Liouville's as well as the Caputo's fractional derivatives and the generalized Mittag-Leffler functions $E_{\beta,\gamma}^{\delta}(x)$ [Haubold et al., 2011], [Miller and Ross, 1993]. A moment type generating function is obtained for the fractional Laguerre functions by using techniques developed in [Dattoli and Migliorati, 2008]. The results are applied in a forthcoming paper to densities of mixtures of normal distributions [Grosu, 2013].
Mathematics Subject Classification (2010): Primary: 33C45, 42C05, 26A33, Secondary: 60E07
Key words: Mittag-Leffler functions, Laguerre polynomials, fractional derivatives, fractional integrals, chi-squared distribution

1. Introduction

In the theory of statistical distributions, particularly in connection with samples from a multivariate normal distribution [Muirhead, 2005], the noncentral chi-squared distribution plays a determinant role. In order to obtain relevant information concerning the moments of a perturbed chi-square distribution relatively to the original one, two functions are customarily used in probability theory: the moment generating function and the characteristic function.

Let us recall that, for an absolutely continuous random variable $X:\Omega\to\mathbf{R}$, with probability density function $f:\mathbf{R}\to\mathbf{R}$, the fractional moment, if it exists [Harvey, 1965]) is defined as $m_{\alpha}=M(X^{\alpha})=\int_{R}x^{\alpha}f(x)dx$

In particular, the fractional moments m_{α} of the noncentral chi-squared distribution $\chi_{\upsilon}^{2}(\delta)$

$$m_{\alpha}=M(X^{\alpha})=\int_{R}x^{\alpha}f(x)dx=2^{\alpha}\sum_{k=0}^{\infty}\frac{\delta^{k}}{2^{k}k!}e^{-\frac{\delta}{2}}\frac{\Gamma(\frac{\upsilon}{2}+k+\alpha)}{\Gamma(\frac{\upsilon}{2}+k)} \quad (1.1)$$

are given by [Harvey, 1965], where Γ is Euler's gamma function.
For non fractional moments, if they exist, one can define the moment generating function

$$G_{X}(t,x)=\sum_{k=0}^{\infty}t^{k}M(X^{k}) \quad (1.2)$$

One of the purposes of this paper is to extend this notion for the case of fractional moments.

The main problem in obtaining a generating function for fractional Laguerre functions is the form of the Taylor formula for fractional derivatives. While there are several forms of such a Taylor formula [Jumarie, 2006], [Munkhammar, 2004], [Osler, 1971], we do not use any of them, since the particular case of the generating function for associated Laguerre polynomials (i.e. $\beta=n-1\in\mathbf{N}$) cannot be obtained from them. In order to obtain such a formula, we shall use a technique similar to the one developed in [Dattoli and Migliorati,

2008] for the associated Laguerre polynomials. Thus, our present result is a generalization of the result obtained there.

2. Special functions and fractional Laguerre polynomials

Let us introduce various classes of functions.

A function $f:[0,\infty)\to\mathbf{R}$ is of class $\mathbf{C}$ if it is piecewise continuous on $(0,\infty)$ and integrable on any finite subinterval $[a,b]\subset[0,\infty)$ [Miller and Ross, 1993].

The Riemann-Liouville fractional integral [Miller and Ross, 1993], [Samko et al.,1993] of order υ, with $\upsilon>0$, of a function f of class $\mathbf{C}$ is defined, for every $0<x<\infty$ as

$$D_x^{-\upsilon}f(x)=\frac{1}{\Gamma(\upsilon)}\int_0^x(x-t)^{\upsilon-1}f(t)dt$$

The Riemann-Liouville fractional derivative [Miller and Ross, 1993], [Samko et al.,1993] of order υ, with $\upsilon>0$, of a function f of class $\mathbf{C}$, is defined, for every $0<x<\infty$ as $D_x^{\upsilon}f(x)=\frac{d^n}{dx^n}(D_x^{-(n-\upsilon)}(f(x))=\frac{1}{\Gamma(n-\upsilon)}\frac{d^n}{dx^n}(\int_0^x\frac{f(t)}{(x-t)^{\upsilon-n+1}}dt)$, where $n-1=\lfloor\upsilon\rfloor$ (or, equivalently, $n-1<\upsilon<n$) is the greatest integer less than υ.

A function $f:[0,\infty)\to\mathbf{R}$ is of class C^n if all the derivatives $f^{(k)}(x)=\frac{d^k}{dx^k}f(x)$ of order $0\le k\le n$ on $(0,\infty)$ exist and are continuous functions (on 0_+ the condition is for right derivatives).

The Caputo fractional derivative [Ishteva et al., 2005] of order υ, with $\upsilon>0$, $n-1<\upsilon<n$, of a function f of class C^n, is defined, for every $0<x<\infty$ as

$$D_{C,x}^{\upsilon}f(x)=\frac{1}{\Gamma(n-\upsilon)}\int_0^x\frac{f^{(n)}(t)}{(x-t)^{\upsilon+1-n}}dt$$

Proposition 2.1. [Ishteva et al., 2005] Let $f:[0,\infty)\to\mathbf{R}$ and $g:[0,\infty)\to\mathbf{R}$ be function of class C^∞. Then, for any $x\in(0,\infty)$ the Leibniz's formula for the Caputo's derivative is valid

$$D_{C,x}^{\upsilon}[f(x)g(x)]=\sum_{k=0}^{\infty}\frac{\Gamma(\upsilon+1)}{\Gamma(\upsilon-k+1)\Gamma(k+1)}[\frac{d^k}{dx^k}g(x)][D_x^{\upsilon-k}f(x)]-\sum_{k=0}^{n-1}\frac{x^{k-\upsilon}}{\Gamma(k+1-\upsilon)}[\frac{d^k}{dx^k}(f(x)g(x)]|_{x=0}$$

Let us consider now Kummer's confluent hypergeometric function [Miller and Ross, 1993]

$${}_1F_1(a,b;x)=\sum_{k=0}^{\infty}\frac{(a)_k}{(b)_k}\frac{x^k}{k!}\qquad(2.1)$$

where $(a)_k=a(a+1)..(a+k-1)$ stands for the Pochhammer symbol and the series is convergent for all x if $b\notin\{0,-1,-2,..,-n,..\}$.

Next we consider the generalized Mittag-Leffler functions [Miller and Ross, 1993], [Haubold et al., 2011]. It is defined, for $\alpha>0,\beta>0$ and $\delta\in\mathbf{R}$ as

$$E_{\alpha,\beta}^{\delta}(x)=\sum_{k=0}^{\infty}\frac{(\delta)_k}{\Gamma(\alpha k+\beta)}\frac{x^k}{k!}\quad(2.2)$$

Both in [El-Sayed, 1999] and [Ishteva et al., 2005] the generalized Laguerre functions of real order $\beta\in\mathbf{R}_+$ were introduced by means of a Rodrigues' type formula:

$$L_\beta^{(\alpha)}(x)=\frac{x^{-\alpha}e^x}{\Gamma(\beta+1)}D_{C.x}^{\beta}(e^{-x}x^{\alpha+\beta})$$

Instead, we choose in our present paper to define the fractional Laguerre functions of positive order $\beta>0$ and $\alpha>-1$ by using the generalized Mittag-Leffler functions

$$L_\beta^{(\alpha)}(x)=\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)}E_{1,\alpha+1}^{-\beta}(x) \quad (2.3)$$

Proposition 2.3. The fractional Laguerre functions introduced in (2.3) are equivalent with the functions introduced in [El-Sayed, 1999], [Ishteva et al., 2005] for positive order β and $\alpha>-1$.

Proof. We shall show that the functions from (2.3) verify the Rodrigues' type formula. But this follows from [Ishteva et al., 2005] where it was proved that

$$L_\beta^{(\alpha)}(x)=\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+1)\Gamma(\beta+1)}{}_1F_1(-\beta,\alpha+1;x)$$

and the connection between the confluent hypergeometric function and the generalized Mittag-Leffler functions.

Let $\beta>0$, $\alpha>-1$. We introduce now the function $G(x,t;\alpha,\beta)$, called the moment generating function for generalized Laguerre functions of positive order:

for $0<\beta<1$

$$G(x,t;\alpha,\beta)=\sum_{k=0}^{\infty}t^{k+\beta}\frac{\Gamma(k+\beta+1)}{\Gamma(k+\alpha+\beta+1)}L_{k+\beta}^{(\alpha)}(x) \quad (3.6)$$

and for $n-1<\beta<n$

$$G(x,t;\alpha,\beta)=\sum_{k=0}^{n-1}t^k\frac{\Gamma(k+1)}{\Gamma(k+\alpha+1)}L_k^{(\alpha)}(x)+\sum_{k=n}^{\infty}t^{k+\beta-n}\frac{\Gamma(k+\beta-n+1)}{\Gamma(k+\alpha+\beta-n+1)}L_{k+\beta-n}^{(\alpha)}(x) \quad (3.7)$$

In order to prove our main result, we shall need the following lemma.

Lemma 2.4. Let $\alpha>-1$ and $k\in\mathbf{N}$. Let $\widehat{D}_{x,\alpha}^{-k}f(x)=\frac{k}{\Gamma(k+\alpha+1)}\int_0^x(x-t)^{k-1}f(t)dt$ be the operator introduced in [Dattoli and Migliorati, 2008]. Then

i) for any $k\in\mathbf{N}$, $E_{1,\alpha+1}^{-k}(x)=[1-\widehat{D}_{x,\alpha}^{-1}]^k(1)$

ii) for any $\beta>0$, $E_{1,\alpha+1}^{-\beta}(x)=[1-\widehat{D}_{x,\alpha}^{-1}]^\beta(1)$

iii) for any $t\in\mathbf{R}$, $[1-tE_{1,\alpha+1}^{-1}]^{-1}=(1-t)^{-1}E_{1,\alpha+1}^1(-\frac{xt}{1-t})$

Proof. For the first affirmation it suffices to notice that

$$[1-\widehat{D}_{x,\alpha}^{-1}]^k(1)=\sum_{j=0}^{k}(-1)^j\binom{k}{j}(\widehat{D}_{x,\alpha}^{-1})^k(1)=\sum_{j=0}^{k}(-1)^j\frac{k!}{j!(k-j)!}\widehat{D}_{x,\alpha}^{-k}(1)=\sum_{j=0}^{k}\frac{(-k)_j}{j!}\frac{x^k}{\Gamma(k+\alpha+1)}=E_{1,\alpha+1}^{-k}(x)$$

Replacing the binomial formula by the binomial series gives the second statement.

For the third statement, we use the result from [Dattoli and Migliorati, 2008] along with a property of $E_{\beta,\gamma}^{\delta}$, hence $[1-tE_{1,\alpha+1}^{-1}]^{-1}=[1-t(1-\widehat{D}_{x,\alpha}^{-1})]^{-1}=(1-t)^{-1}E_{1,\alpha+1}^1(-\frac{xt}{1-t})$

Theorem 2.5. Let $\beta>0$, $\alpha>-1$. The following relations are valid

i) if $0<\beta<1$, then $G(x,t;\alpha,\beta)=(1-t)^{-1}t^\beta E_{1,\alpha+1}^{-\beta}(x)E_{1,\alpha+1}^1(-\frac{xt}{1-t})$

ii) if $n-1<\beta<n$, then $G(x,t;\alpha,\beta)=(1-t)^{-1}[1-t^nE_{1,\alpha+1}^{-n}(x)+t^\beta E_{1,\alpha+1}^{-\beta}(x)]E_{1,\alpha+1}^1(-\frac{xt}{1-t})$

iii) if $\beta \to n$, then $G(t,x;\alpha,\beta) \to G(t,x,\alpha) = (1-t)^{-1} E^{1}_{1,\alpha+1}(-\frac{xt}{1-t})$

Proof. If $0 < \beta < 1$, then

$$G(x,t;\alpha,\beta) = \sum_{k=0}^{\infty} t^{k+\beta} \frac{\Gamma(k+\beta+1)}{\Gamma(k+\alpha+\beta+1)} L^{(\alpha)}_{k+\beta}(x) = \sum_{k=0}^{\infty} t^{k+\beta} \frac{\Gamma(k+\beta+1)}{\Gamma(k+\alpha+\beta+1)} \frac{\Gamma(\alpha+k+\beta+1)}{\Gamma(k+\beta+1)} E^{-(k+\beta)}_{1,\alpha+1} =$$

$$= t^{\beta} E^{-\beta}_{1,\alpha+1} \sum_{k=0}^{\infty} t^{k} E^{-k}_{1,\alpha+1} = t^{\beta} E^{-\beta}_{1,\alpha+1} [1 - t E^{-1}_{1,\alpha+1}]^{-1} = t^{\beta} E^{-\beta}_{1,\alpha+1} (1-t)^{-1} E^{1}_{1,\alpha+1}(-\frac{xt}{1-t})$$

If $n-1 < \beta < n$, then

$$G(x,t;\alpha,\beta) = \sum_{k=0}^{n-1} t^{k} \frac{\Gamma(k+1)}{\Gamma(k+\alpha+1)} L^{(\alpha)}_{k}(x) + \sum_{k=n}^{\infty} t^{k+\beta-n} \frac{\Gamma(k+\beta-n+1)}{\Gamma(k+\alpha+\beta-n+1)} L^{(\alpha)}_{k+\beta-n}(x) =$$

$$= \sum_{k=0}^{n-1} t^{k} E^{-k}_{1,\alpha+1}(x) + \sum_{k=n}^{\infty} t^{k+\beta-n} E^{-(k+\beta-n)}_{1,\alpha+1}(x) = \sum_{k=0}^{n-1} t^{k} E^{-k}_{1,\alpha+1}(x) + \sum_{k=0}^{\infty} t^{k+\beta} E^{-(k+\beta)}_{1,\alpha+1}(x) =$$

$$= (1-t)^{-1} [1 - t^{n} E^{-n}_{1,\alpha+1}(x) + t^{\beta} E^{-\beta}_{1,\alpha+1}(x)] E^{1}_{1,\alpha+1}(-\frac{xt}{1-t})$$

The last statement is an easy consequence of the previous one.

Remark 2.6.

1. Statement iii) of the theorem gives the result from [Dattoli and Migliorati, 2008] as a particular case of our result.
2. The function $G(x,t;\alpha,\beta)$ becomes the generating function for the fractional moment of order β of the noncentral chi-square distribution $\chi^2_{\nu}(\delta)$ (see also [Grosu, 2013]).

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COMPUTING COFACTORS IN N-DIAGONALIZABLE MATRICES

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Abstract: In this paper we study n-matrices A, having constant line and column sums equal to ***k,*** diagonalizable with nonzero eigenvalues, generalizing thus the already known results for **k**=0. We denote by A_{ij} the cofactor of A defined as A_{ij} =(-1) $^{ij}\cdot$det(M_{ij}) where M_{ij} is the (n-1)-matrix obtained from A by removing the i-row and j-column. The main result of this paper is that $A_{ij}\cdot[1-(y_2+y_3+...y_n)]=\frac{\lambda_2\cdot\lambda_3\cdot...\lambda_n}{n}$ where spectrum of A is {k, λ_2, λ_3,...λ_n} and $\{y_2, y_3,...y_n\}$ represents the solution of the (n-1) system of linear equations given by $M_{ij}\cdot\begin{pmatrix} y_2 \\ y_3 \\ \vdots \\ y_n \end{pmatrix}=\frac{k}{n}\cdot\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$. It can also be proved that $(y_2+y_3+...y_n)=1$ if and only if A has 0 as eigenvalue.

Mathematics Subject Classification (2010):
Key words: Cofactor, d-regular graph, generalised stochastic matrices

1.Introduction

It is known that a n-matrix A, with real entries, having constant line and column sums equal to ***0,*** diagonalizable, with {k, λ_2, λ_3,...λ_n} its eigenvalues, has all its cofactors equal. If we denote by A_{ij} the cofactor of A defined as A_{ij} =(-1) $^{ij}\cdot$det(M_{ij}) where M_{ij} is the (n-1)-matrix obtained from A by removing the i-row and j-column, then any A_{ij} is given by the formula $A_{ij}=\frac{\lambda_2\cdot\lambda_3\cdot...\lambda_n}{n}$.

To prove this, consider the matrix $J=\begin{pmatrix} 1 & 1 & ... & 1 & 1 \\ 1 & 1 & ... & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & ... & 1 & 1 \\ 1 & 1 & ... & 1 & 1 \end{pmatrix}$ with its spectrum {n,0,..0,0}and eigenvectors $B=\{(1,1,...1,1), w_2, w_3,...w_n\}$ where $\{w_2, w_3,...w_n\}$ are (n-1) arbitrary vectors, linear independent, included in the subspace L of $\Re^n$ characterized by $L=\{x\in\Re^n / x=(x_1, x_2,...x_n), x_1+x_2+...+x_n=0\}$, dimension of L is (n-1).

If A is a n-matrix with real entries, having line and column sums equal to ***0,*** diagonalizable, then its eigenvectors are given by $B_1=\{(1,1,...1,1), v_2, v_3,...v_n\}$ where $\{v_2, v_3,...v_n\}$ are (n-1) vectors, well defined, linear independent, included in the subspace L. The condition $x_1+x_2+...+x_n=0$ is a necessary condition, but not sufficient. So $\{v_2, v_3,...v_n\}$ is different

from $\{w_2, w_3,...w_n\}$ but both systems are included in L and hence J is also diagonalizable B_1. It results that A and J are simultaneously diagonalizable and det(A+ J) =det(A_D + J_D)= $n\cdot\lambda_2\cdot\lambda_3\cdot...\cdot\lambda_n$. Using another way of calculating det(A+ J), expressed by its cofactors, it results:

$$\det(A+J)=\begin{vmatrix} a_{11}+1 & a_{12}+1 & ... & a_{1n-1}+1 & a_{1n}+1 \\ a_{21}+1 & a_{22}+1 & ... & a_{2n-1}+1 & a_{2n}+1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n-11}+1 & a_{n-12}+1 & ... & a_{n-1n-1}+1 & a_{n-1n}+1 \\ a_{n1}+1 & a_{n2}+2 & ... & a_{nn-1}+1 & a_{nn}+1 \end{vmatrix} = n^2 \begin{vmatrix} 1 & 1 & ... & 1 & 1 \\ 0 & a_{22} & ... & a_{2n-1} & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & a_{n-12} & ... & a_{n-1n-1} & a_{n-1n} \\ 0 & a_{n2} & ... & a_{nn-1} & a_{nn} \end{vmatrix} = n^2 A_{11}$$

and thus we obtain $A_{11}=\dfrac{\lambda_2\cdot\lambda_3\cdot...\lambda_n}{n}$ and this results holds for any cofactor of A.

Remark1 For the above result to hold true, we must have $\sum_{j=1}^{n} a_{ij}=0$ for any $i\in(1,n)$ and $\sum_{i=1}^{n} a_{ji}=0$ for any $j\in(1,n)$.

Remark2 If we multiply A by J, it results $A\cdot J=\mathbf{0_n}$ loosing thus any information contained in A.

2.Study of matrix U

Let us consider the following matrix

$$U=\begin{pmatrix} \frac{1}{n} & \frac{1}{n}-1 & ... & \frac{1}{n}-1 & \frac{1}{n}-1 \\ \frac{1}{n}-1 & \frac{1}{n} & ... & \frac{1}{n}-1 & \frac{1}{n}-1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{1}{n}-1 & \frac{1}{n}-1 & ... & \frac{1}{n} & \frac{1}{n}-1 \\ \frac{1}{n}-1 & \frac{1}{n}-1 & ... & \frac{1}{n}-1 & \frac{1}{n} \end{pmatrix}$$

with its spectrum {2-n,1,..1,1} and eigenvectors $B=\{(1,1,...1,1), w_2, w_3,...w_n\}$ where $\{w_2, w_3,...w_n\}$ are (n-1) arbitrary vectors, linear independent, included in the subspace $L=\{x\in\Re^n / x=(x_1,x_2,...x_n), x_1+x_2+...+x_n=0\}$. This last affirmation holds true as they are all eigenvectors for $\lambda=1$and any linear combination of vectors from L has the same property.

Let us consider now an arbitrary matrix A, having constant line and column sums equal to $k\neq 0$, diagonalizable with nonzero eigenvalues, $\{k, \lambda_2, \lambda_3,...\lambda_n\}$ and $k\neq\lambda_j$ for any $j\in(2,n)$.

For $\lambda=k$ the proper vector is $w_1=(1,1...1,1)$ and denote by $\{w_2, w_3,...w_n\}$ the other (n-1) vectors for some $\lambda\neq k$. So $\{w_2, w_3,...w_n\}$ are in L and U and A are simultaneously diagonalizable and det(A·U) =det(A_D·U_D)= k·(2-n)$\cdot\lambda_2\cdot\lambda_3\cdot...\cdot\lambda_n$. A direct computation of det(A·U)

$$\det(A\cdot U)=\begin{pmatrix} k(\frac{1}{n}-1)+a_{11} & \dots & k(\frac{1}{n}-1)+a_{1n}\\ k(\frac{1}{n}-1)+a_{21} & \dots & k(\frac{1}{n}-1)+a_{2n}\\ \vdots & \vdots\ \vdots\ \vdots & \vdots\\ k(\frac{1}{n}-1)+a_{n-11} & \dots & k(\frac{1}{n}-1)+a_{n-1n}\\ k(\frac{1}{n}-1)+a_{n1} & \dots & k(\frac{1}{n}-1)+a_{nn}\end{pmatrix}= =k(n-2)n\begin{pmatrix} 1 & 1 & \dots & 1 & 1\\ \frac{k}{n} & a_{22} & \dots & a_{2n-1} & a_{2n}\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ \frac{k}{n} & a_{n-12} & \dots & a_{n-1n-1} & a_{n-1n}\\ \frac{k}{n} & a_{n2} & \dots & a_{nn-1} & a_{nn}\end{pmatrix}$$

expanding det(A·U) by the first line it results the following decomposition

$$k(n-2)n\left(\begin{vmatrix} a_{22} & a_{23} & \dots & \dots & a_{2n}\\ a_{32} & a_{33} & \dots & \dots & a_{3n}\\ \dots & \dots & \dots & \dots & \dots\\ a_{n-12} & a_{n-13} & \dots & \dots & a_{n-1n}\\ a_{n2} & a_{n3} & \dots & \dots & a_{nn}\end{vmatrix}-\begin{vmatrix} k/n & a_{23} & \dots & \dots & a_{2n}\\ k/n & a_{33} & \dots & \dots & a_{3n}\\ \dots & \dots & \dots & \dots & \dots\\ k/n & a_{n-13} & \dots & \dots & a_{n-1n}\\ k/n & a_{n3} & \dots & \dots & a_{nn}\end{vmatrix}+\begin{vmatrix} k/n & a_{22} & a_{24} & \dots & a_{2n}\\ k/n & a_{32} & a_{34} & \dots & a_{3n}\\ \dots & \dots & \dots & \dots & \dots\\ k/n & a_{n-12} & a_{n-14} & \dots & a_{n-1n}\\ k/n & a_{n2} & a_{n4} & \dots & a_{nn}\end{vmatrix}\right.$$

$$\left.+(-1)^{j-1}\begin{vmatrix} k/n & a_{23} & a_{2j-1} & a_{2j+1} & a_{2n}\\ k/n & a_{33} & a_{3j-1} & a_{3j+1} & a_{3n}\\ \dots & \dots & \dots & \dots & \dots\\ k/n & a_{n-13} & a_{n-1j-1} & a_{n-1j+1} & a_{n-1n}\\ k/n & a_{n3} & a_{nj-1} & a_{nj+1} & a_{nn}\end{vmatrix}\dots\right)$$

$$=k(n-2)n\left(A_{11}-(y_2A_{11}+y_3A_{11}+\dots+y_nA_{11})\right)=k(n-2)nA_{11}\left(1-(y_2+y_3+\dots+y_n)\right),$$

where $\{y_2,y_3,\dots y_n\}$ represents the solution of the (n-1) system of linear equations given by

$$\begin{pmatrix} a_{22} & a_{23} & \dots & a_{2n-1} & a_{2n}\\ a_{32} & a_{33} & \dots & a_{3n-1} & a_{3n}\\ \dots & \dots & \dots & \dots & \dots\\ a_{n-12} & a_{n-13} & \dots & a_{n-1n-1} & a_{n-1n}\\ a_{n2} & a_{n3} & \dots & a_{nn-1} & a_{nn}\end{pmatrix}\cdot\begin{pmatrix} y_2\\ y_3\\ \vdots\\ y_n\end{pmatrix}=\frac{k}{n}\cdot\begin{pmatrix} 1\\ 1\\ \vdots\\ 1\end{pmatrix}\text{ or }\ M_{11}\cdot\begin{pmatrix} y_2\\ y_3\\ \vdots\\ y_n\end{pmatrix}=\frac{k}{n}\cdot\begin{pmatrix} 1\\ 1\\ \vdots\\ 1\end{pmatrix}.$$

Thus we can state the following result

2.1Theorem Let A be an arbitrary matrix A, having constant line and column sums equal to $k\neq 0$**,** diagonalizable with nonzero eigenvalues, $\{k,\lambda_2,\lambda_3,\dots\lambda_n\}$, $k\neq\lambda_j$ for any $j\in(2,n)$ and denote by A_{ij} the cofactor of A defined as $A_{ij}=(-1)^{ij}\cdot\det(M_{ij})$ where M_{ij} is the (n-1)-matrix obtained from A by removing the i-row and j-column. Then $A_{ij}\cdot[1-(y_2+y_3+\dots y_n)]=\dfrac{\lambda_2\cdot\lambda_3\cdot\dots\lambda_n}{n}$ where spectrum of A is $\{k,\lambda_2,\lambda_3,\dots\lambda_n\}$ and $\{y_2,y_3,\dots y_n\}$ represents the solution of the (n-1) system of linear equations given by $M_{ij}\cdot\begin{pmatrix} y_2\\ y_3\\ \vdots\\ y_n\end{pmatrix}=\frac{k}{n}\cdot\begin{pmatrix} 1\\ 1\\ \vdots\\ 1\end{pmatrix}$.

Proof For computing A_{ij} it is sufficient to follow the above given calculi for det(A·U), but, instead of adding all the lines to line 1, add them to line i, and, then, after extracting the k(2-n)factor, add all the columns to the j column, instead of column 1.

2.2Remark If k=0 the solution of the sistem $M_{ij}\cdot\begin{pmatrix} y_2\\ y_3\\ \vdots\\ y_n\end{pmatrix}=\frac{0}{n}\cdot\begin{pmatrix} 1\\ 1\\ \vdots\\ 1\end{pmatrix}$ is $\{y_2=0,y_3=0,\dots,y_n=0\}$ the

formula becomes $A_{ij}\cdot[1]=\frac{\lambda_2\cdot\lambda_3\cdot\ldots\lambda_n}{n}$, so all cofactors are equal.

2.3Corrolary Under the same hypothesis, if $M_{ij}\cdot\begin{pmatrix}y_2\\y_3\\\vdots\\y_n\end{pmatrix}=\frac{k}{n}\cdot\begin{pmatrix}1\\1\\\vdots\\1\end{pmatrix}$ then $(y_2+y_3+\ldots y_n)=1$ implies A has 0 as eigenvalue.

3.Application

3.1The matrix associated to a d-regular graph

Let the graph G be a d-regular graph, undirected, connected (the multiplicity of $\lambda=d$ equals the number of connected components of the graph).The hypotheses of **Th2.1** are thus fulfilled and the main result for computing the cofactors holds true.

3.2Generalised stochastic matrices

Let $A\in\mathfrak{M}_n(R)$ be a n-stochastic symmetric matrix [3]. Being stochastic means $\sum_{j=1}^{n}a_{ij}=s$ for any $i\in(1,n)$, namely the sum of the elements on each line(and column) is equal to s. Thus A has s as eigenvalue. Since A is symmetric the hypotheses of **Th2.1** hold true.

3.3 Numerical examples

Let A be a 5x5 matrix given by

$A=\begin{pmatrix}1&3&2&2&-2\\3&-1&3&3&-2\\2&3&1&2&-2\\2&3&2&1&-2\\-2&-2&-2&-2&14\end{pmatrix}$ which has eigen values $\lambda_1=6,\lambda_2=16,\lambda_3=-4,\lambda_4=-1,\lambda_5=-1$

and eigenvectors

$u_1=(1,1,1,1,1)\ u_2=(1,1,1,1,-4)\ u_3=(1,-3,1,1,0)\ u_4=(1,0,0,-1,0)\ u_5=(0,0,1,-1,0)$

Let us compute $A_{32}(1-y_2-y_3-y_4-y_5)=\frac{16\cdot(-4)(-1)(-1)}{5}$ where $\{y_2,y_3,y_4,y_5\}$ is the solution of the

sistem $\begin{pmatrix}y_2+2y_3+2y_4-2y_5=\frac{6}{5}\\3y_2+3y_3+3y_4-2y_5=\frac{6}{5}\\2y_2+2y_3+1y_4-2y_5=\frac{6}{5}\\-2y_2-2y_3-2y_4+14y_5=\frac{6}{5}\end{pmatrix}$ with solution $\{y_2=\frac{24}{95},y_3=\frac{72}{95},y_4=\frac{-48}{95},y_5=\frac{15}{95}\}$ thus

$(1-y_2-y_3-y_4-y_5)=\frac{32}{95}$ and it results $A_{32}=\frac{-64\cdot95}{5\cdot32}=-38$.

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SIMULTANEOUS INTERPOLATION AND APPROXIMATION FOR INFINITELY DIFFERENTIABLE FUNCTIONS

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Abstract: By using a suitable set of interpolation, simultaneous interpolation and approximation for infinitely differentiable functions are given. An application of the method for boundary value problems for ODEs is presented.
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Key words: Newton polynomials, boundary value problems

1. Introduction

Let $\mathcal{C}([0,1])$ be the real Banach space of real continuous functions defined on the interval $[0,1]$ with supremum norm $\|\ \|_\infty$. By Weierstrass Theorem it follows that every $f \in \mathcal{C}([0,1])$ may be approximated uniformly by a polynomial. It is also known that we may interpolate to f a finite set of points in $[0,1]$. Combining these methods it follows a problem of simultaneous interpolation and approximation. Walsh proved the following result (see [2], Theorem 6.5.1, p. 121).

Theorem 1 *Let S be a closed bounded point set in the complex plane and suppose that the function* $f(z)$ *is defined on S and it is uniformly approximable by polynomials on S. Let* $z_1, z_2, \ldots, z_m$ *be m distinct points of S. Then f is uniformly approximable by a polynomial P that satisfies the auxiliary conditions*

$$P(z_i) = f(z_i),\ i = 1,2,\ldots,m.$$

We study a problem of simultaneous interpolation and approximation for real infinitely differentiable functions from $\mathcal{C}([0,1])$ by choosing the degrees of polynomials close to the number of interpolating points.

2. Main result and an application

Let $\{\alpha_i\}_{i\geq 1}$ be a sequence of distinct elements from $[0,1]$. We construct the polynomials

$$u_k(x)=\prod_{j=1}^{k}\left(x-\alpha_j\right), \tag{1}$$

and we consider

$$N_m(x)=\sum_{k=0}^{m}\alpha_k u_k(x), \tag{2}$$

the Newton interpolating polynomial with respect to the elements of $\alpha_1,\alpha_2,\ldots,\alpha_{m+1}$. Then its derivative can be expressed in the form

$$N_m'(x)=\sum_{k=0}^{m-1}a_k^{(1)}u_k(x). \tag{3}$$

In this paper we study infinitely differentiable functions from $\mathcal{C}([0,1])$ which are uniformly approximable by polynomials P_m of degree $m=2^n$. We denote

$$I_m := \left\{\alpha_{j+1}=\frac{j}{2^n},\ j=0,1,\ldots 2^n\right\},$$

and we seek the polynomials P_m such that

$$P_m(x)=f(x), \text{ for every } x\in I_m.$$

The expressions of the coefficients $a_k^{(1)}$ from (3), with respect to a_k, are given in the following result.

Lemma 1 *If $m=2^n$, then it follows that*

$$a_k^{(1)}=\sum_{s=0}^{m}C_{s,k}a_s. \tag{4}$$

and

$$C_{s,k}=\begin{cases}0, & \text{if } s<k+1\\ \dfrac{(-1)^{s-k-1}s!}{k!2^{n(s-k-1)}(s-k)}, & \text{if } s\geq k+1.\end{cases} \tag{5}$$

The main result is given in the following theorem.

Theorem 2 *Assume $f\in\mathcal{C}([0,1])$ is an infinitely differentiable function with the property that there exist a positive constant C and a non-negative integer r such that, for every non-negative integer s,*

$$\left\|f^{(s)}\right\|_\infty\leq C(s-r-1)!, \tag{6}$$

where here $t!=1$ if $t<0$. Then, for every positive real number ε, there exists a positive integer n_1 such that, for every $n\geq n_1$,

$$\left\|f^{(\gamma)}-N_m^{(\gamma)}\right\|_\infty\leq\varepsilon, \tag{7}$$

and

$$f^{(\gamma)}(x)=N_m^{(\gamma)}(x), \tag{8}$$

for every $x\in I_m$, where $\gamma=0,1,\ldots,r$ and $m=2^n$.

Corollary 1 *(see [3], Theorem 3.2) Consider the linear differential equation*

$$y^{(n)}(x) = \sum_{i=1}^{n} b_i(x) y^{(n-i)}(x) + c(x), \tag{9}$$

where $b_i, c \in \mathcal{C}([0,1])$ *are infinitely differentiable functions such that there exists a positive constant* α *and, for every non-negative integer* k *,*

$$\max\left\{\left\|b_i^{(k)}\right\|_\infty, i = 1,2,\ldots n, \left\|c^{(k)}\right\|_\infty\right\} \le \alpha^{k+1}. \tag{10}$$

Then, for every non-negative integer r, every infinitely differentiable solution $y \in \mathcal{C}([0,1])$ *of the equation* (9) *and its derivatives* $y^{(j)}, j = 1,2,\ldots,r,$ *are uniformly approximable by polynomials* $N_m(x)$ *and their derivatives* $N_m^{(j)}(x)$*, with* $m = 2^u$*, u a positive integer, such that, for every* $j = 0,1,\ldots,r,$

$$y^{(j)}(\alpha_k) = N_m^{(j)}(\alpha_k), k = 1,2,\ldots,m+1-j. \tag{11}$$

The method which follows from Corollary 1 is useful to approximate solutions of boundary value problems for linear ODEs.

Example 1 Consider the two-point boundary value problem ([1], p. 140)

$$y''(x) - 2500y(x) = 2500\cos^2 \pi x + 2\pi^2 \cos 2\pi x, \quad x \in [0,1], \quad y(0) = y(1) = 0. \tag{12}$$

The two-point boundary value problem has the exact solution:

$$y(x) = \frac{e^{50(x-1)} + e^{-50x}}{1+e^{-50}} - \cos^2 \pi x.$$

By Corollary 1, we can construct the approximate solution:

$$\tilde{y}(x) = N_m(x) = \sum_{i=0}^{m} a_i u_i(x), \; a_i \in \mathbf{R},$$

where the derivatives $N_m'(x)$ and $N_m''(x)$ approximate the derivatives y' and y'' of y. By Lemma 1 we can represent the coefficients of $N_m'(x)$ and $N_m''(x)$ by means of a_i. Thus by using (12), the coefficients $a_i, i = 0,1,\ldots,m,$ are determined by solving a linear system of $m+1$ equations. If the solution is approximated by the polynomial

$$N_{64}(x) = \sum_{i=0}^{64} a_i u_i(x),$$

then the boundary conditions imply

$$a_0 = 0, \; a_1 = -\sum_{i=2}^{64} a_i u_i(1).$$

Table 1 lists the absolute errors in *y*. In this case $\left\|y - N_{64}(x)\right\|_\infty < 0.63 \times 10^{-14}$. The computations were performed on a computer with a 40-hexadecimal-digit mantissa. Note that the errors are clearly unacceptable in simple shooting method.

Table 1

x	simple shooting ([1], p.141)	Newton polynomial N_{64}
0.1	$.19\cdot10^{-7}$	$.19\cdot10^{-14}$
0.2	$.28\cdot10^{-5}$	$.13\cdot10^{-16}$
0.3	$.41\cdot10^{-3}$	$.87\cdot10^{-19}$
0.4	$.61\cdot10^{-1}$	$.59\cdot10^{-21}$
0.5	$.90\cdot10$	$.90\cdot10^{-23}$
0.6	$.13\cdot10^{4}$	$.19\cdot10^{-20}$
0.7	$.20\cdot10^{6}$	$.29\cdot10^{-18}$
0.8	$.29\cdot10^{8}$	$.42\cdot10^{-16}$
0.9	$.44\cdot10^{10}$	$.63\cdot10^{-14}$
1	$.65\cdot10^{12}$	0

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SCHUR MULTIPLIERS ON A CERTAIN SPACE OF INFINITE MATRICES

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Abstract: In this paper we study some Schur multipliers on the space.
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Key words: Infinite matrices, Toeplitz matrices, Schur multipliers, discrete Sawyer duality principle, quasi-monotone sequences, bounded operators, Fourier coefficient.

1. Introduction

The Schur product appeared in an old paper of Schur in 1911 (see [9]). In the literature sometimes this product is called Hadamard product. Schur multipliers (for infinite matrices) are studied from 70's and one of the first papers on this field is the paper [5] of G. Bennett. Since than, Schur multipliers appeared in different fields of mathematics.

In this paper we study the Schur multipliers on the space $B_w^\alpha(\ell^2)$, for $\alpha \geq 0$. The space $B_w^\alpha(\ell^2)$ is defined below:

$$B_w^\alpha(\ell^2) = \{A \text{ infinite matrix}; Ax \in \ell^2 \text{ for every}$$

$$x = (x_n)_n \in \ell^2 \text{ with } \frac{|x_n|}{n^\alpha} \searrow 0,\ \alpha \geq 0\}.$$

This space of infinite matrices is a natural generalization of the Banach space $B_w(\ell^2)$ introduced in [7] and studied in [8].

In the following we recall some definitions and notations that we use through this paper (see [1], [2], [3], [4], [5]).

The *Schur product* of two matrices is defined by $A * B = (a_{ij} \cdot b_{ij})_{i,j\geq 1}$, where $A = (a_{ij})_{i,j\geq 1}$ and $B = (b_{ij})_{i,j\geq 1}$. We denote by

$$M(\ell^2) = \{M : M * A \in B(\ell^2) \text{ for every } A \in B(\ell^2)\}$$

the *space of all Schur multipliers* equipped with the following norm

$$\|M\| = \sup_{\|A\|_{B(\ell^2)}\leq 1} \|M * A\|_{B(\ell^2)}.$$

More generally if X and Y are two Banach spaces of matrices we call Schur multipliers from X to Y the space $M(X,Y) = \{M : M * A \in Y \text{ for every } A \in X\}$, equipped with the natural norm

$$\| M \| = \sup_{\| A \|_X \le 1} \| M * A \|_Y .$$

If $X = Y$ we make the following notation: $M(X) := M(X, X)$.

For an infinite matrix $A = (a_{ij})$ and an integer k we denote by $A_k = (a'_{ij})$, where

$$a'_{ij} = \begin{cases} a_{ij} & \text{if } j - i = k, \\ 0 & \text{otherwise,} \end{cases}$$

i.e. we have that

$$A_k = \begin{pmatrix} 0 & 0 & \dots & a_{1k} & 0 & \dots & 0 & \dots \\ 0 & 0 & \dots & 0 & a_{2k+1} & \dots & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 0 & \dots & a_{k\,2k-1} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}.$$

Finally, a Toeplitz matrix is a matrix $A = (a_{ij})_{i,j \ge 1}$ such that $a_{ij} = a_{i-j}$ for every $i, j \ge 1$. In the next section we present the results obtained regarding Schur multipliers on $B_w^\alpha(\ell^2)$.

2. Schur multipliers

The first theorem prove that the classical space $B(\ell^2)$ is included in $M(B_w^\alpha(\ell^2))$.

Theorem 2.1 *Let $\alpha \ge 0$. Then all matrices from the space $B(\ell^2)$ are Schur multipliers on $B_w^\alpha(\ell^2)$.*

Proof. Let us take $A \in B(\ell^2)$ and $B \in B_w^\alpha(\ell^2)$. Then we have that

$$\sum_j | \sum_k a_{jk} b_{jk} x_k |^2 \le \sum_j \left(\sum_k | a_{jk} \| b_{jk} \| x_k | \right)^2$$
$$\le \sum_j \left(\sum_k | a_{jk} |^2 \right) \sum_k \left(| b_{jk} |^2 | x_k |^2 \right)$$
$$\le \sup_j \left(\sum_k | a_{jk} |^2 \right) \sum_j \left(\sum_k | b_{jk} |^2 | x_k |^2 \right).$$

Using Rademacher functions $r_k(t) = sgn \sin(2^n \pi t)$ on $[0,1]$, for $k \ge 1$ and the equality (see e.g. [6]) $\sum_k | y_k |^2 = \int_0^1 | \sum_k y_k r_k(t) |^2 \, dt$, it follows that

$$\sum_j \left(\sum_k | b_{jk} |^2 | x_k |^2 \right) \le \| B \|^2_{B_w^\alpha(\ell^2)} \| x \|_2^2 .$$

Thus, we obtain that

$$\| A * B \|_{B_w^\alpha(\ell^2)} \le \| A \|_{B(\ell^2)} \cdot \| B \|_{B_w^\alpha(\ell^2)} .$$

and the proof is complete.

Next theorem gives us the characterization of diagonals of Schur multipliers on $B_w^\alpha(\ell^2)$.

Theorem 2.2 *Let M be an infinite matrix of a diagonal form. Then $M \in M(B_w^\alpha(\ell^2))$, if and only if the sequence from the diagonal is bounded.*

Proof. It is enough to prove only for the main diagonal. Let $m = (m_k)_{k\geq 1}$ a bounded sequence. We remark that if $A = (A_{jk})_{jk}$ is an infinite matrix and A_0 is the matrix defined as $A_0 = diag(a_{kk})$, then

$$\|A_0\|_{B_w^\alpha(\ell^2)} \leq \|A\|_{B_w^\alpha(\ell^2)}.$$

Indeed, in the case of positive matrices is easy to see just computing the norms and in the case of arbitrary matrices using the same techniques with Rademacher functions as in the previous theorem.

Let us take a infinite matrix $A \in B_w^\alpha(\ell^2)$ and M a diagonal matrix having on the main diagonal the sequence $m = (m_k)$. Then

$$\begin{aligned}
&\|(M * A)x\|_2^2 = \|(M * A_0)x\|_2^2 = \|MA_0x\|_2^2 \\
&\leq \|m\|_\infty^2 \cdot \|A_0x\|_2^2 \leq \|m\|_\infty^2 \cdot \|A_0\|_{B_w^\alpha(\ell^2)}^2 \cdot \|x\|_2^2 \\
&\leq \|m\|_\infty^2 \cdot \|A\|_{B_w^\alpha(\ell^2)}^2 \cdot \|x\|_2^2,
\end{aligned}$$

for every $x = (x_k)_{k\geq 1} \in \ell^2$ such that $\frac{|x_k|}{k^\alpha} \searrow 0$.

For the converse result we assume that $M \in M(B_w^\alpha(\ell^2))$. Then $M * A \in B_w^\alpha(\ell^2)$, for every $A \in B_w^\alpha(\ell^2)$. In particular, $M * A_0 \in B_w^\alpha(\ell^2)$, for A_0 given by the sequence $a = (a_k)_{k\geq 1}$. We have that $A_0x \in \ell^2$ for every $x \in \ell^2$ with $\frac{|x_k|}{k^\alpha} \searrow 0$. Since $(M * A_0)x \in \ell^2$ it implies that $m = (m_k)_{k\geq 1}$ is bounded and the proof is complete.

Moreover, in the case of Toeplitz matrices we have the following characterization:

Theorem 2.3 *Let M be a Toeplitz matrix from $M(B_w^\alpha(\ell^2))$. Then there exists a bounded, complex, Borel measure μ on the circle group $\mathbb{T}$ with* $\hat{\mu}(n) = c_n$ for $n = 0, \pm 1, \pm 2, \cdots$.

Proof. Let us take $M = (m_{jk})_{jk} \in M(B_w^\alpha(\ell^2))$ be a Toeplitz matrix of the form

$$m_{jk} = c_{j-k}, j, k = 0,1,2,\cdots \tag{2.1}$$

Using the same arguments as in [5] we obtain the result.

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ON A CLASS OF LINEAR OPERATORS ON A QUASI-MONOTONE SEQUENCES

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Abstract: In this paper we investigate a new class $B_w^{\alpha}(\ell^2)$, $\alpha \geq 0$ of linear operators consisting of infinite matrices A such that $Ax \in \ell^2$ for every sequence $x = (x_n)_n \in \ell^2$ such that $|x_n|/n^{\alpha} \downarrow 0$.

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Key words: Infinite matrices, Banach spaces.

1. Introduction

In the paper [6] the authors introduced and studied the Banach space of infinite matrices $B_w(\ell^2)$, consisting in all infinite matrices $A: \ell^2_{dec} \to \ell^2$, where the spaces ℓ^2 and ℓ^2_{dec} are the classical spaces of sequences with real or complex numbers. This space appeared in the study of matriceal harmonic analysis (see e.g. [1], [2] and [3]). In this paper, our aim is to study the more general space $B_w^{\alpha}(\ell^2)$ with $\alpha \geq 0$, consisting in all infinite matrices A, such that $Ax \in \ell^2$ for every sequence $x = (x_n)_n \in \ell^2$ such that $|x_n|/n^{\alpha} \downarrow 0$.

In this paper we will denote with T the space of all Toeplitz matrices. We recall the definition of Toeplitz matrices. A Toeplitz matrix is a matrix of the form

$$T = \begin{pmatrix} a_0 & a_1 & a_2 & \ddots \\ a_{-1} & a_0 & a_1 & \ddots \\ a_{-2} & a_{-1} & a_0 & \ddots \\ \ddots & \ddots & \ddots & \ddots \end{pmatrix},$$

where $a = (a_n)_n$ is a sequence of real or complex numbers (see e.g. [4], [7]).

In the last section are presented two theorems. First theorem is a characterization of infinite matrices of diagonal form. This theorem is useful to construct different infinite matrices from $B_w^{\alpha}(\ell^2)$ with desired properties. The last theorem shows the coincidence between the classical space $B(\ell^2)$ and the studied space in this paper.

2. Main results

Theorem 2.1 *Let $\alpha \geq 0$ and A be a diagonal matrix having the nonzero elements given by the sequence $a = (a_n)_n$. Then*

$$A \in B_w^{\alpha}(\ell^2) \text{ if and only if } \sup_{n\geq 1}\left(\frac{\sum_{k=1}^{n} |\, a_k \,|^2\, k^{2\alpha}}{\sum_{k=1}^{n} k^{2\alpha}}\right)^{\frac{1}{2}} < \infty.$$

Moreover, the norm equivalent with $\|A\|_{B_w^{\alpha}(\ell^2)} = \sup_{n\geq 1}\left(\frac{\sum_{k=1}^{n} |\, a_k \,|^2\, k^{2\alpha}}{\sum_{k=1}^{n} k^{2\alpha}}\right)^{\frac{1}{2}}$.

Proof. First we remark that is enough to prove the equivalence if A is a infinite diagonal matrix having nonzero elements only on the main diagonal. Since

$$\sup_{\frac{|x_n|_{\downarrow}}{n^{\alpha}}} \frac{\|Ax\|_{\ell^2}}{\|x\|_{\ell^2}} = \sup_{\frac{|x_n|_{\downarrow}}{n^{\alpha}}} \frac{\left(\sum_{n=1}^{\infty} |\, a_n \,|^2 |\, x_n \,|^2\right)^{\frac{1}{2}}}{\left(\sum_{n=1}^{\infty} |\, x_n \,|^2\right)^{\frac{1}{2}}}$$

we have for $y_n = \frac{x_n}{n^{\alpha}}$,

$$\sup_{\frac{|x_n|_{\downarrow}}{n^{\alpha}}} \frac{\|Ax\|_{\ell^2}}{\|x\|_{\ell^2}} = \sup_{|y_n|^{\downarrow}} \frac{\left(\sum_{n=1}^{\infty} |\, a_n y_n n^{\alpha} \,|^2\right)^{\frac{1}{2}}}{\left(\sum_{n=1}^{\infty} |\, y_n n^{\alpha} \,|^2\right)^{\frac{1}{2}}}.$$

We denote with S the following supremum.

$$S := \sup_{|y_n|^{\downarrow}} \frac{\left(\sum_{n=1}^{\infty} |\, a_n y_n n^{\alpha} \,|^2\right)^{\frac{1}{2}}}{\left(\sum_{n=1}^{\infty} |\, y_n n^{\alpha} \,|^2\right)^{\frac{1}{2}}}.$$

Applying now Sawyer's duality principle (see e.g. [5], [8]) we obtain that for every non negative integer n,

$$V(n) = \sum_{k=1}^{n} v(k) = \sum_{k=1}^{n} k^{2\alpha} \,|\, a_k \,|^2$$

and

$$W(n) = \sum_{k=1}^{n} w(k) = \sum_{k=1}^{n} k^{2\alpha}.$$

It follows that

$$S = \sup_{n\geq 1}\left(\frac{\sum_{k=1}^{n} k^{2\alpha} \,|\, a_k \,|^2}{\sum_{k=1}^{n} k^{2\alpha}}\right)^{\frac{1}{2}}$$

and

$$\|A\|_{B_w^\alpha(\ell^2)} = \sup_{n\geq 1}\left(\frac{\sum_{k=1}^{n} k^{2\alpha}\,|\,a_k\,|^2}{\sum_{k=1}^{n} k^{2\alpha}}\right)^{\frac{1}{2}}.$$

In the case of diagonals above the main diagonal, following the same ideas we obtain the same norm. However, in the case of diagonal below the main diagonal, the norm is equivalent with the above supremum. The proof is complete.

We remark that the inclusion between $B(\ell^2)$ and $B_w^\alpha(\ell^2)$ is proper. For example, we can construct infinite matrices from $B_w^\alpha(\ell^2)$ that do not belong to $B(\ell^2)$. One way of constructing this kind of matrices is by using the above theorem. The last theorem shows that in the case of Toeplitz matrices these two spaces coincides.

Theorem 2.2 *The spaces* $B_w^\alpha(\ell^2)$ *and* $B(\ell^2)$ *coincides in the case of Toeplitz matrices.*

Proof. It is easy to observe that

$$B(\ell^2)\subset B_w^\alpha(\ell^2)\subset B_w(\ell^2).$$

Since $B_w(\ell^2)\cap\mathsf{T} = B(\ell^2)\cap\mathsf{T}$ (see e.g. [6]) it follows the desired result.

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HARDY'S INEQUALITY. A BRIEF HISTORY AND RELATED RESULTS

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Abstract: In this paper we briefly present the Hardy inequality, about its history and some related results.
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Key words: Hardy's inequality, factorization, Lorentz sequence spaces.

1. Hardy's inequality

The Hardy inequality (in the discreet and in the continuous form) has been generalized and applied in analysis and in the theory of differential equations. This inequality has a fascinating history. We start with the discrete version of Hardy's inequality which is:

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}a_k\right)^p \le \left(\frac{p}{p-1}\right)^p \sum_{n=1}^{\infty}a_n^p,$$

where $p>1$ and $(a_k)_{k\ge 1}$ is a real sequence of positive numbers. The Hardy inequality has been extensively studied by many authors.

The continuous case: suppose that $p>1$ and f is a positive p - integrable function on $(0,\infty)$, then f is integrable on $(0,x)$, for every positive x and

$$\int_0^{\infty}\left(\frac{1}{x}\int_0^x f(t)dt\right)^p dx \le \left(\frac{p}{p-1}\right)^p \int_0^{\infty} f(x)^p dx.$$

The constant that appears both in the discrete and the continuous case is sharp.

The original motivation of Hardy was to find an elementary proof of the Hilbert inequality, that is:

$$\sum_{n.m=1}^{\infty}\frac{a_m b_n}{m+n} < \frac{\pi}{\sin\left(\frac{\pi}{p}\right)}\left(\sum_m a_m^p\right)^{1/p}\left(\sum_n b_n^{p'}\right)^{1/p'}.$$

Although that H. Weyl, F.W. Wiener, I. Schur proved by different methods Hilbert's inequality, Hardy, however, considered that none of these proofs was as simple and elementary as he desired. Therefore, Hardy added a new proof, which was simpler in his opinion, than the earlier proofs. The continuous case of Hilbert's inequality is

$$\int_0^{\infty}\int_0^{\infty}\frac{f(x)g(y)}{x+y}dxdy < \frac{\pi}{\sin\left(\frac{\pi}{p}\right)}\left(\int_0^{\infty} f(x)^p dx\right)^{1/p}$$

$$\times\left(\int_0^{\infty} g(y)^{p'}\,dy\right)^{1/p'}.$$

A detailed description of the prehistory of Hardy's inequality can be found in [4]. Further results and generalizations of Hardy's inequality can be found in [4] and [6] and in the references given in these books.

2. Hardy's inequality with weights

Several authors have studied Hardy's inequality with general weights. We recall here just a few of them: Muckenhoupt (1972), Talenti (1969), Tomaselli (1969), Maz'ya (1985).

Let us denote by $Sf(x)=\frac{1}{x}\int_0^x f(t)dt$. Then

$$S: L^p(u(x)dx)\to L^p(v(x)dx)$$

if and only if

$$sup_{r>0}\left(\int_r^{\infty}\frac{v(x)}{x^p}dx\right)^{1/p}\left(\int_0^r u(x)^{1/(1-p)}dx\right)^{1/p'}<\infty.$$

The Hardy inequality can also be written in the following equivalent form, known as differential form of Hardy's inequality,

$$\left(\int_a^b |g(x)|^q\,u(x)dx\right)^{1/q}\le C\left(\int_a^b |g'(x)|^p\,v(x)dx\right)^{1/p}$$

Here $g(a)=0$ and g is a derivable function. In the n-th dimensional case, the inequality in this form is useful in partial differential equations.

3. Factorizations of sequences

Also in connection with the Hardy inequality, in 1996, G. Bennett (see [2]) studied the sequence spaces denoted by

$$d(p)=\{x:\sum_{n=1}^{\infty}\sup_{k\ge n}|x_k|^p<\infty\},\ p\ge 1,$$

$$g(p)=\{x:|x_1|^p+\cdots+|x_n|^p=O(n)\},$$

and

$$ces(p)=\{x=(x_k)_k\ \text{with}\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}|x_k|\right)^p<\infty\}.$$

It can be seen that Hardy's inequality is equivalent with the following inclusion $\ell^p\subset ces(p)$. As an improvement of Hardy's inequality, G. Bennett proved the next result (we present here the short variant of Bennett's theorem, the original one is more general).

Theorem 3.1 *Let* $\frac{1}{p}+\frac{1}{p'}=1$ *and* $p>1$. *Then we have that:*

$$ces(p)=\ell^p\cdot g(p').$$

Here $z\in\ell^p\cdot g(p')$ if and only if there exists $x\in\ell^p$ and $y\in g(p')$ such that $z=x\cdot y$.

4. Lorentz sequence spaces

For a sequence $x = (x_n) \in c_0$, the decreasing rearrangement x^* of x is $(|x_n|)$ arranged in decreasing order. Let us recall the definition of Lorentz spaces with weights,

$$d(w, p) = \{x : \|x\|_{p,w} := \left(\sum_{n=1}^{\infty} (x_n^*)^p w_n \right)^{\frac{1}{p}} < \infty\}.$$

It is proved in [3] that $\|\cdot\|_{p,w}$ is a norm if and only if w is decreasing. Moreover, $d(w, p)$ can be endowed with an equivalent norm if and only if

$$\sum_{k=0}^{n} \left(\frac{1}{W_k} \right)^{1/p} \leq C \frac{n+1}{W_n^{1/p}}, n = 0,1,2,\ldots \tag{4.1}$$

where $W_n = \sum_{k=0}^{n} w_k$. This condition characterizes the boundedness of the discrete Hardy operator

$$A_d(x_n) = \frac{1}{n+1} \sum_{k=0}^{n} x_n,\ n \in \mathbb{N}$$

from $d(w, p)$ to $l^p(w)$.

As a consequence of the fact that $\|\cdot\|_{p,w}$ is equivalent to a norm, it is easy to see that it is a quasi-norm satisfying the triangle inequality uniformly in the numbers of terms expressed as follows: there exists a constant $C_{p,w} > 0$ such that, for every finite collection $\{x^{(k)}\} \subset d(p, w)$, it yields that

$$\left\| \sum_{k=1}^{N} x^{(k)} \right\|_{p,w} \leq C_{p,w} \sum_{k=1}^{N} \left\| x^{(k)} \right\|_{p,w}. \tag{4.2}$$

The best constant in triangle inequality can be found in [1]. The result which gives us the best constant is the following theorem.

Theorem 4.1 *Let* $1 < p < \infty$ *and an increasing weight* w *which satisfy (4.1). Assume that* $x^{(k)} = (x_n^{(k)})_n \in d(p, w)$, $k = 1, \ldots, N$. *Then we have the following inequality*

$$\left\| \sum_{k=1}^{N} x^{(k)} \right\|_{p,w} \leq C_{p,w} \sum_{k=1}^{N} \left\| x^{(k)} \right\|_{p,w}, \tag{4.3}$$

where $C_{p,w}$ is sharp.

$$C_{p,w} = \sup_{n \in \mathbb{N}} \left(\frac{1}{n} \sum_{k=1}^{n} w_k \right) \left(\frac{1}{n} \sum_{k=1}^{n} w_k^{1-p'} \right)^{p-1}. \tag{4.4}$$

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ON THE CONTINUITY AND BOUNDEDNESS OF NEMYTSKIJ OPERATORS IN LEBESGUE SPACES WITH A VARIABLE EXPONENT

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Abstract: In this paper we prove a result concerning sufficient conditions for the continuity of the general nonlinear superposition operator (generalized Nemytskij operator) acting in Lebesgue spaces with a variable exponent. We also provide an application to the study of the Fréchet-differentiability of the gradient norm on a Sobolev space with a variable exponent.
Mathematics Subject Classification (2010): 47H30, 49J50
Key words: Nemytskij operators; Lebesgue spaces with a variable exponent; Fréchet-differentiability of the gradient norm.

1. Introduction

Suppose that $\Omega \subset \mathbf{R}^N$ is a bounded domain. Let $f : \Omega \times \mathbf{R}^M \to \mathbf{R}$ be a function satisfying the *Carathéodory conditions*:

(i) for each $s \in \mathbf{R}^M$, the function $x \to f(x,s)$ is Lebesgue measurable in Ω;

(ii) for almost all $x \in \Omega$, the function $s \to f(x,s)$ is continuous in $\mathbf{R}^M$.

To such a function we associate the *Nemytskij operator*

$$(N_f u)(x) := f(x,u(x)) \text{ for each } x \in \Omega,$$

defined on classes of vector functions $u : \Omega \to \mathbf{R}^M$, $u(x) = \left(u_1(x), u_2(x), \ldots, u_M(x)\right)$.

Let us make the following convention for the Carathéodory function, the assertion "$x \in \Omega$" is to be understood in the sense "almost all $x \in \Omega$".

It is well known that, for any measurable function $u : \Omega \to \mathbf{R}^M$, the function $\Omega \ni x \mapsto f\left(x,u(x)\right) \in \mathbf{R}$ is also measurable.

We now review some definitions and properties related to Lebesgue spaces with variable exponents needed throughout the paper. For proofs and references see [3].

Given a function $p(\cdot) \in L^\infty(\Omega)$ that satisfies

$$1 \le p^- := \operatorname{ess\,inf}_{x \in \Omega} p(x) \le \operatorname{ess\,sup}_{x \in \Omega} p(x) =: p^+ < \infty,$$

the *Lebesgue space* $L^{p(\cdot)}(\Omega)$ *with variable exponent* $p(\cdot)$ is defined as

$$L^{p(\cdot)}(\Omega) := \left\{ v : \Omega \to \mathbf{R};\ v \text{ is measurable and } \rho_{p(\cdot)}(v) := \int_\Omega |v(x)|^{p(x)}\, dx < \infty \right\}.$$

Equipped with the norm

$$u \in L^{p(\cdot)}(\Omega) \to \|u\|_{p(\cdot)} := \inf\left\{\lambda > 0; \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} \le 1\right\},$$

the space $L^{p(\cdot)}(\Omega)$ is a separable Banach space.

Given $p(\cdot) \in L^\infty(\Omega)$ such that $p^- > 1$, let $p'(\cdot) \in L^\infty(\Omega)$ be defined by

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = 1 \text{ for almost all } x \in \Omega.$$

For any $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$, the following inequality holds:

(1) $$\int_\Omega |u(x)v(x)|\,dx \le \left(\frac{1}{p^-} + \frac{1}{(p')^-}\right)\|u\|_{p(\cdot)}\|v\|_{p'(\cdot)}.$$

If $v, w \in L^{p(\cdot)}(\Omega)$, then:

(2) $$\rho_{p(\cdot)}(v+w) \le 2^{p^+}\left(\rho_{p(\cdot)}(v) + \rho_{p(\cdot)}(w)\right).$$

The following theorem summarizes the relations between the norm $\|\cdot\|_{0,p(\cdot)}$ and the convex modular $\rho_{p(\cdot)}$.

Theorem 1. *Let* $p(\cdot) \in L^\infty(\Omega)$ *be such that* $p^- \ge 1$ *and let* $u \in L^{p(\cdot)}(\Omega)$. *Then:*

(a) *If* $u \ne 0$, *then* $\|u\|_{p(\cdot)} = a$ *if and only if* $\rho_{p(\cdot)}(a^{-1}u) = 1$.

(b) $\|u\|_{p(\cdot)} < 1$ *(resp.* $= 1$ *or* > 1*) if and only if* $\rho_{p(\cdot)}(u) < 1$ *(resp.* $= 1$ *or* > 1*).*

(c) $\|u\|_{p(\cdot)} > 1$ *implies* $\|u\|_{p(\cdot)}^{p^-} \le \rho_{p(\cdot)}(u) \le \|u\|_{p(\cdot)}^{p^+}$.

(d) $\|u\|_{p(\cdot)} < 1$ *implies* $\|u\|_{p(\cdot)}^{p^+} \le \rho_{p(\cdot)}(u) \le \|u\|_{p(\cdot)}^{p^-}$.

(e) *Let* $u \in L^{p(\cdot)}(\Omega)$ *and* $u_n \in L^{p(\cdot)}(\Omega)$, $n = 1, 2, \ldots$. *The following statements are equivalent:*

(i) $\|u - u_n\|_{p(\cdot)} \to 0$ *as* $n \to \infty$.

(ii) $\rho_{p(\cdot)}(u_n - u) \to 0$ *as* $n \to \infty$.

(iii) $(u_n)_n$ *converges to* u *in measure and* $\rho_{p(\cdot)}(u_n) \to \rho_{p(\cdot)}(u)$ *as* $n \to \infty$.

2. The main result

The main result of this paper states sufficient conditions to ensure the Nemytskij operator that maps $\left[L^{p_1(\cdot)}(\Omega)\right]^M$ into $L^{p_2(\cdot)}(\Omega)$ is continuous and bounded.

On $\left[L^{p_1(\cdot)}(\Omega)\right]^M$ consider the norm

$$\|u\| := \left\|\sqrt{T[u,u]}\right\|_{p_1(\cdot)},$$

where $u = (u_1, u_2, \ldots, u_M)$, $T[u,u] := \sum_{i=1}^{M} u_i^2$.

Theorem 2. *Let* $f:\Omega\times\mathbf{R}^M\to\mathbf{R}$ *be a Carathéodory function which satisfies the growth condition*

(3) $$|f(x,u)|\le c_1(x)+c(x)\sum_{i=1}^{M}|u_i|^{p_1(x)/p_2(x)},\ x\in\Omega,\ u\in\mathbf{R}^M,$$

where $c_1\in L^{p_2(\cdot)}(\Omega)$ *and* c *is a non-negative* $L^\infty(\Omega)$*-function. Then* N_f *is a well-defined, bounded, continuous operator from* $\left[L^{p_1(\cdot)}(\Omega)\right]^M$ *into* $L^{p_2(\cdot)}(\Omega)$.

For details of the proof see [4].

For $M=1$ we obtain:

Corollary 3. *Let* $f:\Omega\times\mathbf{R}\to\mathbf{R}$ *be a Carathéodory function which satisfies the growth condition*

$$|f(x,u)|\le c_1(x)+c(x)|u|^{p(x)-1},\ x\in\Omega,\ u\in\mathbf{R},$$

where $c_1\in L^{p'(\cdot)}(\Omega)$ *and* c *is a non-negative* $L^\infty(\Omega)$*-function. Then* N_f *is a well-defined, bounded, continuous operator from* $L^{p(\cdot)}(\Omega)$ *into* $L^{p'(\cdot)}(\Omega)$.

Note that this corollary is contained in Theorem 1.16, Fan and Zhao [3].

3. Fréchet differentiability of the gradient norm on a Sobolev space with a variable exponent

In this section, the above results are used to prove the Fréchet differentiability of a norm on a Sobolev space with a variable exponent.

Given a function $p(\cdot)\in L^\infty(\Omega)$ that satisfies $p^-\ge 1$, the *Sobolev space* $W^{1,p(\cdot)}(\Omega)$ *with variable exponent* $p(\cdot)$ is defined as

$$W^{1,p(\cdot)}(\Omega):=\left\{v\in L^{p(\cdot)}(\Omega);\ \partial_i v\in L^{p(\cdot)}(\Omega),\ 1\le i\le N\right\},$$

where, for each $1\le i\le N$, ∂_i denotes the distributional derivative operator with respect to the *i*-th variable. $W^{1,p(\cdot)}(\Omega)$ is a Banach space with respect to the norm

$$\|u\|_{1,p(\cdot),\nabla}:=\|u\|_{0,p(\cdot)}+\sum_{i=1}^{N}\|\partial_i u\|_{0,p(\cdot)}.$$

Consider the space (see [2] for details)

$$U_{\Gamma_0}:=\left\{u\in W^{1,p(\cdot)}(\Omega);\ \mathrm{tr}\,u=0\ \text{on}\ \Gamma_0\right\},\ \Gamma_0\subset\Gamma=\partial\Omega,\ \mathrm{d}\Gamma-meas\Gamma_0>0.$$

The map

$$u\in U_{\Gamma_0}\to\|u\|_{0,p(\cdot),\nabla}:=\left\|\|\nabla u\|\right\|_{p(\cdot)}$$

is a norm on U_{Γ_0}, equivalent to the norm $\|u\|_{1,p(\cdot),\nabla}$ ([2], Theorem 6 (b))

Moreover ([2], Lemma 1), the norm $\|u\|_{0,p(\cdot),\nabla}$ is Gâteaux-differentiable at any nonzero $u \in U_{\Gamma_0}$ and the Gâteaux-differential of this norm at any nonzero $u \in U_{\Gamma_0}$ is given for any $h \in U_{\Gamma_0}$ by

$$\left\langle \|\cdot\|'_{0,p(\cdot),\nabla}(u), h \right\rangle = \frac{\int_{\Omega \setminus \Omega_{0,u}} p(x) \frac{|\nabla u(x)|^{p(x)-2} \langle \nabla u(x), \nabla h(x) \rangle}{\|u\|^{p(x)-1}_{0,p(\cdot),\nabla}} dx}{\int_{\Omega} p(x) \frac{|\nabla u(x)|^{p(x)}}{\|u\|^{p(x)}_{0,p(\cdot),\nabla}} dx}, \tag{13}$$

where $\Omega_{0,u} := \{x \in \Omega;\ |\nabla u(x)| = 0\}$.

By using Theorem 2 and Corollary 3, we will prove:

Theorem 4. *The map*

$$u \in U_{\Gamma_0} \setminus \{0\} \to \|\cdot\|'_{p(\cdot)}$$

is continuous.

For details of the proof see [4].

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THE STUDY OF MATHEMATICAL LOGIC AT FUTURE TENSE

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Abstract: Mathematical logic develops a fundamental way of thinking and planning, that is needed by scientists, engineers and programmers who are designing technical devices or software for such devices, in order to have end-user friendly, logical technical applications. This creates an urgent need for workers with mathematical logic skills in European enterprises and this was the main undertaking of MALog project. The aim of the project has been to provide learning materials (LMs) in a variety of mathematical logic and applied logic topics, to develop an ontology of mathematical logic, and most important, to improve the suitability of LMs for individual learners. This paper presents some of the results of the project, more exactly an e-learning pilot course and the end-user feedback, which was the students viewpoints from the TUCEB's pilot course.

Mathematics Subject Classification (2010): 03-02
Key words: mathematical logic, e-learning

1. Introduction. Motivation of the project

Mathematics develops a fundamental way of thinking and planning that is a basic skill of scientists, engineers and programmers who are designing complex technical devices or the software for technical applications. Thus, there is a need in the industry of personnel with very good mathematical and logical skills, and this was the major challenge for the partners in the MALog international project.

The motivation of the project was the fact that the mathematical and logical knowledge is on a decreasing learning curve. This reality was pointed out in a number of occasions and it became a concern it the entire European Community because mathematics and logic are the base of fundamental disciplines in the area of computer science and programming, engineering and even medical sciences.

2. The project objectives

The main objective of the project was to develop materials and resources that can help in the process of learning mathematical logic and in the process of improving the existing mathematical logic skills. These resources are meant not only for those in the educational and

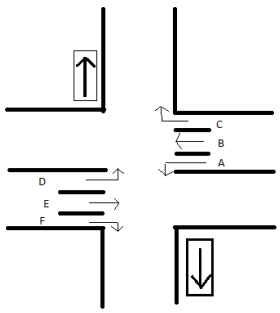
academic field, but also for those from industry or business environment, who can use and can benefit from such knowledge.
Some of the project specific objectives and new approaches were:

a. the detailed analysis of the current situation and of the role that mathematical logic and logical reasoning are playing in specific fields of industry/economy;
b. finding real life problems that can make use of abstract concepts of mathematical logic skills;
c. the development of a mathematical logic ontology;
d. planning and designing the necessary learning materials;
e. designing test pilot courses and gathering the feedback for improving the existing materials.

In order to accomplish the goals of the project, a deep collaboration with industry partners was established. The main challenge was to find the real problems that make use of mathematical logic skills and to find the concepts used in solving such problems. As a result of this collaboration with industry/business partners the project has combined a series of different perspectives in an innovating manner of presenting the concepts of mathematical logic. The materials developed and designed as a result of this research, are meant to allow each student to follow his own learning path, are flexible and suited for an adaptive learning process.
The ontology developed during the project is the mechanism to structure the information related to different subjects and to establish different connections between them.

3. Project results

The main accomplishment of the project is the creation of a database with learning materials that cover problems from simple to complicated ones, from abstract to real ones. Some of these materials are designed for the learning process, and some of them for testing the skills the students should posses after following such courses.
The information gathered at the beginning of the project about the real life problems and the mathematical concepts associated with them were used to improve the quality of individual materials and of the database. These real life problems are the main asset in implementing any course of mathematical logic based on the materials database. They are also the means for the students to realize the implication of logic in every day life.
An example of such a problem is presented below:



Design a traffic lights backup system, according to the instructions given below. The vehicles are entering the intersection from two directions, as you can see from the picture:

Fig1. Traffic lights backup system

When the green lights of intersecting roads are simultaneously on, and the collision is possible, the safety device of the system becomes active the lights turn to red everywhere. Do not take into account the yellow light. Both directions of the road are with only one sense and one lane. That means that two vehicles cannot move in the same direction simultaneously.

When the red light is on the vehicle cannot enter the intersection. When the green light is on, the vehicle is allowed to enter the intersection. There are also traffic lights with an arrow (direction) to allow vehicles to turn left or right.

a. Find the combination to activate the backup system;
b. Simplify the problem;
c. Design the system with circuits with two logical gates.

4. Evaluations of the results

The evaluation of the learning materials was realized by means of some pilot courses implemented at high school and university level. The content of these courses was established depending on the level of knowledge of the students and on their field of study, in case of university courses. The feedback from students and teachers was used to improve the quality of the materials and a statistic of their opinions can be found on the project site.

Some of the feedback to the course held in our university is presented in the charts and tables below, along with the questions that the students replied to.

a. Evaluate on a scale of 1 to 5 learning materials related to each topic

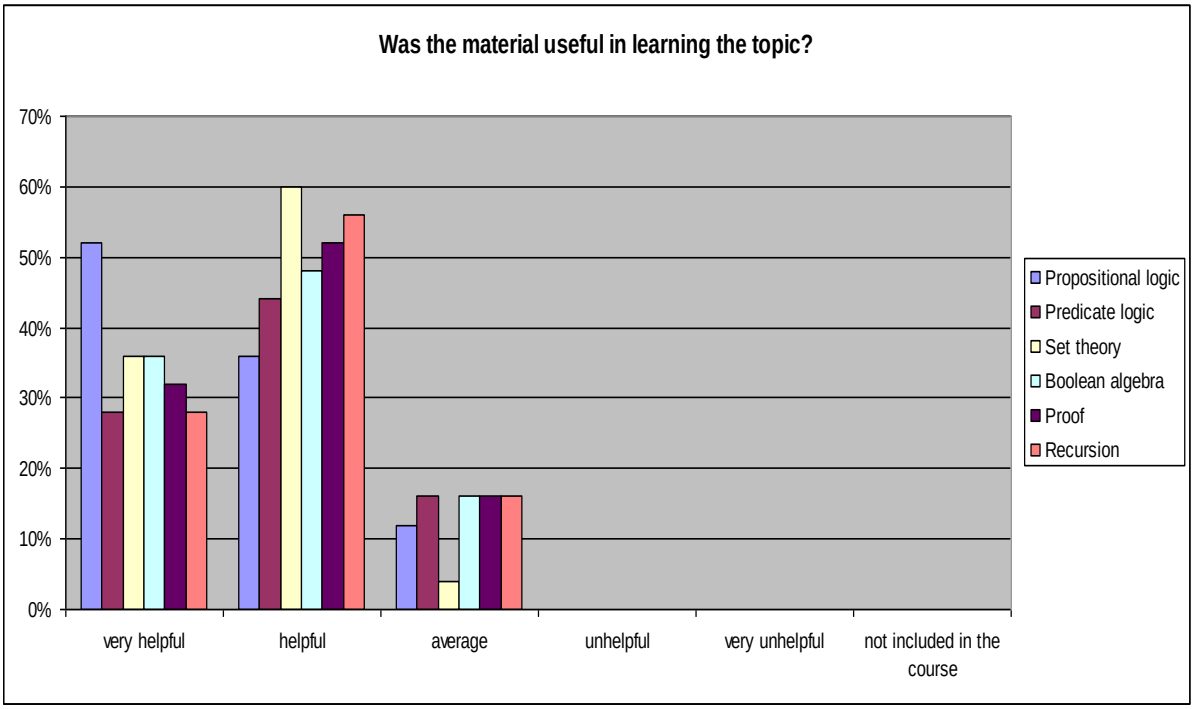
	Propositional logic	Predicate logic	Set theory	Boolean algebra	Proof (axiom systems, natural deduction)	Recursion
1 (weak)	0%	0%	0%	0%	0%	0%
2	0%	0%	0%	0%	0%	0%
3	28%	12%	4%	32%	4%	0%
4	32%	36%	52%	28%	52%	44%
5 (excellent)	40%	48%	40%	36%	40%	52%
6 (don't know / not heard of)	0%	0%	0%	0%	0%	0%

b. How easy do you find it is to learn mathematical logic in a particular setting?

	Lectures/ Classroom teaching	Exercise/ problem classes	Working with the computer	Individual study with learning materials	Individual study with example problems	Preparing for a test/exam
very easy	32%	20%	12%	8%	16%	16%
easy	28%	36%	28%	52%	32%	36%
average	36%	40%	48%	40%	28%	40%
difficult	4%	4%	8%	0%	24%	8%
very difficult	0%	0%	0%	0%	0%	0%

not tried / don't answer	0%	0%	0%	0%	0%	0%

c. Was the material useful in learning the topic?



5. Conclusions

The results of the pilot courses and of evaluation that was conducted in our university showed that most students can improve their mathematical and logic skills if they choose to follow this course. They gain also a better understanding of the mathematical concepts involved in real life problems. The team participating in this project found also new problems of study that have implications in both logic and engineering fields, but they are related to fuzzy logic and systems and they will be the starting point of a future project.

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NEW CONNECTIONS DERIVED FROM SEMI-SYMMETRIC METRIC CONNECTIONS

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Abstract: The notion of a semi-symmetric metric connection on a Riemannian manifold was introduced by Hayden [4]. Later, in 1970, Yano studied in [14] some properties of a Riemannian manifold endowed with a semi-symmetric metric connection. Recently, the subject of manifolds endowed with semi-symmetric metric and non-metric connections and their submanifolds became a topic of interest for many geometers.
In this paper we construct examples of different kind of connections starting from a semi-symmetric metric connection, for example a connection which is a symmetric metric connection with respect to a conformally related metric, but symmetric non-metric with respect to the initial metric. We formulate an open problem: to find a parallel complex structure on a Kaehler manifold with respect to such a new connection.
Math. Subject Classification (2010): 53C05, 53C55.
Key words: metric connection, symmetric connection, semi-symmetric connection.

1. Introduction

The notion of a *connection* is one of the most important in Geometry. Its history starts with the work of famous mathematicians: Christoffel, Ricci, Levi-Civita, Cartan, Darboux, Koszul (see, for example, [7]). In 1924, Friedmann and Schouten ([3]) introduced the notion of a *semi-symmetric linear connection* on a differentiable manifold. After that, in 1932, Hayden, in [4], introduced the idea of metric connection with torsion on a Riemannian manifold.

Semi-symmetric metric connection plays an important role in the study of Riemannian manifolds [13] and a systematic study of the semi-symmetric metric connections on a Riemannian manifold was published by Yano in 1970 [14].

Regarding the physical problems which involve the semi-symmetric metric connections, in [13] the following two interesting examples are given:

If a man is moving on the surface of the earth always facing one definite point, say Jerusalem or Mekka or the North Pole, then this displacement is semi-symmetric and metric.

During the mathematical congress in Moscow in 1934, *one evening, mathematicians invented the Moscow displacement: the streets of Moscow are approximately straight lines through the Kremlin and concentric circles around it. If a person walks in the street always facing the Kremlin, then this displacement is semi-symmetric and metric.*

In this paper, as we already mentioned in the abstract, we construct examples of different kind of connections starting from a semi-symmetric metric connection $\tilde{\nabla}$ on a Riemannian manifold, for example a connection which is a symmetric metric connection with respect to a conformally related metric g^*, but symmetric non-metric with respect to the initial metric g.

The results from this paper were presented at the 12-th Workshop of Scientific Communications, Department of Mathematics and Computer Science, Technical University of Civil Engineering Bucharest, Romania, May 24-25, 2013. Complete proofs of the enounced results can be found in the article [9], submitted for publication.

We formulate an open problem: to find a parallel complex structure on a Kaehler manifold with respect to such a new connection.

Next, we recall basic definitions of different kind of connections on Riemannian manifolds.

Let $(\tilde{N}, g)$ be an *n*-dimensional Riemannian manifold endowed with the Riemannian metric g and $\tilde{\nabla}$ be a linear connection on $\tilde{N}$, i.e

i) $\tilde{\nabla}_{fX+hY}Z = f\tilde{\nabla}_X Z + h\tilde{\nabla}_Y Z$,

ii) $\tilde{\nabla}_X(fY) = (Xf)Y + f\tilde{\nabla}_X Y$, for any $f, h \in C^\infty(\tilde{N})$ and $X, Y, Z \in \Gamma(T\tilde{N})$.

Let $\tilde{T}$ be the torsion tensor of $\tilde{\nabla}$, i.e

$$\tilde{T}(X,Y) = \tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X,Y],\ \forall X,\ Y \in \Gamma(T\tilde{N}).$$

If $\tilde{T} = 0$, $\tilde{\nabla}$ is a *symmetric (torsion-free) connection.* $\tilde{\nabla}$ is *Levi-Civita connection* if $\tilde{T} = 0$ and $\tilde{\nabla} g = 0$. If $\tilde{T}$ satisfies $\tilde{T}(X,Y) = \phi(Y)X - \phi(X)Y$, for a 1-form ϕ, then $\tilde{\nabla}$ is called a *semi-symmetric connection*. If, moreover, $\tilde{\nabla} g = 0$, then $\tilde{\nabla}$ is called a *semi-symmetric metric connection*. If $\tilde{\nabla} g \neq 0$, then $\tilde{\nabla}$ is called a *semi-symmetric non-metric connection*.

K. Yano [14] showed that a semi-symmetric metric connection $\tilde{\nabla}$ is given by

$$(*) \qquad \tilde{\nabla}_X Y = \nabla^\circ_X Y + \phi(Y)X - g(X,Y)P,$$

where ∇° is the Levi-Civita connection on $\tilde{N}$ with respect to *g* and *P* is a vector field defined by $P = \phi^\#$, equivalent with $g(P, X) = \phi(X)$, for any vector field *X*.

So, the previous relation can be written as

$$\tilde{\nabla}_X Y = \nabla^\circ_X Y + g(P,Y)X - g(X,Y)P.$$

It is easy to prove that $\tilde{\nabla}$ is semi-symmetric: denoting by $\tilde{T}$ the corresponding torsion, one has $\tilde{T}(X,Y) = \phi(Y)X - \phi(X)Y$. To prove that $\tilde{\nabla}$ is metric it follows immediately from $\tilde{\nabla} g = 0$, i.e. $(\tilde{\nabla}_X g)(Y,Z) = 0$.

2. New Connections Derived From $\tilde{\nabla}$

Let us consider only a part of formula (*) and define

$$\nabla'_X Y = \nabla^\circ_X Y + \phi(Y)X,$$

with ϕ a 1-form.

By straightforward calculations, one can prove that ∇' is a semi-symmetric connection, but not metric (to become semi-symmetric metric connection we have to add the term $-g(X,Y)P$ and ∇' will be exactly $\tilde{\nabla}$).

So, the following result holds:

Proposition 2.1. *Let $(\tilde{N}, g)$ be an n-dimensional Riemannian manifold and ∇° be the Levi-Civita connection with respect to the Riemannian metric g. Then the connection ∇' defined by*

$$\nabla'_X Y = \nabla^\circ_X Y + \phi(Y)X,$$

with ϕ a 1-form on $\tilde{N}$, is a semi-symmetric non-metric connection on $\tilde{N}$.

To provide another type of connection starting from $\tilde{\nabla}$, denote by $\Omega^1(\tilde{\nabla})$ the space of *1*-form on $\tilde{N}$. Following Yano [14], to any 1-form $\phi \in \Omega^1(\tilde{\nabla})$ corresponds a connection

$$\tilde{\nabla}_X Y = \nabla^\circ_X Y + \phi(Y)X - g(X,Y)\phi^\#.$$

There are two cases:

i) ϕ is closed (i.e. $d\phi = 0$)

ii) ϕ is exact ($\exists\, f \in C^\infty(\tilde{N})$ such that $\phi = df$; then $g(P, X) = \phi(X) = df(X) = Xf$, $P = grad\, f$)

Obviously, second case implies the first and then one can show that

$$g(\nabla^\circ_X grad\, f, Y) = g(\nabla^\circ_Y grad\, f, X).$$

For an exact *1*-form, i.e. $\exists\, f \in C^\infty(\tilde{N})$ such that $\phi = df$, we define a *conformally related metric* g^* such that $g^* = e^{2f} g$ and denote by ∇^* its Levi-Civita connection (on $(\tilde{N}, g)$).

One proves that

$$\nabla^*_X Y = \nabla^\circ_X Y + \phi(Y)X - g(X,Y)\phi^\# + \phi(X)Y,$$

i.e. $\nabla^*_X Y = \tilde{\nabla}_X Y + \phi(X)Y$, where $\tilde{\nabla}$ is the semi-symmetric metric connection with respect to *g* (see formula (*) from section 1).

The previous formula can be proved by using the Christoffel coefficients. Moreover, ∇^* is symmetric and also it is a metric connection with respect to g^*, but it is not metric with respect to *g*.

We have proved the following:

Proposition 2.2. *Let $(\tilde{N}, g)$ be an n-dimensional Riemannian manifold and $g^* = e^{2f} g$ a conformally related metric to g, with $f \in C^\infty(\tilde{N})$. Let ∇^* be the Levi-Civita connection with respect to g^*. Then:*

i) $\nabla^*_X Y = \nabla^\circ_X Y + \phi(Y)X - g(X,Y)\phi^\# + \phi(X)Y$, *i.e.* $\nabla^*_X Y = \tilde{\nabla}_X Y + \phi(X)Y$;

ii) *The connection ∇^* is symmetric metric connection with respect to g^*;*

iii) *The connection ∇^* is symmetric non-metric connection with respect to g.*

3. The Kaehler Case

A Kaehler manifold is one of the most interesting manifold from the class of complex manifolds. A *complex manifold* of dimension *n* is a pair (M, A) of a non-empty set *M* and a family of applications *A* defined on an open set of the complex space of dimension *n* to subsets of *M* satisfying certain properties. Any complex manifold *M* admits an almost complex structure, i.e. an anti-involutive endomorphism *J* of the tangent bundle *TM*, $J^2 = -I_{TM}$. A manifold endowed with an almost complex structure is called an *almost complex manifold.*

A *Kaehler manifold* is a complex manifold endowed with a Hermitian metric *g*, $g(JX, JY) = g(X,Y)$, for any tangent vectors *X* and *Y*, with *g* a Riemannian metric on *M*, with the fundamental 2-form ω closed, i.e. $d\omega = 0$, where, by definition $\omega = g(JX, Y)$. So, a Kaehler manifold *M* is well determined by its metric *g* and its almost complex structure *J* and then is usualy denoted by (M, g, J). A Hermitian manifold (M, g, J) is Kaehler if and only if its almost complex structure *J* is parallel with respect to the Levi-Civita connection associated to the Riemannian metric *g*, $\nabla^\circ J = 0$.

Let $\tilde{\nabla}$ be the semi-symmetric metric connection with respect to *g* on a Kaehler manifold (M, g, J). By straightforward calculations and using the previous definitions, we get

$$J\phi(Y)X + J\phi(X)Y = \phi(JY)X + \phi(JX)Y.$$

Denoting by $(J*\phi)(X,Y)$ the left side and by $(\phi*J)(X,Y)$ the right side, the last formula means that *J* and ϕ commute, $J*\phi=\phi*J$. This condition comes natural, because we need a relationship between the Kaehler structure and the *1*-form.

Starting from the semi-symmetric connection $\tilde{\nabla}$ on the Kaehler manifold (M,g,J) we can derive, as in the previous section, another connection, ∇^*.

We formulate the following:

Open Problem:

Under the assumption $J*\phi=\phi*J$, find another almost complex structure J^* on the Kaehler manifold (M,g,J) such that J^* is parallel with respect to ∇^*.

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ON MECHANICS PROBLEMS SOLVED WITH MATHCAD

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Abstract: Mathcad is a software for engineering calculations. Some mechanical problems are solved using Mathcad. For some points in 3D, we compute vectors, scalar product, vector product, measure of the angle, surface of a triangle and the volume of a tetrahedron. Tetrahedron tips are plotted. Equations of the edges are written and used for graphycal representation. Using a Maple plottool emulation, the tetrahedron is graphycally represented. Kinematics problems are solved and represented with animation.
Mathematics Subject Classification (2010): 51P04, 70B04.
Key words: Mathcad, Kinematics, Linear Algebra.

1. Introduction in Mathcad

Mathcad was conceived and originally written by Allen Razdow (of MIT), co-founder of Mathsoft. It is a computer software primarily intended for the verification, validation, documentation and re-use of engineering calculations. It was the first to introduce live editing of typeset mathematical notation, its automatic computation. Mathcad today includes some of the capabilities of a computer algebra system but remains oriented towards ease of use and numerical engineering applications**.** Mathcad is oriented around a worksheet, in which equations and expressions are displayed graphically, as opposed to plain text.

2. Vectorial calculus

Let us consider four points in space A(2,-1,1), B(5,5,4), C(3,2,-1), D(4,1,3). Compute
a) $\overrightarrow{AC}\cdot\overrightarrow{AD},\ \overrightarrow{BA}\times\overrightarrow{BD}$;
b) Measure of the angle ABD, surface of triangle ABD;
c) Volume of the tetrahedron ABCD.
Solution. In Mathcad, we define an array for each point

$$A := \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \qquad B := \begin{pmatrix} 5 \\ 5 \\ 4 \end{pmatrix} \qquad C := \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} \qquad D := \begin{pmatrix} 4 \\ 1 \\ 3 \end{pmatrix}$$

We compute vectors

$$AB := B - A \qquad BA := -AB$$

$$AB = \begin{pmatrix} 3 \\ 6 \\ 3 \end{pmatrix} \qquad BA = \begin{pmatrix} -3 \\ -6 \\ -3 \end{pmatrix}$$

$AC := C - A$ $\quad AC = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}$ $\quad BC := C - B$ $\quad BC = \begin{pmatrix} -2 \\ -3 \\ -5 \end{pmatrix}$

$AD := D - A$ $\quad AD = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$ $\quad BD := D - B$ $\quad BD = \begin{pmatrix} -1 \\ -4 \\ -1 \end{pmatrix}$

Scalar product and vectorial product can be computed using Matrix Toolbar.

$AC \cdot AD = 4$ $\qquad BA \times BD = \begin{pmatrix} -6 \\ 0 \\ 6 \end{pmatrix}$

Measure of the angle and surface of triangle are evaluated symbolically and numerical, using Algebra formulas

$$cosABD := \frac{AB \cdot BD}{|AB| \cdot |BD|} \qquad cosABD = -0.962 \qquad cosABD \rightarrow -\frac{5 \cdot \sqrt{2} \cdot \sqrt{6}}{18}$$

$$SABD := \frac{1}{2} \cdot |AB \times AD| \qquad SABD = 4.243 \qquad SABD \rightarrow 3 \cdot \sqrt{2}$$

Now we find the volume of the tetrahedron.

$$M := augment(AB, AC, AD) \qquad M = \begin{pmatrix} 3 & 1 & 2 \\ 6 & 3 & 2 \\ 3 & -2 & 2 \end{pmatrix} \qquad |M| = -18$$

$$VABCD := \frac{1}{6} \cdot ||M|| \qquad VABCD = 3 \qquad VABCD \rightarrow 3$$

3. Graphycal representation of the tetrahedron

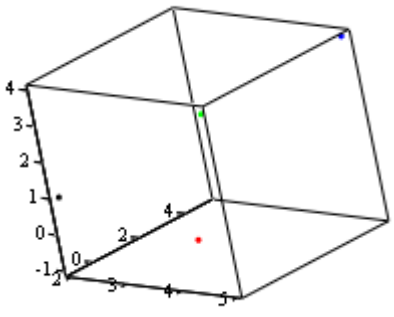
We represent the four tips of the tetrahedron.

$X_1 := A_1$ $\quad X = (2)$

$Y_1 := A_2$ $\quad Y = (-1)$

$Z_1 := A_3$ $\quad Z = (1)$

$$AA := \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \qquad AA = \begin{pmatrix} \{1,1\} \\ \{1,1\} \\ \{1,1\} \end{pmatrix}$$



AA , BB, CC, DD

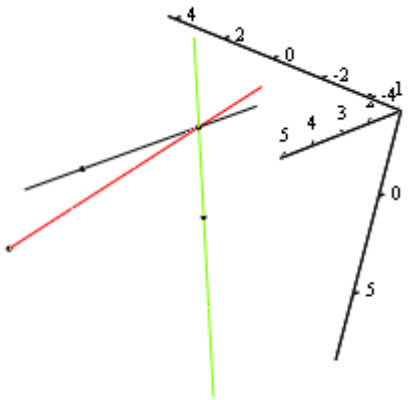
Fig. 1 Plot of the tips of the tetrahedron

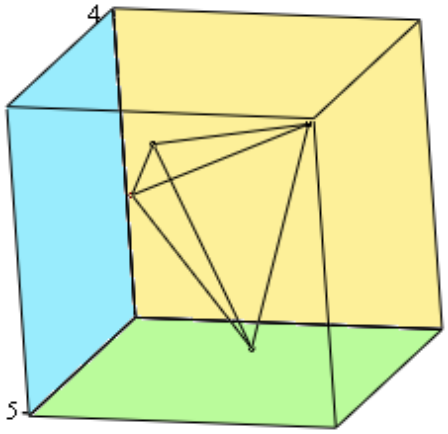
If we pass the plot an array it plots the row and column numbers as x and y, and the value as z. We need to pass it arrays for each coordinate, even if they are just single element arrays.

Thus, we define a new matrix *AA*, which contains the coordinates of point *A* and similarly for *B*, *C* and *D*. In Fig. 1 we plot the tips of the tetrahedron.

Equations of the edges of the tetrahedron that pass through A are written and used for graphical representation in Fig. 2.

AB: $\frac{x - A_1}{B_1 - A_1} = \frac{y - A_2}{B_2 - A_2} = \frac{z - A_3}{B_3 - A_3} \rightarrow \frac{x}{3} - \frac{2}{3} = \frac{y}{6} + \frac{1}{6} \wedge \frac{y}{6} + \frac{1}{6} = \frac{z}{3} - \frac{1}{3}$





$(x, y, z), (xx, yy, zz), (xxx, yyy, zzz), AA, BB, CC, DD$

$ln1, ln2, ln3, ln4, ln5, ln6, ptA$

Fig. 2. Tetrahedron edges passing through A

Fig. 3. Tetrahedron plot using a Maple plottool emulation

One different method for the tetrahedron plot is to use a Maple plot tool emulation, in Fig. 3.

4. Kinematics problems

1. Let us study the motion of a fixed point P on a spinning wheel of radius R, which rolls without slip on a line from a plane with constant velocity (uniform motion).

Using the parametric equations of the trajectory, the graphycal representation of the cycloid is in Fig. 4.

$t := 0, 0.01 .. 4\cdot\pi \qquad R := 1$

$x(t) := R\cdot(t - \sin(t))$

$y(t) := R\cdot(1 - \cos(t))$

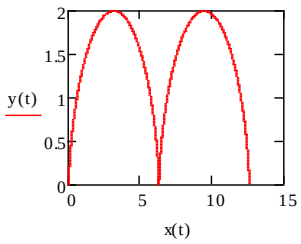


Fig. 4. Cycloid

For graphycal represention with animation, variable FRAME has to be used.

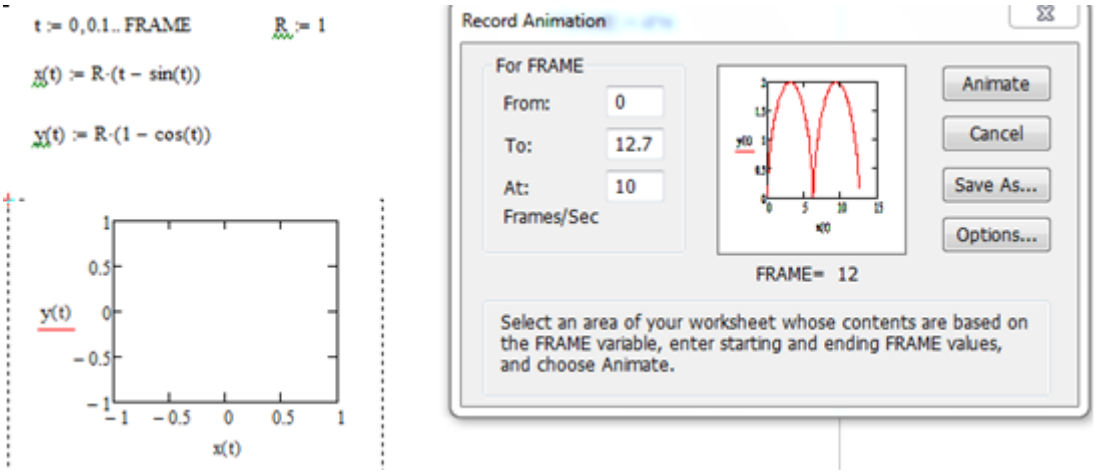


Fig. 5. Animation for cycloid

2. A submarine travels with constant speed *v* on a path given by the following equation

$$r(t) = vt + \frac{at^2}{2}.$$

Figure the trajectory in the first 2 seconds from departure, knowing the components of the acceleration $a_x = 0\,m/s^2, a_y = 0\,m/s^2, a_z = -32\,m/s^2$ and the initial velocity $v_{Ox} = 2m/s$, $v_{Oy} = 3m/s$ and $v_{Oz} = 10m/s$.

$a_0 := 0$ $a_1 := 0$ $a_2 := -32$ $v0_0 := 2$ $v0_1 := 3$ $v0_2 := 10$

$n := 10$

$$\text{traiectorie}(t) := \left| \begin{array}{l} \text{for } i \in 0..t\cdot n \\ \quad t \leftarrow \frac{i}{n} \\ \quad r_{0,i} \leftarrow v0_0\cdot t + \frac{a_0\cdot t^2}{2} \\ \quad r_{1,i} \leftarrow v0_1\cdot t + \frac{a_1\cdot t^2}{2} \\ \quad r_{2,i} \leftarrow v0_2\cdot t + \frac{a_2\cdot t^2}{2} \\ \text{return } r \end{array} \right.$$

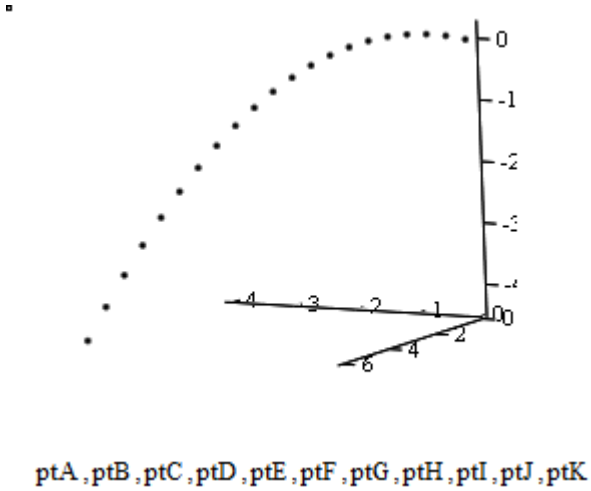


Fig. 6. The trajectory of the submarine in the first 2 seconds

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VECTORIAL INVARIANT MEASURES ASSOCIATED WITH AN ITERATED FUNCTION SYSTEM

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Abstract: In this paper is given a generalization of the Hutchinson measure associated with an iterated function system. This measure is the only fixed point of the Markov operator, which is defined and takes values in the set of Borel normalized measures on a compact metric space. We get a similar result in the case of vectorial measures, that is, in the situation when the Borel measures takes values in a Hilbert space. In the case of a finite -dimensional space, we use the Monge-Kantorovich norm on the liniar space of vectorial measures with bounded variation, and in the case of an infinite-dimensional Hilbert space, we use the variation norm on the same space.
Mathematics Subject Classification (2010): 28C20**,** 46G12.
Key words: iterated function system, Hutchinson measure, vectorial invariant measures.

1. Hutchinson metric and measure

Let (T, *d*) a compact metric space. We denote by: $\boldsymbol{B}$ = { μ - normalized Borel measure: $\mu(\mathrm{T}) = 1$} and $Lip_1(\mathrm{T})$= { $f : \mathrm{T} \to \mathbf{R}$: $|f(x) - f(y)| \le d(x,y)$ }. One can define $d_H : \boldsymbol{B} \times \boldsymbol{B} \to \mathbf{R}$, $d_H(\mu,\tau) = \sup\{ \int f d\mu - \int f d\tau : f \in Lip_1(\mathrm{T})\}$.

Proposition 1. d_H is a metric on $\boldsymbol{B}$ and ($\boldsymbol{B}$, d_H) is a compact metric space. d_H is called the *Hutchinson metric.*

Let now { ω_1 , ω_2 **,...,** ω_N }be an *iterated function system*, that is : the functions ω_i are contractions of ratios $r_i < 1$, defined on T and taking values in T, for any *i* between 1 and *N*. Let, also, the numbers $p_i > 0$, such that $\sum_1^N p_i = 1$. One can define *the Markov operator*: $\mathrm{M}(\mu)(A) = \sum_1^N p_i \mu(\omega_i^{-1}(A))$ for any $\mu \in \boldsymbol{B}$ and any Borel set $\mathrm{A} \subset \mathrm{T}$.

Theorem 2. The Markov operator is a contraction on ($\boldsymbol{B}$, d_H). Consequently, there is a single measure $\mu \in \boldsymbol{B}$, such that $\mathrm{M}(\mu) = \mu$. This measure is called the *Hutchinson measure* associated with the iterated function system before mentioned.

2. Vectorial invariant measures associated with an iterated function system

Now, we will get a generalization of the results given in [1]**,** working with vectorial measures. We will work in the following frame :

- (T, d) is a compact metric space, as before;
- X is a Hilbert space;
- $\boldsymbol{L}(\mathrm{X})$ is the set of linear and continuous operators on X; for $H \in \boldsymbol{L}(\mathrm{X})$, we denote by $\|H\|$ the operatorial norm;

- cabv(X) is the set of σ - additive measures, defined on the Borel subsets of T, taking values in X, having bounded variation;
- for any *i* between 1 and *N*, $\omega_i : T \to T$ is a contraction of ratio r_i;
- for any *i* between 1 and *N*, $R_i \in \boldsymbol{L}(X)$.

One can define the operator H : cabv(X) $\to$ cabv(X), $H(\mu) = \sum_1^N R_i \circ \omega_i(\mu)$, that is: $H(\mu)(A) = \sum_1^N R_i(\mu(\omega_i^{-1}(A)))$, for any Borel set $A \subset T$ (we will see that, for $\mu \in$ cabv(X), $H(\mu) \in$ cabv(X)).

We recall that, for any Borel subset *A* of T, the variation of *A* is : $|\mu|(A) = \sup\{\sum \|\mu(A_i)\|\}$, the supremmum being computed with respect to all the partitions of *A* with finite families of Borel sets.

If $|\mu|(T) < \infty$, we say that the measure μ has *bounded variation.* In this case, denoting by $\|\mu\| = |\mu|(T)$, the application $\|.\|$ is a norm on cabv(X) and (cabv(X), $\|.\|$) is a Banach space.

Remark. We can work with a close subset **A** of cabv(X), but only if the condition $H(\mathbf{A}) \subset \mathbf{A}$ holds.

Theorem 3. Let us consider the Banach space (cabv(X), $\|\ \|$). Then, $H \in \boldsymbol{L}$(cabv(X)) and, besides, $\|H\| \le \sum_{i=1}^N \|R_i\|$.

We will denote, in the following, by *C*(X) the set of continuous functions, defined on T, taking values in X. Let, also, *S*(X) the set of the simple functions defined on T, taking values in X.

Definition 1. Let $f \in S(X)$, $f = \sum_{i=1}^m \varphi_{A_i} x_i$, $A_i \subset T$ being Borel sets and $x_i \in X$. Let $\mu \in$ cabv(X). The number $\sum_{i=1}^m \langle x_i, \mu(A_i) \rangle$ is called the integral of f with respect to μ and is denoted by $\int f d\mu$ (is easy to prove that the value of the integral doesn't depend on the representation of f).

If $f \in$ TM(X) (that is: f is totally measurable) , there is a sequence $(f_n) \subset S(X)$ that converges uniformly to *f*. We define $\int f d\mu = \lim_{n\to\infty} \int f_n d\mu$.

We will use the following definitions and results :

1) $B_a(X) = \{\mu \in \text{cabv(X)}, \|\mu\| \le a\}$;

2) BL(X) = { $f : T \to X, f$ Lipschitz function};

3) On the space BL(X) we have the norm : $\|f\|_{BL} = \|f\|_\infty + \|f\|_L$, where $\|f\|_L$ is the Lipschitz constant of f;

4) $BL_1(X) = \{f \in \text{BL(X)}, \|f\|_{BL} \le 1\}$;

5)We denote by: $\|\mu\|_{MK} = \sup\{\left|\int f d\mu\right|, f \in BL_1(X)\}$.

Lemma 1. (change of variable formula) For any f $\in C$(X), we have $\int f dH(\mu) = \int g d\mu$, where g = $\sum_{i-1}^N R_i^* \circ f \circ \omega_i$.

Lemma 2. a) The application $\|.\|_{MK}$ is a norm on cabv(X) and we have : $\|\mu\|_{MK} \le \|\mu\|, \forall \mu \in \text{cabv(X)}$;

b) Let a>0. If X = K^n (where K = **R** or **C**) then the topology generated by $\|\mu\|_{MK}$ is the same with the weak-* topology on $B_a(K^n)$.

c) $B_a(K^n)$ equipped with the metric generated by $\|\ \|_{MK}$ is a compact metric space, and, consequently, is complete.

Definition 2. The norm defined in lemma 2 is called the *Monge-Kantorovich norm.*

A) THE CASE X= K^n

Theorem 4. a) Let us consider $f \in BL_1(\text{X})$. Let $g = \sum_{i=1}^{N} R_i^* \circ f \circ \omega_i$, as in Lemma 1. Then $\|g\|_L \le \sum_{i=1}^{N} \|R_i\| r_i$;

b)Let H as before. We consider the space (cabv(X), $\|\ \|_{MK}$). Then $H \in \boldsymbol{L}(\text{cabv(X)})$ and $\|H\| \le \sum_{i=1}^{N} \|R_i\|(1+r_i)$.

Proof a) $\forall x, y \in \text{T}$, we have :

$$\|g(x)-g(y)\| = \left\| \sum_{i=1}^{N} R_i^* (f(\omega_i(x)) - f(\omega_i(y))) \right\| \le \sum_{i=1}^{N} \|R_i\| \|\omega_i(x) - \omega_i(y)\| \le (\sum_{i=1}^{N} \|R_i\| r_i) d(x,y).$$

From these relations, we get $\|g\|_L \le \sum_{i=1}^{N} \|R_i\| r_i$.

b) $\forall t \in \text{T}$, $\|g(t)\| \le \sum_{i=1}^{N} \|R_i^*\| \|f(\omega_i(t)\| \le \sum_{i=1}^{N} \|R_i^*\| = \sum_{i=1}^{N} \|R_i\|$. Hence $\|g\|_\infty \le \sum_{i=1}^{N} \|R_i\|$. Adding this inequality to the one from a),we get: $\|g\|_{BL} \le \sum_{i=1}^{N} \|R_i\|(1+r_i)$. Using Lemma 1, for any $f \in BL_1(\text{X})$, we can write : $\left| \int f dH(\mu) \right| = \left| \int g d\mu \right| \le \|g\|_{BL} \|\mu\|_{MK} \le (\sum_{i=1}^{N} \|R_i\|(1+r_i)) \|\mu\|_{MK}$. Taking the supremmum with $f \in BL_1(\text{X})$, we obtain $\|H(\mu)\|_{MK} \le (\sum_{i=1}^{n} \|R_i\|(1+r_i)) \|\mu\|_{MK}$, and this proves b).

Theorem 5. Let us suppose that the hypothesis of Theorem 4 are fulfilled and $\sum_{i=1}^{N} \|R_i\|(1+r_i)$<1. Let a>0, $\mu^0 \in$ cabv(X) ; we define P:cabv(X)$\to$cabv(X), $P(\mu) = H(\mu) + \mu^0$. Let $\mathbf{A} \subset B_a(\text{X})$, not empty, weak-* close, such that $P(\mathbf{A}) \subset \mathbf{A}$. We denote by Π the restriction of P to **A**. Then, there is a single measure $\mu^* \in \mathbf{A}$, such that $\Pi(\mu^*) = \mu^*$. If $\mu^0 = \Theta$, (the measure zero) then $\mu^* = \Theta$. (the proof uses Theorem 4 and Lemma 2)

B) THE CASE X INFINITE DIMENSIONAL HILBERT SPACE

Theorem 6. Let us consider the space (cabv(X), $\|\ \|$) and suppose that $\sum_{i=1}^{N} \|R_i\|$<1. Let $\mu^0 \in$cabv(X). We define P : cabv(X)$\to$cabv(X), $P(\mu) = H(\mu) + \mu^0$.Then, P is a contraction

with the ratio r < $\sum_{i=1}^{N} \|R_i\|$. Consequently, there is a single measure $\mu^* \in$cabv(X) such that $P(\mu^*) = \mu^*$.

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DOUBLY WARPED PRODUCTS IN *S*-SPACE FORMS

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Abstract: Recently, the author established a general inequality for doubly warped products in arbitrary Riemannian manifolds [9]. In the present paper, we obtain a similar inequality for doubly warped products isometrically immersed in *S*-space forms. As applications, we derive certain obstructions to the existence of minimal isometric immersions of doubly warped product integral submanifolds in *S*-space forms.
Mathematics Subject Classification (2010): 53C40, 53C25.
Key words: doubly warped product, warping functions, integral submanifold, *S*-space form.

1. Introduction

1.1 Doubly warped products. Let (M_1, g_1) and (M_2, g_2) be two Riemannian manifolds and let $f_1 : M_1 \to (0,\infty)$ and $f_2 : M_2 \to (0,\infty)$ be differentiable functions.

The *doubly warped product* $M =_{f_2} M_1 \times_{f_1} M_2$ is the product manifold $M_1 \times M_2$ endowed with the metric $g = f_2^2 g_1 + f_1^2 g_2$. More precisely, if $\pi_1 : M_1 \times M_2 \to M_1$ and $\pi_2 : M_1 \times M_2 \to M_2$ are natural projections, the metric g is defined by $g = (f_2 \circ \pi_2)^2 \pi_1^* g_1 + (f_1 \circ \pi_1)^2 \pi_2^* g_2$.

The function f_1 and f_2 are called *warping functions*. If either $f_1 \equiv 1$ or $f_2 \equiv 1$, but not both, then we obtain a warped product. If both $f_1 \equiv 1$ and $f_2 \equiv 1$, then we have a Riemannian product manifold. If neither f_1 nor f_2 is constant, then we have a *non-trivial* doubly warped product [10].

Let $x :_{f_2} M_1 \times_{f_1} M_2 \to \tilde{M}$ be an isometric immersion of a doubly warped product ${}_{f_2} M_1 \times_{f_1} M_2$ into a Riemannian manifold $\tilde{M}$. We denote by *h* the second fundamental form of *x* and by $H_i = \frac{1}{n_i} trace\, h_i$ the partial mean curvatures, where *trace* h_i is the trace of *h* restricted to M_i and $n_i = \dim M_i \, (i = 1,2)$. The immersion *x* is said to be *mixed totally geodesic* if $h(X, Z) = 0$, for any vector fields *X* and *Z* tangent to D_1 and D_2, respectively, where D_i are the distributions obtained from the vectors tangent to M_i.

1.2 Motivation of the study. Recently, in [8], the present author studied warped product submanifolds in generalized Sasakian space forms. In [6], M. K. Dwivedi and J.-S. Kim considered warped product submanifolds in *S*-space forms. In [9], the present author established a general inequality for arbitrary isometric immersions of doubly warped product manifolds in arbitrary Riemannian manifolds.

In Section 3, we present a similar inequality for doubly warped products in *S*-space forms.

On the other hand, the concept of framed metric structure unifies the concepts of almost Hermitian and almost contact metric structures. In particular, an *S*-structure

generalizes Kaehler and Sasakian structure. In [2], Blair discusses principal toroidal bundles and generalizes the Hopf fibration to give a canonical example of an *S*-manifold playing the role of complex projective space in Kaehler geometry and the odd-dimensional sphere in Sasakian geometry. An *S*-manifold of constant *f*-sectional curvature *c* is called an *S-space form* $\tilde{M}(c)$ [4], which generalizes the complex space form and Sasakian space form.

1.3 Basic definitions and formulas. Let M be a Riemannian *n*-manifold isometrically immersed in a Riemannian *m*-manifold $\tilde{M}^m$. We choose a local field of orthonormal frame $e_1,....,e_n,e_{n+1},...,e_m$ in $\tilde{M}^m$ such that, restricted to *M*, the vectors $e_1,....,e_n$ are tangent to *M* and $e_{n+1},...,e_m$ are normal to *M*.

Let $K(e_i \wedge e_j)$, $1 \le i < j \le n$, denote the sectional curvature of the plane section spanned by e_i and e_j. Then the scalar curvature of *M* is given by $\tau = \sum_{1 \le i < j \le n} K(e_i \wedge e_j)$. The equation of Gauss is given by

$$\tilde{R}(X,Y,Z,W) = R(X,Y,Z,W) + g(h(X,W),h(Y,Z)) - g(h(X,Z),h(Y,W)),$$

for any vectors *X, Y, Z, W* tangent to *M,* where *h* is the second fundamental form and *R* the Riemann curvature tensor of *M*.

The mean curvature vector *H* is defined by $H = \frac{1}{n} trace h = \frac{1}{n}\sum_{i=1}^{n} h(e_i,e_i)$. As is known, *M* is said to be *minimal* if *H* vanishes identically.

Let *M* be a Riemannian *p*-manifold and $\{e_1,....,e_p\}$ be an orthonormal basis of *M*. For a differentiable function *f* on *M*, the Laplacian Δf of *f* is defined by $\Delta f = \sum_{j=1}^{p} \{(\nabla_{e_j} e_j) f - e_j e_j f\}$.

2. *S*-space forms

Let $\tilde{M}$ be a (*2m+s*)-dimensional *framed metric manifold* [12] with a framed structure $(f, \xi_\alpha, \eta^\alpha, \tilde{g})$, $\alpha \in \{1,...,s\}$, that is, f is a (1,1) tensor field defining a *f*-structure of rank *m*; $\xi_1,...,\xi_s$ are vector fields; $\eta^1,...,\eta^s$ are 1-forms and $\tilde{g}$ is a Riemannian metric on $\tilde{M}$ such that for all $X, Y \in T\tilde{M}$ and $\alpha, \beta \in \{1,...,s\}$

$$f^2 = -I + \eta^\alpha \otimes \xi_\alpha,\ \eta^\alpha(\xi_\beta) = \delta^\alpha_\beta,\ f(\xi_\alpha) = 0,\ \eta^\alpha \circ f = 0,$$

$$\langle fX, fY \rangle = \langle X,Y \rangle - \sum_\alpha \eta^\alpha(X)\eta^\alpha(Y), \quad \Omega(X,Y) \equiv \langle X, fY \rangle = -\Omega(Y,X),\ \langle X, \xi_\alpha \rangle = \eta^\alpha(X),$$

where $\langle , \rangle$ denotes the inner product of the metric $\tilde{g}$. A framed metric structure is an *S*-structure if the Nijenhuis tensor of *f* equals $-2d\eta^\alpha \otimes \xi_\alpha$ and $\Omega = d\eta^\alpha$, for all $\alpha \in \{1,...,s\}$.

When *s=1*, a framed metric structure is an almost contact metric structure, while an *S*-structure is a Sasakian structure. When *s=0*, a framed metric structure is an almost Hermitian structure, while an *S*-structure is a Kaehler structure.

A plane section in $T_p\tilde{M}$ is a *f-section* if there exists a vector $X \in T_p\tilde{M}$ orthogonal to $\xi_1,...,\xi_s$ such that $\{X, fX\}$ span the section. The sectional curvature of a *f*-section is called a *f-sectional curvature*. It is known that [4] in an *S*-manifold of constant *f*-sectional curvature *c*

$$\tilde{R}(X,Y)Z = \sum_{\alpha,\beta}\{\eta^\alpha(X)\eta^\beta(Z)f^2Y - \eta^\alpha(Y)\eta^\beta(Z)f^2X - \langle fX, fZ\rangle\eta^\alpha(Y)\xi_\beta + \tag{2.1}$$

$$+\langle fY, fZ\rangle\eta^\alpha(X)\xi_\beta\} + \frac{c+3s}{4}\{-\langle fY, fZ\rangle f^2X + \langle fX, fZ\rangle f^2Y\} +$$

$$+\frac{c-s}{4}\{\langle X, fZ\rangle fY - \langle Y, fZ\rangle fX + 2\langle X, fY\rangle fZ\}$$

for all $X,Y,Z \in T\tilde{M}$ where $\tilde{R}$ is the curvature tensor of $\tilde{M}$. An *S*-manifold of constant *f*-sectional curvature *c* is called an *S*-space form $\tilde{M}(c)$.

When s=1, an *S*-space form $\tilde{M}(c)$ reduces to a Sasakian space form $\tilde{M}(c)$ [3].

3. Doubly warped product integral submanifolds in *S*-space forms

Let $\tilde{M}$ be an *S*-manifold equipped with an *S*-structure $(f,\xi_\alpha,\eta^\alpha,\tilde{g})$. A submanifold *M* of $\tilde{M}$ is an *integral submanifold* if $\eta^\alpha(X)=0$, $\alpha \in \{1,...,s\}$, for every tangent vector *X*. A submanifold *M* of $\tilde{M}$ is *anti-invariant* submanifold if $f(TM) \subseteq T^\perp M$. An integral submanifold is identical with an anti-invariant submanifold normal to the structure vector fields $\xi_1,...,\xi_s$. In particular case of *s=1*, an integral submanifold *M* of a Sasakian manifold is a *C-totally real submanifold* [11].

Next, we investigate doubly warped product integral submanifolds in an *S*-space form $\tilde{M}(c)$.

Theorem 3.1 *Let x be an integral isometric immersion of an n-dimensional doubly warped product* ${}_{f_2}M_1 \times_{f_1} M_2$ *into a (2m+s)-dimensional S-space form* $\tilde{M}(c)$*.Then:*

$$n_2\frac{\Delta_1 f_1}{f_1} + n_1\frac{\Delta_2 f_2}{f_2} \le \frac{n^2}{4}\|H\|^2 + n_1n_2\frac{c+3s}{4}, \tag{3.1}$$

where $n_i = \dim M_i$, $n = n_1 + n_2$ *and* Δ_i *is the Laplacian operator of* M_i, $(i=1,2)$.

Moreover, the equality case of (3.1) *holds if and only if x is a mixed totally geodesic immersion and* $n_1H_1 = n_2H_2$ *where* H_i*, i=1,2, are the partial mean curvature vectors.*

Remark 3.2 *If either* $f_1 \equiv 1$ *or* $f_2 \equiv 1$, *then the inequality* (3.1) *is exactly the inequality* (4.15) *from* [6] *for warped products.*

Putting *s=1* in (3.1), we have the following

Corollary 3.3 [9] *Let x be a C-totally real isometric immersion of an n-dimensional doubly warped product* ${}_{f_2}M_1 \times_{f_1} M_2$ *into a (2m+1)-dimensional Sasakian space form* $\tilde{M}(c)$ *. Then*

$$n_2\frac{\Delta_1 f_1}{f_1} + n_1\frac{\Delta_2 f_2}{f_2} \le \frac{n^2}{4}\|H\|^2 + n_1n_2\frac{c+3}{4}, \tag{3.2}$$

where $n_i = \dim M_i$ *, and* Δ_i *is the Laplacian operator of* M_i, $(i=1,2)$.

Moreover, the equality case of (3.2) *holds if and only if* x *is a mixed totally geodesic immersion and* $n_1 H_1 = n_2 H_2$ *where* H_i, *i=1,2, are the partial mean curvature vectors.*

As an application, we obtain certain obstructions to the existence of minimal doubly warped product integral submanifolds in *S*-space forms.

Corollary 3.4 *Let* ${}_{f_2}M_1 \times_{f_1} M_2$ *be a doubly warped product whose warping functions are harmonic. Then* ${}_{f_2}M_1 \times_{f_1} M_2$ *admits no minimal integral immersion into an S- space form* $\tilde{M}(c)$ *with c<-3s.*

Corollary 3.5 *If the warping functions* f_1 *and* f_2 *of a doubly warped product* ${}_{f_2}M_1 \times_{f_1} M_2$ *are eigenfunctions of the Laplacian on* M_1 *and* M_2, *respectively, with corresponding eigenvalues* $\lambda_1 > 0$ *and* $\lambda_2 > 0$, *respectively, then* ${}_{f_2}M_1 \times_{f_1} M_2$ *admits no minimal integral immersion into an S-space form* $\tilde{M}(c)$ *with* $c \le -3s$.

Corollary 3.6 *Let* ${}_{f_2}M_1 \times_{f_1} M_2$ *be a doubly warped product. If one of the warping functions is harmonic and the other one is an eigenfunction of the Laplacian with corresponding eigenvalue* $\lambda > 0$, *then* ${}_{f_2}M_1 \times_{f_1} M_2$ *admits no minimal integral immersion into an S-space form* $\tilde{M}(c)$ *with* $c \le -3s$.

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DYNAMICAL AND STATISTICAL PREDICTIONS OF CO_2 LEVELS FOR INDOOR POLLUTION

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Abstract: This work is about how to apply existing theories to analyze and predict evolution of indoor pollutants. More specific, indoor levels of CO_2 measured at a specific time stamps are analyzed and predicted by dynamical and statistical methods.
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Key words: Numerical chaos, ARIMA model, pollution

1. Introduction

Indoor measurements of CO_2 levels between 11/1/2010 0:00 and 10/30/2011 23:50 are at our disposal with a 10 min. time stamp. That leads to an amount of 52416 data as in the following picture:

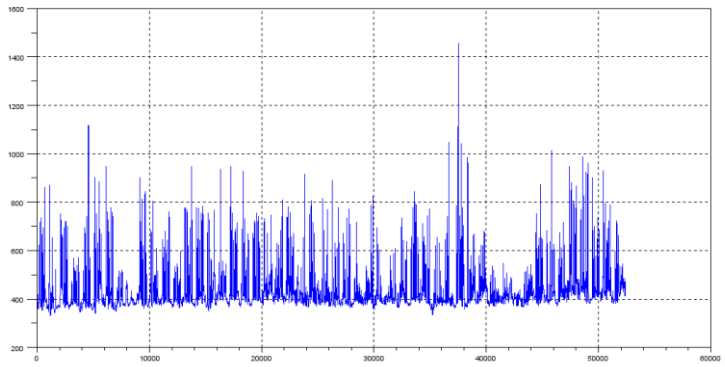

Fig 1

The problem is to find some patterns in the data an to predict the evolution for the next few moments. For outdoor air pollution the problem is studied from many points of view (see [1], [2] but many other more comprehensive works are disponible in the literature). Indoor air pollution is in the state of beginning. In this work we study the possibilty to predict the evolution of CO_2 by dynamical and statistical techniques using the same tequniques as for outdoor pollution.

2. Mathematical preliminaries

The idea of dynamical study is that we have a deterministic but complicated evolution in the form of a dynamical system

$$\bar{x}' = \bar{X}(\bar{x}) \quad (1)$$

with $\bar{x}$ belonging to a finite or infinite normed space E. After a certain amount of time the orbit stabilizes around an irreducible attractive set A of finite Hausdorff dimension n. A theorem of F. Takens or another of E. Manes(see [3]) says that the evolution can be captured by the values of the sequence $f(\bar{x}_0), f(\bar{x}_1), f(\bar{x}_2), \ldots f(\bar{x}_k), \ldots$ for almost any function $f : A \to R \quad \bar{x}_0 \in A,\ \bar{x}_1 = \Phi(\bar{x}_0, \tau),\ \ldots\ x_k = \Phi(\bar{x}_{k-1}, \tau) \ldots$, where Φ is the flux $\Phi : E \times R \to E$ of the dynamical system (1). In our case f is the level of CO_2 and the sequence $f(\bar{x}_0), f(\bar{x}_1), f(\bar{x}_2), \ldots f(\bar{x}_k), \ldots$ is the sequence of measured levels. More precisely the theorem of F. Takens says that for $m \geq 2n+1$ where n is the Hausdorff dimension of A, the application $S : A \to R^m, \quad S(\bar{y}) = (f(\bar{y}_0), f(\bar{y}_1), \ldots f(\bar{y}_{m-1}))$ where $\bar{y}_0 = \bar{y}, \bar{y}_1 = \Phi(\bar{y}_0, \tau), \ldots \bar{y}_k = \Phi(\bar{y}_{k-1}, \tau)$ is an embedding of A into R^m. For a given τ this is true for almost any f. Because A is an irreductible attractor the set $(\bar{x}_n)_{n \in N}$ is dense in A. For enough large N and $\varepsilon > 0$ any point of A is ε close to a point of the finite set $\{\bar{x}_0, \bar{x}_1, \ldots \bar{x}_N\}$. That means the set $\{(f(\bar{x}_k), f(\bar{x}_{k+1}), \ldots f(\bar{x}_{k+m-1})), 0 \leq k \leq N-m\}$ for large N is a good approximation of $S(A) \subset R^m$ and this is the basis of prediction for the next p values $(f(\bar{x}_{m+i}))_{1 \leq i \leq p}$. For details see [3] or a short version in [1].

Concerning statistical predictions we consider the measured values as empirical realisations of random variables $X_1, X_2, \ldots X_N$. After the extraction of the tendency we consider residuals connected in an ARIMA process (see [5]). The predictions are given by continuation of the tendency and ARIMA process after the last measured value. We skip these details which reader may found in classical books as [5].

3. Methods and results

To realize numerically the procedures decribed in section [2] we used the program R and particulary the TISEAN package (see [4]} and ARIMA modelling routines. The reason to use a dynamical modelling may be summarized as follows:

a. If the concentrations are governed by a system of PDE, then by discretizations we get a system of ODE.

b. If we think in an engineering fashion assume the indoor space is divided into a number of cells. Some transfer coefficients for CO_2 exist between neighboring cells and some sources exist in some cells. The evolution of concentrations in each cell depends on the values on the neighboring cells and this hypothesis leads to an unknown system of ODE.

So the description of the concentrations of CO_2 by a system of ODE is based on some assumptions and only the concordance of the consequences with practical observations motivates these assumptions.

In the next figure we see how dynamical prediction is in accordance with practical measurements. We used 50000 data to predict next 1000 data.

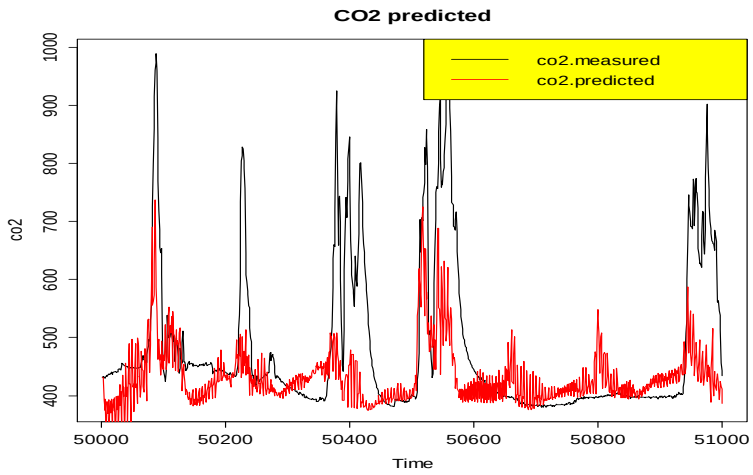

Fig. 2 CO_2 prediction

We found that among the best predictions are those obtained with the embedding dimension m=200 with a delay $\tau = 50\,\text{min}$ or five times the time stamp of 10 min. That perhaps means the number of unknowns of the unknown system of ODE is around 100. That's all we can tell. For a given place we can empirically find the parameters which give better predictions inside a given set of data and use them to predict next future values. In our case we predicted 1000 next values (approx. a week) based on a year mesured values.

For statistical predictions we used the first 50000 data to buld a model and the next 1000 data to compare the predictions given by the model with the actual data. The Holt-Winters procedure was used to extract the tendency with a period T=1008 which corresponds to a week for a delay of 10 min between measurements. The residuals were modelled as ARIMA(2,0,2). A graphical comparison between predicted data and measured data we can see in the next figure.

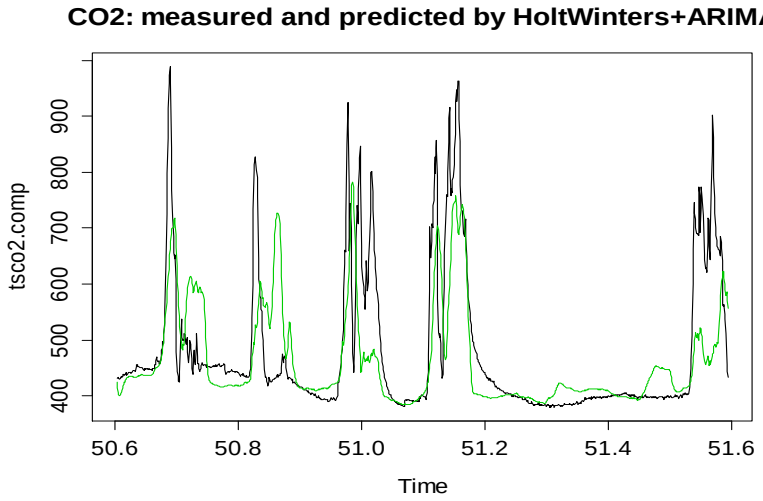

Fig. 3
The unit of time on the horizontal axis is a week

.

4. Conclusions

1. Dynamical and statistical methods give comparable results.
2. For a given place by numerical experiments we can find parameters for the mathematical model (dynamical or statistical) to use for predictions. If we change the location of the measurement then new parameters has to be found.

3. For indoor pollution it needs to take into account new parameters as human presence and possible new kind of modelisation.

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PARTICLE SYSTEMS IN QUASIHOMOGENEOUS FIELDS

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Abstract: The quasihomogeneous fields appear in problems of nonlinear particle dynamics belonging mainly to physics and astronomy. In this frame, we distinguish some classes of post-Newtonian models: Schwarzschild, Fock, Seeliger, Manev, gravito-elastic etc. We present some results in the study of the quasihomogeneous potentials, which cover all the above models and many others.
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Key words: *n*-body problem, quasi-homogeneous fields, collision

1. Introduction

The *n*-body problem of celestial mechanics is to determine the possible motions of *n* point particles of masses $m_i > 0,\ i = \overline{1,n}$, which attract each other according to Newton's law of inverse squares. Although the one and two body problems were completely solved by the time of Newton by means of elementary functions, no similar solution to the *n*-body problem exists for $n \geq 3$. The works of Euler, Clairaut, d'Alembert and others in the XVIII century constituted the Newtonian *n*-body problem as the description of solutions of a system of ordinary differential equations. The problem was given a major impulse when Lagrange transformed mechanics and dynamics into a branch of mathematical analysis, laying the foundations of differential geometry. The *n*-body problem is linked to questions such as the nature of universal attraction and the stability of the Solar System.

The study of *n*-body problem in other fields than Newtonian one constitutes a challenge. The goal is to determine the properties of the Keplerian motion which are preserved within a new framework. We are interested to emphasize the non-Keplerian motions which are characteristic to such models.

Many concrete physical and astronomical phenomena can be studied via the mathematical model of the *n*-body problem associated to quasihomogeneous potentials. Newton was the first to study a quasihomogeneous model in classical celestial mechanics. He considered a gravitational force deriving from a potential of the form

$$U = A|\mathbf{q}|^{-1} + B|\mathbf{q}|^{-2}$$

This model was resumed by Clairaut. Other models based on a slightly modified gravitational law were proposed by Hall and Newcomb. They considered potentials of the form

$$U = A|\mathbf{q}|^{-n},\ n = 1 + \varepsilon.$$

Many other potentials were studied (we denote $r = |\mathbf{q}|$):

- Schwarzschild potential

$$U = A/r + B/r^3;$$

- Fock potential

$$U = \sum_{n=1}^{4} A_n / r^n;$$

- Manev potential

$$U = A/r + B/r^2$$

- Gravito-elastic potential

$$U = A/r + Br^2.$$

In our days, Delgado et al. ([1]), Diacu et al. ([2]), Mioc and Stoica ([9]) or Mioc and Stavinschi ([7], [8]) have considered general quasihomogeneous models, which cover all the above quoted models and many others. A quasihomogeneous potential has the form of a sum of homogeneous potentials:

$$U(\mathbf{q}) = \sum_{k=1}^{N} U_k(\mathbf{q}), \quad U_k(\mathbf{q}) = \frac{A_k}{|\mathbf{q}|^{\gamma_k}}, \tag{1.1}$$

where the parameters A_k have different analytical expressions according to the field they characterize (but they depend neither on **q**, nor explicitly on time), γ_k are real numbers

$$\gamma_k < \gamma_{k+1}, \quad k = \overline{1, N-1},$$

whereas **q** stands for the radius vector of one particle with respect to another in the force field generated by this potential. As far as our knowledge goes, potential *U* is much more general than the above quoted ones for the following reasons: (i) γ_k may run all along the real axis; (ii) such a model allows the study of particle dynamics under hybrid forces of totally different nature; (iii) we also consider here the case $N = \infty$ for generality, even if in studies of above concrete situations is finite. Thus, our results provide a unifying viewpoint (physical and mathematical) for a lot of problems of particle dynamics.

2. Equations of motion

Let us consider a system of *n* interacting particles $m_i > 0$, $i = \overline{1, n}$. Let $\mathbf{r}_i = (x_i, y_i, z_i) \in \mathbf{R}^3$ be their position vectors with respect to an arbitrary origin and $\mathbf{r} = (\mathbf{r}_1, \mathbf{r}_2, ..., \mathbf{r}_n) \in \mathbf{R}^{3n}$ be the configuration of the system. Let the motion of the system be ruled by a quasi-homogeneous force deriving from a potential function of the form

$$U(\mathbf{r}) = \sum_{k=1}^{N} U_k(\mathbf{r}), \quad U_k(\mathbf{r}) = \sum_{1 \le i < j \le n} \frac{A_{k,ij}}{r_{ij}^{\gamma_k}}. \tag{2.1}$$

Here $U_k : (\mathbf{R}^{3n} \setminus \Delta) \to \mathbf{R}$ for $\gamma_k > 0$, and $U_k : \mathbf{R}^{3n} \to \mathbf{R}$ for $\gamma_k \le 0$; $r_{ij} = |\mathbf{r}_i - \mathbf{r}_j|$; $\Delta = \bigcup_{1 \le i < j \le n} \{\mathbf{r} \,|\, \mathbf{r}_i = \mathbf{r}_j\}$ is the collision set, whereas $A_{k,ij} : \mathbf{R}^2 \to \mathbf{R}$, are symmetric functions (mainly of masses, but not only, as we shall see in the last section): $A_{k,ij} = A_{k,ji}$.

The dynamics of this *n*-body system in such a field is described by the vectorial equation

$$m_i\ddot{\mathbf{r}}_i = \frac{\partial U(\mathbf{r})}{\partial \mathbf{r}_i} = -\sum_{1\le i<j\le n}(\mathbf{r}_i-\mathbf{r}_j)\sum_{k=1}^{N}\gamma_k\frac{A_{k,ij}}{r_{ij}^{\gamma_k+2}}. \tag{2.2}$$

To be able to tackle the case $N=\infty$ too, we state that the series of functions $\sum_{k=1}^{\infty}\gamma_k A_{k,ij}/r_{ij}^{\gamma_k+2}$ converges uniformly on $\mathbf{R}^{3n}\setminus\Delta$. Because the series $\sum_{k=1}^{\infty}A_{k,ij}/r_{ij}^{\gamma_k}$ is simply convergent to $U(\mathbf{r})$ and the series of derivatives $\sum_{k=1}^{\infty}\gamma_k A_{k,ij}/r_{ij}^{\gamma_k+2}$ is uniformly convergent, then, by the Theorem of differentiation term by term of the series of functions, the series of derivatives tends to $\partial U(\mathbf{r})/\partial\mathbf{r}_i$ and is continuous on $\mathbf{R}^{3n}\setminus\Delta$.

Denoting $\mathbf{q}_i=\mathbf{r}_i$, $\mathbf{q}=\mathbf{r}$ (the configuration vector), $\mathbf{p}_i=m_i\dot{\mathbf{r}}_i$, $\mathbf{p}=(\mathbf{p}_1,\mathbf{p}_2,...,\mathbf{p}_n)\in\mathbf{R}^{3n}$ (the momentum vector), and defining $H(\mathbf{q},\mathbf{p})=T(\mathbf{p})-U(\mathbf{q})$ as the Hamiltonian function (where *T* is the kinetic energy), equations (2.2) can be transposed into a canonical form.

Results of the theory of differential equations ensure, for given initial conditions $(\mathbf{r},\dot{\mathbf{r}})(t=0)$, the existence and uniqueness of an analytic solution of the system (2.2), defined on an interval (t^-,t^+), $t^-<0<t^+$. This can be analytically extended to a maximal interval $(\tilde{t}^-,\tilde{t}^+)$, $-\infty\le\tilde{t}^-\le t^-<0<t^+\le\tilde{t}^+\le+\infty$. If $\tilde{t}^{\pm}=\pm\infty$, the solution is regular; else, it encounters a singularity.

There is no difficulty to prove that ten classical first integrals exist for the system (2.2): the integrals of momentum $\sum_{i=1}^{n}m_i\dot{\mathbf{r}}_i=\boldsymbol{\alpha},\boldsymbol{\alpha}\in\mathbf{R}^3$; the integrals of mass centre $\sum_{i=1}^{n}m_i\mathbf{r}_i-(\sum_{i=1}^{n}m_i\dot{\mathbf{r}}_i)\,t=\boldsymbol{\beta},\boldsymbol{\beta}\in\mathbf{R}^3$; the integrals of angular momentum $\sum_{i=1}^{n}(m_i\mathbf{r}_i\times\dot{\mathbf{r}}_i)=\mathbf{C},\quad\mathbf{C}\in\mathbf{R}^3$, and the integral of energy $T(\dot{\mathbf{r}})-U(\mathbf{r})=h,\quad h\in\mathbf{R}$, where $\boldsymbol{\alpha},\boldsymbol{\beta},\mathbf{C}$ and *h* are integration constants. In the last relation, the kinetic energy of the system has the expression

$$T:\mathbf{R}^{3n}\to[0,+\infty),\quad T(\dot{\mathbf{r}})=\frac{1}{2}\sum_{i=1}^{n}m_i|\dot{\mathbf{r}}_i|^2.$$

The moment of inertia $J(\mathbf{r})$ of the system is defined by $J(\mathbf{r})=\frac{1}{2}\sum_{i=1}^{n}m_i|\mathbf{r}_i|^2$.

The moment of inertia represents a physical measure of the distribution (scattering) of the bodies (particles) in space. The following theorem is called the relation Lagrange-Jacobi.

THEOREM 2.1. *In the n-body problem associated to a quasi-homogeneous field, the following relation holds:*

$$\ddot{J}(\mathbf{r})=\sum_{k=1}^{N}(2-\gamma_k)U_k(\mathbf{r})+2h,$$

where $\ddot{J}(\mathbf{r})$ *is the second derivative of* $[J(\mathbf{r})]$ *with respect to the time.*

Within the Newtonian model, the inequalities of Sundman connect the moment of inertia and the angular momentum (of course, under the respective potential).

THEOREM 2.2. *In a quasi-homogeneous field, the following inequality holds:*

$$|\mathbf{C}|^2 \le 2J(\mathbf{r})[\ddot{J}(\mathbf{r}) + \sum_{k=1}^{N} \gamma_k U_k(\mathbf{r})].$$

These are the analogs of the Lagrange-Jacobi relation and Sundman's inequalities corresponding to our much more general framework.

3. Nature of singularities

Due to the time-reversibility of motion equations, we can study the properties of the solution only in the past **,** $(t^*,0]$, or only on $[0,t^*)$, in the future. We shall focus on the motion in the past $(t^*,0]$**.** We denote $\Delta(r) := \min\{r_{ij} | 1 \le i < j \le n\}$**.** Then *t** is a singularity of the solution of the equation (2.2) with $U(\mathbf{r})$ given by (2.1), defined on the maximal interval $(t^*,0]$**,** if and only if $\lim_{t \to t^*} \Delta(r(t)) = 0$**.** The following results are valid also for $N = \infty$.

A singularity t* is due to a *collision* if **r** has a finite limit when $t \to t^*$. *t** will be called a *collision singularity*, else we shall call a *noncollision singularity*.

THEOREM 3.1. *In the n-body problem associated to the potential (2.1) singularities may appear only if at least one of the coefficients* $A_{N,ij}$ *is nonnegative.*

THEOREM 3.2. *Let n=3 and consider (r, r') be a solution of the equation (2.2) defined on* $(t^*,0]$ *with* $t^* > -\infty$. *Then the singularity t* is due to a collision.*

THEOREM 3.3. *The necessary and sufficient condition for the simultaneous total collision is* $J(r(t)) \to 0$ *when* $t \to t^*$.

THEOREM 3.4. *No solution of the motion equations leads to simultaneous total collision in infinite time.*

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A PARTICULAR NEWELL-WHITEHEAD-SEGEL EQUATION SOLVED BY VARIOUS METHODS

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Abstract In this paper, we consider two methods for finding exact solutions of a particular Newell-Whitehead-Segel (NWS) equation: Laplace Method and Differential Transform Method (DTM). The methods described here can be applied to partial differential equations with variable coefficients arising in physical and engineering applications.
Mathematics Subject Classification (2010): 35E05
Key words: Newell-Withehead-Segel equation, Differential transform method, Taylor series.

1. Introduction

The equation NWS is part of the reaction-diffusion equations (diffusion equation complemented with an additional source term which can be a positive or negative term, usually called the "production" or "source" term). Some concrete physical examples:

Heat equation in a medium with internal heat sources, the variable beeing the temperature T and the source term may be due to (for example) electrical resistance embedded in the environment of tubes through which a hot fluid (or cold), chemical reactions that occur in this material (e.g. "strengthening" of concrete), nuclear reactions (in bars with nuclear fuels), in the materials that undergo radiation occurs a phenomenon of absorption of radiation.

The equation of substance conservation, with applications to problems of pollution or biological problems,in which the variable is the concentration of pollutants (or organisms) with the source term due to chemical reactions (or production, or extinction based on growth laws).

Moreover, such equations appear in electrical and other propagation phenomena.

In this paper, we present two methods for finding approximate and exact solution of a particular NWS equation. The NWS equation is written as:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + au - bu^q \tag{1}$$

Where a, b and k are real numbers with $k > 0,$ and q is a positive integer. In equation (1) the first term on the left hand side, $\frac{\partial u}{\partial t},$ expresses the variations of $u(x,t)$ with time at a fixed location, the first term on the right hand side, $\frac{\partial^2 u}{\partial x^2},$ expresses the variations of $u(x,t)$ with

spatial variable x at a specific time and the remaining terms on the right hand side, $au - bu^q$ takes into account the effect of the source term. In equation (1) $u(x,t)$ is a function of the spatial variable x and the temporal variable t with $x \in R$ and $t \geq 0$. The function $u(x,t)$ may be thought of as the (nonlinear) distribution of temperature in an infinitely thin and long rod or as the flow velocity of a fluid in an infinitely long pipe with small diameter.
In this paper we proposed to solve a particular case of equation (1):

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - 4u\,. \tag{2}$$

with the initial condition:

$$u(x,0) = 6\sin x - 4\sin 2x \tag{3}$$

and the boundary conditions:

$$u(0,t) = u(\pi,t) = 0\,. \tag{4}$$

For starters we solve equation (2) in Section 2 with the Laplace Method, and then in Section 3 with Differential Transform Method.

2. Laplace Method

Solving equation (2) Laplace Method is much faster than the method of separation of variables commonly used in educational applications.

We write $L(u(x,t)) = U(x,p)$ and then $L\left(\frac{\partial u}{\partial t}\right) = \int_0^\infty e^{-pt}\frac{\partial u}{\partial t}dt = pU(x,p) - u(x,0)$.

Problem (2) is transformed into the equation

$$\frac{d^2U}{dx^2} - (p+4)U = -6\sin x + 4\sin 2x \tag{5}$$

with general solution

$$U(x,p) = C_1 e^{\sqrt{p+4}x} + C_2 e^{-\sqrt{p+4}x} + \frac{6}{p+5}\sin x - \frac{4}{p+8}\sin 2x\,. \tag{6}$$

Applying boundary conditions $U(0,p) = 0$ and $U(\pi,p) = 0$, we obtain $C_1 = C_2 = 0$.
So we can write

$$U(x,0) = \frac{6}{p+5}\sin x - \frac{4}{p+8}\sin 2x\,. \tag{7}$$

Applying the inverse transform we obtain

$$u(x,t) = 6e^{-5t}\sin x - 4e^{-8t}\sin 2x\,. \tag{8}$$

3. Differential Transform Method

In [5], Raslan introduced a reliable technique in order to solve partial differential equations with variable coefficients. The technique is called differential transform method, which is based on Taylor series expansion. But, it differs from the traditional high order Taylor series method by the way of calculating coefficients. The technique and construct an analytical solution is in the form of a polynomial. The concept of differential transform was first introduced by Pukhov [4], who solved linear and nonlinear initial value problems in electric circuit analysis. Chen and Ho [3] developed this method for PDEs and obtained closed form series solutions for some linear and nonlinear initial value problems. In the following years, Raslan extended the DTM method to solve partial differential equation with variable coefficients [1].
In this section, we apply the differential transform method to solve equation (2).

To illustrate the basic idea of the DTM, we considered $u(x,t)$ is analytic and differentiated continuously in the domain of interest, then let

$$U_k(x) = \frac{1}{k!}\left[\frac{\partial^k u(x,t)}{\partial t^k}\right]_{t=t_0} \tag{9}$$

where the spectrum $U_k(x)$ is the transformed function, which is called T-function in brief. The differential inverse transform of $U_k(x)$ is defined as follows

$$u(x,t) = \sum_{k=0}^{\infty} U_k(x)(t-t_0)^k \ . \tag{10}$$

Combining (9) and (10), it can obtained that

$$u(x,t) = \sum_{k=0}^{\infty} \frac{1}{k!}\left[\frac{\partial^k u(x,t)}{\partial t^k}\right]_{t=t_0} (t-t_0)^k \ , \tag{11}$$

when t_0 are taken as $t_0 = 0$ then equation (11) is expressed as

$$u(x,t) = \sum_{k=0}^{\infty} \frac{1}{k!}\left[\frac{\partial^k u(x,t)}{\partial t^k}\right]_{t=0} t^k \tag{12}$$

and equation (10) is shown as

$$u(x,t) = \sum_{k=0}^{\infty} U_k(x) t^k \ . \tag{13}$$

In real application, the function $u(x,t)$ by a finite series of equation (13) can be written as

$$u(x,t) = \sum_{k=0}^{n} U_k(x) t^k \ . \tag{14}$$

We intend to solve problem (2)-(4) with initial condition $u(x,0) = 6\sin x - 4\sin 2x$.
By using the basic properties of the reduced differential transformation, we can find the T-function

$$(k+1)U_{k+1}(x) = \frac{d^2 U_k(x)}{dx^2} - 4U_k(x). \tag{15}$$

According to the hypothesis we have that

$$U_0(x) = 6\sin x - 4\sin 2x. \tag{16}$$

Now, substituting equation (16) into (15), we obtain the following $U_k(x)$ values successively

$$\begin{aligned} U_1 &= 6\cdot(-5)\sin x - 4\cdot(-8)\sin 2x \\ U_2 &= \left[6\cdot(-5)^2 \sin x - 4\cdot(-8)^2 \sin 2x\right]\cdot\frac{1}{2} \\ U_3 &= \left[6\cdot(-5)^3 \sin x - 4\cdot(-8)^3 \sin 2x\right]\cdot\frac{1}{2}\cdot\frac{1}{3}. \end{aligned} \tag{17}$$

Then, we get the general form of the T-function

$$U_k = \left[6\cdot(-5)^k \sin x - 4\cdot(-8)^k \sin 2x\right]\cdot\frac{1}{k!}. \tag{18}$$

Finally the differential inverse transform of $U_k(x)$ gives

$$u(x,t) = \sum_{k=0}^{\infty} U_k(x) t^k = 6\cdot e^{-5t}\sin x - 4\cdot e^{-8t}\sin 2x. \tag{19}$$

The exact solution is given as

$$u(x,t) = 6\cdot e^{-5t}\sin x - 4\cdot e^{-8t}\sin 2x. \tag{20}$$

4. Conclusion

The methods described above can be applied to partial differential equations with variable coefficients arising in physical and engineering applications. Laplace Method is a quick and easy to apply. It gives students the opportunity to put into practice the knowledge acquired during the Special Mathematics. The Differential Transform Method has been successfully applied for solving partial differential equations with variable coefficients. The solution obtained by differential transform method is an infinite power series for appropriate initial condition, which can in turn express the exact solutions in a closed form. The results show that the differential transform method is a powerful mathematical tool for solving partial differential equations with variable coefficients. By both methods the same exact solution can be found. Thus, we conclude that above methods can be applied by students who complete the second year or by the engineers.

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ON LAGRANGE MULTIPLIERS METHOD IN TEACHING CALCULUS

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Abstract A short, elementary and elegant proof is supplied for the classical Lagrange Multipliers Method. Since its strong intuitive geometric support, this proof can be successfully be used in a reasonable teaching of Advanced Calculus for students in Mathematics, Engineering, Economy, etc.
Mathematics Subject Classification (2010): 26B12, 26B99, 00A99.
Key words: functions of many variables, Lagrange Multipliers Method, optimal computations.

Let D be a domain in the n-dimensional real vector space $\mathbf{R}^n$ and let $f, g_1, g_2, ..., g_m$ be $m+1$ functions ($m \le n$) defined on D, with real values, and of class $C^1(D)$, i.e. they have continuous partial derivatives on D. Let $\Sigma = \{x \in D : g_i(x) = 0, i = \overline{1,m}\}$ be the set of zeros of the system $g_i(x) = 0, i = \overline{1,m}$ (the constraints) in D. Let $a \in \Sigma$ be a local conditional maximum (say) point for the function f, i.e. there exists an open ball $B(a,r)$ in $\mathbf{R}^n$ such that for any x in $U = B(a,r) \cap \Sigma$, one has $f(x) \le f(a)$. The following theorem is classic and extremely useful in the optimal problem solutions.

Theorem 1 (Lagrange theorem [1]) With the previously introduced notions and hypotheses we also assume that the gradient vectors $\nabla g_1(a), \nabla g_2(a), ..., \nabla g_m(a)$ are linear independent vectors. Then there exist uniquely defined real numbers $\lambda_1, \lambda_2, ..., \lambda_m$ such that $\nabla f(a) = \sum_{i=1}^{m} \lambda_i \nabla g_i(a)$.

Proof The case $m = n$ is trivial, because in this last case $\nabla g_1(a), \nabla g_2(a), ..., \nabla g_m(a)$ generates the entire space $\mathbf{R}^n$. Let now $m < n$. The idea is to take an arbitrary nonzero vector $v \in M^{\perp}$, the orthogonal subspace of the vector space M generated by the gradient vectors $\nabla g_1(a), \nabla g_2(a), ..., \nabla g_m(a)$ and to prove that $\nabla f(a) \cdot v = 0$, i.e. $\nabla f(a) \in \left(M^{\perp}\right)^{\perp} = M$ (see [2]).
Let $P = a + Sp\{M, v\}$, the affine linear variety generated by the vectors $\nabla g_1(a), \nabla g_2(a), ..., \nabla g_m(a)$ and v, and which passes through a. A vector from P looks like: $x = a + \sum_{i=1}^{m} t_i \nabla g_i(a) + t_{m+1} v$, where $t_1, t_2, ..., t_{m+1}$ are free parameters in $\mathbf{R}$. Let us consider the intersection of P with $U = B(a,r) \cap \Sigma$. We shall prove that this intersection contains a parametric curve γ which passes through a and that v is the tangent vector to it at a. Let $x \in P \cap U$. Then the parameters $t_1, t_2, ..., t_{m+1}$ of x are connected by the following constraints: $h_j(t_1, t_2, ..., t_{m+1}) = g_j(a + \sum_{i=1}^{m} t_i \nabla g_i(a) + t_{m+1} v),\ j = \overline{1,n} = 0$. If one computes the functional

determinant $\frac{D(h_1, ..., h_m)}{D(t_1, t_2, ..., t_m)}(a) = \det\left(\nabla g_i(a) \cdot \nabla g_j(a)\right)$, one obtains exactly the Gram determinant of the linear independent set of vectors $\nabla g_1(a), \nabla g_2(a), ..., \nabla g_m(a)$, which is not zero (see [2]). Use now the implicit function theorem around the point $(0, 0, ..., 0) \in \mathbf{R}^{m+1}$ and find that $t_1 = t_1(t_{m+1}), t_2 = t_2(t_{m+1}), ..., t_m = t_m(t_{m+1}), t_{m+1} \in (-\varepsilon, \varepsilon)$, $t_i(0) = 0, t_i'(0) = 0, i = 1,2,...,m.$ Thus, the parametric equation of the searched curve $\gamma \subset P \cap U$, is: $x(t_{m+1}) = a + \sum_{i=1}^{m} t_i(t_{m+1}) \nabla g_i(a) + t_{m+1} v, t_{m+1} \in (-\varepsilon, \varepsilon)$. We see that $x(0) = a, x'(0) = v.$ Let now $F\left(t_{m+1}\right) = f\left(x(t_{m+1})\right), t_{m+1} \in (-\varepsilon, \varepsilon)$ be the restriction of f to the curve γ. Since $F'(0) = \nabla f(a) \cdot v$ and since zero is a local maximum point for F, one use Fermat theorem to obtain that $\nabla f(a) \cdot v = 0,$ i.e. $\nabla f(a) \in \left(M^{\perp}\right)^{\perp} = M$, i.e. the statement of the theorem.

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SOME ASPECTS OF DISCRETIZATION OF TRANSPORT AND DIFFUSION EQUATION

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Abstract: The goal of this work is to present a few discretization methods for the transport and diffusion equation, which is used to describe the process that is taking place in the variation of a concentration for a chemical constituent in a fluid flow. The transport and diffusion equation can be used to model the variation of residual chlorine in a water distribution network. This methods of discretization it will be used to build an algorithm for a program to model the variation of residual chlorine in a water distribution network.
Mathematics Subject Classification (2010): 3904
Keywords: transport and diffusion equation, discretization, model the variation of residual chlorine.

1. Introduction

Development and improvement of public water supply systems represent an objective necessity worldwide. One important aspect of optimization of water distribution network operation is based on developing computer programs for controlling hydraulic parameters and water quality. For that, a mathematical model is essential. To model a fluid flow thru an water distribution network one of the most used equations is the transport and diffusion equation. With this equation we can model the variation of residual chlorine.
The components of the transport and diffusion equation are: the concentration of residual chlorine, time, time distance, fluid velocity, length section, the diffusion constant and the transport constant. To be able to solve this equation we should know the boundary conditions. We tried different methods of discretization starting from initially and boundary conditions.

2. Solving the transport equation and diffusion.

The transport equation and diffusion can be put in the following form:

$$\begin{cases} \dfrac{\partial u}{\partial t} + v\dfrac{\partial u}{\partial x} + k \cdot u - D\dfrac{\partial^2 u}{\partial x^2} = 0 \\ (x,t) \in (0,L) \times (0,\infty) \end{cases} \qquad (1)$$

with: u = concentration of residual chlorine (mg/dm^3);
t = time (s);
x = time distance (m);
v = fluid velocity (m/s);
L = length section (m);
D = diffusion constant;
k = transport constant.

Obs. This equation it is correlate from the point of view of the measure units, also it wasn't used the SI units.

Solving the equation (1) can be done in one unique way, if we have the initial condition *u(x,0)* and the boundary conditions: *u(0,t)* , *u(L,t)* , $\frac{\partial u}{\partial x}(0,t)$, $\frac{\partial u}{\partial x}(L,t)$.

The initial condition *u(x,0)* can be taken constant positive at *x=0* , witch decreases fast at zero if *x > 0.*

As boundary conditions, must be taking:

- first condition: *u(0,t)= constant=0,5* mg /dm^3 (in case of chlorine);
- the second boundary condition: on what follows it analyzes the many variations.

2.1. VERSION 1 :

We know *u(L,t),* and establish a minimum concentration for residual chlorine .In this case the equation (1) can be discretized classic.

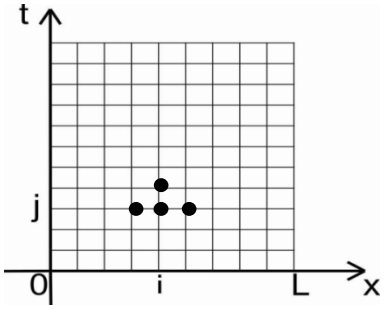

Fig.1: Discretization grid for eq. (1)

On a grid with step *h* on *Ox* and τ on *Ot* (fig. 1), we have in the point

$x_i = (i-1)h$, $i \geq 1$

$t_j = (j-1)\tau$, $j \geq 1$: $$\frac{\partial u}{\partial t}(x_i,t_j) = \frac{u(x_i,t_j) - u(x_i,t_{j-1})}{\tau} + \tau \cdot O(1) \qquad (2)$$

and $$\frac{\partial^2 u}{\partial x^2}(x_i,t_j) = \frac{u(x_{i-1},t_j) - 2u(x_i,t_j) + u(x_{i+1},t_j)}{h^2} + h^2 O(1). \qquad (3)$$

Replace in equation (1), with u_i^j the approximation for $u(x_i, t_j)$ and eliminating the terms of form $\tau \cdot O(1)$ and $h \cdot O(1)$,result :

$\frac{u_i^j - u_i^{j-1}}{\tau} + v\frac{u_i^j - u_i^j}{h} + ku_i^j - D\frac{u_{i-1}^j - 2u_i^j + u_{i+1}^j}{h^2} = 0$, witch after the replacement became: *for i=2,3,…n.*

$$\begin{cases} (-\frac{V\tau}{h} - \frac{D\tau}{h^2})u_{i-1}^j + (1 + \tau \cdot k + \frac{V\tau}{h} + \frac{2D\tau}{h^2})u_i^j + (-\frac{D\tau}{h^2})u_{i+1}^j = u_i^{j-1} \\ u_1^j = dat = const \\ u_n^j = dat = const \end{cases}, \qquad (4)$$

The system (4) is stabile (it have a dominant diagonal) and it can indicate the concentrations thru the length of the pipe in time.

This version it is not sufficiently accurate. We should know how the concentration *u* behave at *x = 0,* to get at destination *x = L,* the concentration we need. But in the process of resolving the system (4), it must be used only the boundary conditions for *x = 0.* Of all this certain is only *u(0,t).*

2.2 VERSION 2 :

Because in practice, the global diffusion coefficient has an approximate value: $D \approx 10^{-5}$ (in SI), it is expected that the contribution of the term $D\frac{\partial^2 u}{\partial x^2}$ to be insignificant. In these conditions, we can neglect this term, leading to a transport equation:

$$\begin{cases} \dfrac{\partial u}{\partial t} + v\dfrac{\partial u}{\partial x} + ku = 0 \\ u(0,t) = dat = u_0(x) \\ u(x,0) = dat = \varphi_1(t) \end{cases} \quad \text{with } v>0 \text{ , } v=\text{constant and } k>0 \text{ , } k=\text{constant.} \qquad (5)$$

Equation (5) it can be explicit resolved, like it is been shown in the following:

$$\frac{\partial u}{\partial t} + v\frac{\partial u}{\partial x} + ku = 0 \quad \text{with} \quad (x,t) \in (0,L)\times(0,\infty). \qquad (6)$$

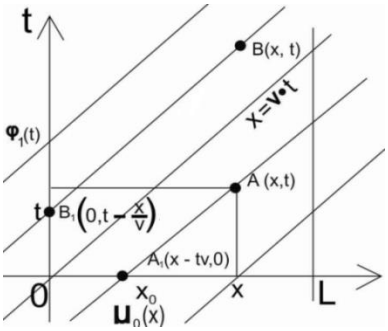

Fig. 2: Discretization grid for eq. (6)

Are given the lines $x=vt+b$ with $b \in \Re$, $v > 0$, in the plane xOt . If $u(x,t)$ is the solution of the transport equation (1), then along the parameterized line we have :

$$\frac{d}{dt}u(x(t),t) = \frac{\partial u}{\partial t}(x(t),t) + \frac{\partial u}{\partial x}(x(t),t)\frac{dx(t)}{dt} = \frac{\partial u}{\partial t} + v\frac{\partial u}{\partial x} = -k\cdot u(x(t),t) \qquad (7)$$

So : $\dfrac{d}{dt}u(x(t),t) = -k\cdot u(x(t),t)$.

For a point $A(x,t)$ like in figure 2, the intersection between a line $x=vt+b$ and the ax Ox it is a point $A_1(x_0,t_0) = A_1(x-tv,0)$.

For a point $B_1(x,t)$, the intersection between a line $x=vt+b$, witch cross thru B , and the line $t=0$ it is a point $B_1(x_0,t_0) = B_1(0,t-\dfrac{x}{v})$.

From equation $\dfrac{du(x(t),t)}{dt} = -k\cdot u(x(t),t)$ result $u(x(t),t) = u(x(t_0),t_0)\cdot e^{-k(t-t_0)}$. (8)

So, for a point $A(x,t)$, with $t < x/v$, placed beneath the line $x=tv$ (like in the figure 2), we have : $u(x,t) = u(A_1)\cdot e^{-k(t-t_0)} = u_0(x-tv)\cdot e^{-kt}$. (9)

For a point $B(x,t)$,with $t>x/v$,we have: $u(x,t) = u(B_1)\cdot e^{-k(t-t_0)} = \varphi_1(t-\dfrac{x}{v})\cdot e^{-k(\frac{x}{v})}$. (10)

So, the exact solution for problem (6), is : $u(x,t) = \begin{cases} u_0(x-tv)\cdot e^{-kt}, daca & x-tv \geq 0 \\ \varphi_1(t-\dfrac{x}{v})\cdot e^{-kx/v}, daca & x-tv < 0 \end{cases}$ (11)

2.3. VERSION 3

Another version it is provide by the attempt to separate the process of transport and diffusion, on the interval $(t , t + dt)$. In this way, we consider that initially is taking place a process of transport witch is unique determinate by the concentration u at time t and the concentration at beginning $\phi_1(t)$. This process it is followed by an process of diffusion witch doesn't change the ending concentrations. There for, the process of transport and diffusion it is, on the same interval, split in 2 processes that came one after the other.

We tried to resolve it, on the interval $(t, t + \alpha\cdot t)$.

First, we resolve the equation:

$$\begin{cases} \dfrac{\partial u}{\partial t} + V\dfrac{\partial u}{\partial x} + k \cdot V = 0, x \in [0,L], t \in [t_{j-1}, t_{j+1}] \\ u(x, t_{j-1}) = cunoscut \\ u(0,t) = \varphi_1(t) \end{cases} \quad (12)$$

where $u(x,t_{j-1})$ it is known from the previous estimation (at $j=2$ we have $u(x,t_1) = u_0(x)$).

The solution to this problem it is given in 2nd version.

Let $\tilde{u}(x,t)$ be the solution for this problem.

In the second part, we tried to resolve the dispersion equation:

$$\begin{cases} \dfrac{\partial u}{\partial t} - D\dfrac{\partial^2 u}{\partial x^2} = 0, x \in [0,L], t \in [t_{j-1}, t_j] \\ u(x, t_{j-1}) = \tilde{u}(x, t_j) \\ u(0,t) = \tilde{u}(0,t) \\ u(L,t) = \tilde{u}(L,t) \end{cases}, \quad (13)$$

where $\tilde{u}(x,t)$ is the solution from the previous part.

The value resulting from (13), $\approx{u}(x,t)$ is for $t = t \cdot j$ the result for this two processes taken separately.

We have $\dfrac{u_i^j - u_i^{j-1}}{\tau} - D\dfrac{u_{i-1}^j - 2u_i^j + u_{i+1}^j}{h^2} = 0$ and $u_i^j(1+2D\dfrac{\tau}{h^2}) - D\dfrac{\tau}{h^2}u_{i-1}^j - D\dfrac{\tau}{h^2}u_{i+1}^j = u_i^{j-1}$.

$$\begin{cases} u_1^j = \tilde{u}(0,\tau) \\ -D\dfrac{\tau}{h^2}u_{i-1}^j + (1+2D\dfrac{\tau}{h^2})u_i^j - D\dfrac{\tau}{h^2}u_{i+1}^j = \tilde{u}(x_i,\tau), \text{ for } i = 2,3,...,n-1 \\ u_n^j = \tilde{u}(L,\tau) \end{cases} \quad (14)$$

3. Conclusions

These methods describe earlier for discretization of the transport and diffusion equation, will help us to determinate the best way to approach an mathematical model to simulate the variation of residual chlorine in an water distribution network. To be able to model this complex physical phenomenon we should know all the aspects that describe the fluid flow and the network characteristics. The methods have different approaches because the initially and the boundary conditions of the equation are not the same. We tried many conditions and we kept only the natural one. So, the final version seems to be the most natural and the most appropriate to what is happening in a water distribution network.This can be an mathematical model to simulate the variation of residual chlorine or for another contaminator that travel thru a water distribution network. To do that, the boundary and the initially conditions must be known, and also the diffusion constant and other specific values for each network that will be modeled.

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APPLICATION OF THE EnKF AND THE TRUNCATED GAUSSIAN METHOD IN THE GEOLOGICAL UNCERTAINTY QUANTIFICATION OF THE CHANNELIZED RESERVOIRS

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Abstract: In this study we present an application in which the ensemble Kalman filter (EnKF), as history matching method, and a form of the truncation plurigaussian scheme, as geological simulation model, are coupled for geological uncertainty quantification of the channelized reservoirs.
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Key words: Data assimilation, EnKF, geological uncertainty, plurigaussian truncation, probabilistic approach

1. Introduction

The truncated Gaussian method and his general form, the truncated plurigaussian method, have been widely used as geological simulation model, especially because of theirs ability of generating various shapes. However, for complex bodies, as channels, these methods do not provide always distributions in the field that keep a perfect continuity and/or directions, but, an appropriate truncation generates geological shapes that are almost channels. In this paper, we present an application in which, using EnKF [1] as history matching method, we are able to generate an initial ensemble constrained to all available measurements (facies observations and production observations) and to provide an updated ensemble that offers a good quantification of the geological uncertainty. The observation operator of the facies is probabilistic, introduced in a previous study [3]. The reference field used (the "truth") is not generated with the same method as the ensemble, being extracted from a training image, using a multipoint geostatistical tool (method).

2. The geological simulation model

The uncertainty in the hydrocarbons reservoirs is especially related with the unknowing of the geology. Even though in the exploration phase a lot of information (usually noisy) are gathered (prior information), a complete description of the geology is never carried out. Therefore, some additional information from the exploitation phase (production data) should be used (data assimilation), in order to improve the subsurface knowledge and further to help in the optimization process of the hydrocarbons recovery. The prior information used in this study refers to the existence of a channelized reservoir (two facies types). The channel, as a facies type, may be defined as a narrow band with relative constant width, having certain direction. The first step of the geology description (uncertainty quantification) is to define a geological simulation model. This is a mathematical model with which we are able to simulate the distribution of the body rock formation (facies) in the reservoir domain. The geo-model used in this study is the truncated Gaussian method [2]. The method consists in the truncation of a Gaussian field defined on the reservoir domain using a truncation map (scheme). The truncation map used in this study is not a priory defined, being based on the method introduced in a previous study [3]. Let's consider a reservoir where two facies types occur, of which one is channel (high permeability) and the other is background (small values for permeability). Also, we have additional information from the cores extracted at the drilled

wells (production and injection), information's related to the type of the facies that exists at that locations. We use probabilities fields to estimate the binary fields defined on the reservoir domain by the facies types. One of the probabilities field (associated with channel) are defined as projection in [0,1] interval of a Gaussian field defined on the reservoir domain, using a projection function (introduced in [3]).

The projection function used is $\varphi_m(t) = \begin{cases} -\frac{|t|}{m} + 1 & if \quad |t| \le m \\ 0 & if \quad |t| > m \end{cases}$, where m represents a truncation parameter. The parameter m is initially chosen based on geological prior knowledge about the facies proportions in the certain case and estimated in the process of history matching. Let's consider the Gaussian field y defined on the reservoir domain, and the probabilities field associated to the channel, denoted α, where $\alpha^{i,j} = \varphi_m(y^{i,j})$, (i,j) being the indicative for the grid cell. For the non-channel facies type the associated probabilities field is $1-\alpha$. The geological simulation model consists on defining in each grid cell of a facies type. This is carried out using a maximization criterion applied to the probabilities fields. At each location (i,j) we assign the facies type with the highest value of the probabilities fields. This geo-model is equivalent with truncation of the Gaussian field with the truncation map presented in Figure 1.

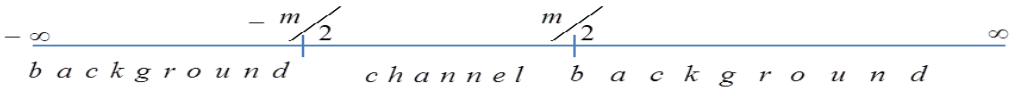

Figure 1. The truncation map of the Gaussian field

In order that, the geo- model to generate realistic reservoirs, we have to connect the Gaussian field geostatistical properties with the channel characteristics (the direction, the width, the curvature). This truncation not always could provide perfect channel distribution in the field, especially for the meandering channel; therefore the method is limited to channels with small sinuosity. Also, the randomness of the Gaussian fields has an impact in not keeping a relative constant length of the channel, but it provides facies that are almost channels (Figure 5).

3. The Ensemble Kalman Filter (EnKF)

The discrete model for a single simulation step of a reservoir dynamical system from time t_{i-1} to time t_i can be described by an equation of the form $u(t_i) = M_i(u(t_{i-1}), \theta), i = \overline{1, N}$, where $u(t_i) \in \Re^n$ denotes the state of pressures and saturations at time t_i, θ denotes the vector of uncertain parameters. The operator M represents the reservoir simulator, which is a nonlinear function. The relationship between measured production data $d(t_i)$ and the state $x(t_i)$ can be described using a nonlinear operator h. Considering the observation as being imperfect we can write the relation as $d(t_i) = h_i(u(t_i), \theta) + v_i$ where $v_i \sim N(0, R_i)$ is the observation error.

Defining the augmented state vector $x_i \overset{\Delta}{=} x(t_i) = \begin{bmatrix} u_i^T & \theta^T & h_i^T \end{bmatrix}$ the relation between the new state and the measurement become linear as $d_i \overset{\Delta}{=} d(t_i) = Hx_i + v_i$.

The filter solution of the problem is to find the conditional pdf of the state given the observations, which using a Bayesian inversion it may be written as

$$p(x_n | d_{n:1}) \propto p(d_n | y_n) \cdot p(y_n | d_{n-1:1}) \qquad (1)$$

The basic idea of the ensemble Kalman filter is to solve the equation (1) using a Monte Carlo integration in time [1]. Hence, an ensemble of N_e states $\{x_1, x_2, \ldots x_{N_e}\}$ is generated to represent the uncertainty in the initial state $x_0^t = x(t_0)$. In the second step (forecast step), the stochastic model propagates each ensemble member through the model equations according to $x_j^f(t_i) = M(x_j^a(t_{i-1}))$ (at the initial step $x_0^a = x_0$). From the forecasted ensemble we calculate the mean $\bar{x}^f(t_i) = \frac{1}{N_e}\sum_{k=1}^{N_e} x_k^f(t_i)$ and the covariance $C^f(t_i) = \frac{1}{N_e - 1} E^f(t_i) E^f(t_i)^T$, where $E^f(t_i) = \left[x_1^f(t_i) - \bar{x}^f(t_i) \quad x_2^f(t_i) - \bar{x}^f(t_i) \quad \ldots \quad x_{N_e}^f(t_i) - \bar{x}^f(t_i) \right]^T$. When the measurements become available values of each ensemble member are adjusted based on the Kalman equation $x_j^a(t_i) = x_j^f(t_i) + K(t_i)[d_{obs}(t_i) - H(t_i)x_j^f(t_i) + v_j(t_i)]$, where $K(t_i) = C^f(t_i)H(t_i)^T[H(t_i)C^f(t_i)H(t_i)^T + R(t_i)]^{-1}$ is the Kalman gain, $R(t_i)$ is the covariance matrix of the measurements error and $v_j(t_i)$ is the realization of the noise added to observed measurements. At the end of the assimilation period we have an estimator for each parameter, defined by the ensemble mean together with his uncertainty represented by the ensemble.

4. Ensemble Kalman Filter implementation for facies update

The state vector for the j[th] ensemble member at the i[th] assimilation step is:

$x_j^i = \left[y^T \quad m^T \quad BHP^T \quad {q_0}^T \quad {q_w}^T \quad \alpha^T \right]_j^T$, where y is the Gaussian field, m is the truncation parameter, BHP is the pressure measured at the injectors, q_w, q_o are the water and oil rates measured at the producers and α represents the simulated facies measurements at the well locations. The facies measurements at the well location are written in probability terms. The values for α are 0 or 1 depending on if the channel facies type occurs or not at the well location. The uncertainty in the initial ensemble is given by the choice of the Gaussian field and the choice of the truncation parameter. The dynamical variables (pressure and saturation) are not in the state vector because we have used the rerun of the simulator from time 0 after each assimilation step. The Gaussian field *y* is generated with constraints given by the type of facies find in the grids where the wells are situated. If in a grid with a well located we have observation about the existence of the channel, then the value in this grid for y is 0 (such that the probability is 1). Also at the grids where the channel does not occurs, the Gaussian field projection has a value in the neighbor of 0(we have set a threshold of 0.05). After each assimilation step we ensure that each ensemble member to have the correct facies observation at the well location (to keep the geological realism) using an extra procedure of constraining.

4. Synthetic example

The simulation model is a 6-spot water flooding 2D-reservoir, black oil model with $50 \times 50 \times 1$active grid blocks. The dimension of each grid block was set at $30 \times 30 \times 1$ ft. and there's three injectors situated at the left side of the reservoir domain and three producers situated at the right (blue dots in Figure 2, right). The values of the permeability (k) and porosity (ϕ), corresponding to each facies type, are: for channel k=500 md, $\phi = 0.3$ and for background k=50 md, $\phi = 0.1$.The reference field is not generated with the truncation Gaussian method and it was generated using a multi-point geostatistical algorithm named SNESIM and a training image provided by the experts (Figure 2). The Gaussian fields have been generated with anisotropic geostatistical characteristics, with long length correlation of

30 grid cells, short length correlation of 15 grid cells, and horizontal principal direction. The truncation parameters are generated Gaussian with mean 0.8 and standard deviation 0.2.

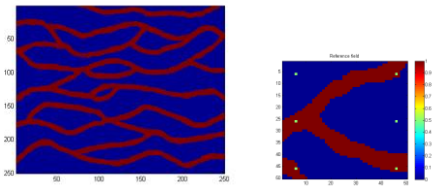

Figure 2: The training image and the reference field

The value of 0.8 is obtained taking into account the prior information of the facies proportions (0.3 for channel facies type) and the standard deviation is the uncertainty associated with it. We have used 20 assimilation type steps at each 10 days and 120 ensemble members. The measurement errors have been set at 3% standard deviation for production data. Even though the facies observations are perfect we have assigned a small error of 0.01 standard deviation in order that the Kalman gain to exists. In Figures 3 and 4 are presented the prior mean of the probabilities fields together with the associated estimated field and respectively the result of the estimation process (the estimated probabilities fields and the estimated field). In Figure 5 are first four ensemble members in initial and in updated ensemble. With a visual inspection of the Figures 2 and 4 one can observe that the updated ensemble of the facies fields captured the main characteristics of the reference field. Also, the updated ensemble has enough left variability to quantify the posterior uncertainty (Figures 4 and 5).

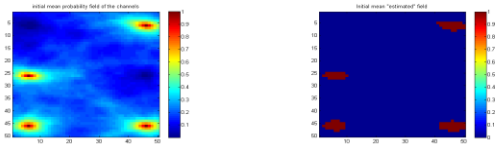

Figure 3: The prior mean of probabilities field and the prior estimated field of channel type

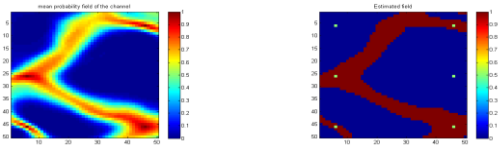

Figure 4: The posterior mean probabilities field and the prior estimated field of channel type

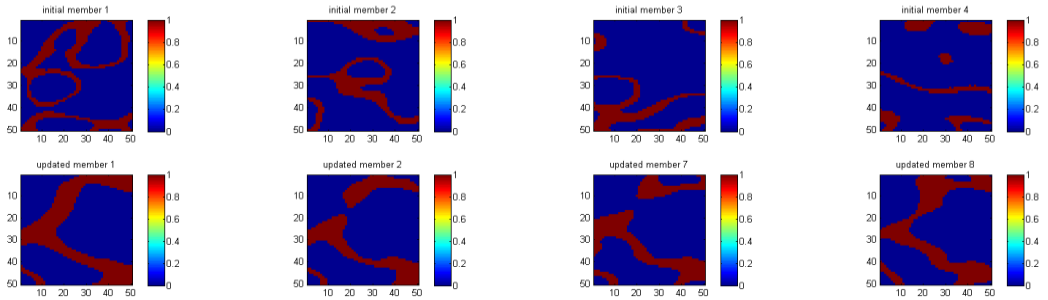

Figure 5: First four members in initial ensemble (line 1) and in updated ensemble (line 2)

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THE ESTIMATION OF THE PARAMETERS OF DISTRIBUTIONS FOUND IN HYDROLOGY USING THE MAXIMUM LIKELIHOOD METHOD AND THE MINIMUM CHI SQUARE METHOD

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Abstract: In this paper we will estimate the parameters of some distributions that are currently used in hydrology based on the maximum likelihood method or the minimum chi square method.
Because for the second method we can not find an analytical solution, and the same thing we can say about the maximum likelihood method for some distributions, like Pareto, we have to apply a numerical method, a Monte Carlo Method, or an heuristic method to estimate the parameters of the distribution. We will use the Nelder-Mead simplex algorithm (we do not write "SIMPLEX" in order to not be confused with the well-known linear programming algorithm), which is a heuristic method. Applying these methods we estimate the parameters of Pareto and Gamma distributions. Finally we compare our results with the results obtained using other methods found in literature.
Mathematics Subject Classification (2010): 62F10, 68T20, 90C59
Key words: Maximum likelihood method, minimum chi squared method, simplex, Pareto, Gamma, discharges.
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1. Introduction

Consider the continuous variable X having the pdf (probability density function) $f(x;\theta)$ depending on vector parameter $\theta \in R^k$. Consider also the sample X_1, X_2,..., X_n from a population characterized by the random variable X.

For the maximum likelihood method we have to compute first (see [1]) the likelihood function

$$V(X_1,...,X_n;\theta)=\prod_{i=1}^{n} f(X_i;\theta). \quad (1)$$

Because the logarithm is an increasing function, we maximize $\ln V$ instead of maximizing V. We do this transformation because the product becomes sum, and for almost

all classical distributions (as normal, gamma, Poisson) the exponential that arises in the pdf formula is reduced by the logarithm. Finally we obtain the system

$$\frac{\partial \ln V}{\partial \theta_j} = 0. \tag{2}$$

The minimum chi square method is as follows (see [1]). First we divide the support of the distribution (the minimal domain *D* such that $X \in D$ with probability one) in *k* intervals. Denoting by $p_i(\theta)$ the probability to have $X \in I_i$, we compute $X^2(\theta) = \sum_{i=1}^{k} \frac{\left(n_i - n \cdot p_i(\theta)\right)^2}{n \cdot p_i(\theta)}$ where n_i is the number of the values X_j that belong to I_i. The estimation of θ according the minimum chi square method is the value θ such that the above value of $X^2(\theta)$ is minimum. We have to solve the system

$$\frac{\partial X^2}{\partial \theta_j} = 0. \tag{3}$$

For classical distributions, as normal, gamma or Poisson we have an analytical solution of the system (2), but for the system (3) we have not an analytical solution, even for normal distribution. But for some distributions that arise in hydrology, we have neither analytical solution for the system (2). Therefore we need a numerical, Monte Carlo or an heuristic method. Consider for instance the Pareto distribution, for which the pdf and cdf are given in [3]. Considering a sample of size *n* from a population characterized by a Pareto random variable, the parameters *a*, *b* and *c* must fulfill the following constraints: $b > 0$, $c \le X_i$ for $a \le 0$, and $c \le X_i \le \frac{b}{a}$ for $a > 0$. It is proved (see [3]) that the likelihood function is increasing with respect to *c*, and for fixed *c* the other two coefficients are obtained by solving a nonlinear system and a nonlinear system arise also for the minimum chi squared method.

If we consider the generalized Gamma distribution, we have not an analytical formula for cdf. Therefore we cannot find an analytical solution in the case of minimum chi squared method. Moreover, we can not find such solution even for the maximum likelihood method, because we have not closed formula for the derivative and the value of special function Γ.

The algorithm simplex was elaborated by Nelder and Mead (see [2,5]), and solves for some real function f the minimization problem $\min_{x \in R^k} f(x)$, or the maximization problem $\max_{x \in R^k} f(x)$. The first step is to compute coordinates of $k+1$ possible solutions $x_0^{(1)}$, $x_0^{(2)}$,..., $x_0^{(k+1)}$ such that they build a regular hyper-tetrahedron (the distance between each two distinct initial solution is the same). This is done taking $\begin{cases} x_0^{(1)} = 0_{R^k} \\ x_{i;0}^{(i-1)} = c_1 \quad \text{for } i = \overline{2,k+1} \\ x_{j;0}^{(i-1)} = c_2 \quad \text{for } i = \overline{2,k+1},\ j \ne i-1 \end{cases}$, where $\begin{cases} c_1 = \frac{t \cdot \left(\sqrt{k+1} + k - 1\right)}{k\sqrt{2}} \\ c_2 = \frac{t \cdot \left(\sqrt{k+1} - 1\right)}{k\sqrt{2}} \end{cases}$, and *t* is the given common distance between two distinct points.

At the step *p* we do the following:

1) Find among the $(k+1)$ points at the previous step the worst solution, $x_{w;\,p-1}$ and the gravity center of the other points, $x_{G;\,p-1}$. Compute also the best solution, $x_{B;\,p-1}$.
2) Compute the reflection of the worst solution with the reflection coefficient $\alpha > 0$ through

the gravity center of the other points: $x_{R,\alpha;p} = x_{G;p-1} + \alpha \cdot \left(x_{G;p-1} - x_{w;p-1} \right)$.

3) If $x_{R,\alpha;p}$ is a better solution than $x_{B;p-1}$ we assume that the modification direction is favorable, and we try to extend the reflection. We compute in this case $x_{E,\gamma;p} = x_{R,\alpha;p} + \gamma \cdot \left(x_{R,\alpha;p} - x_{G;p-1} \right)$, with $\gamma > 1$. If $x_{E,\gamma;p}$ is better than $x_{B;p-1}$ we replace $x_{w;p-1}$ by $x_{E,\gamma;p}$; otherwise we replace $x_{w;p-1}$ by $x_{R,\alpha;p}$.
4) If $x_{R,\alpha;p}$ is not better than $x_{B;p-1}$, but there exists at least another point worse than $x_{R,\alpha;p}$, but better than $x_{w;p-1}$, we replace $x_{w;p-1}$ by $x_{R,\alpha;p}$.
5) If $x_{w;p-1}$ is the only solution worse than $x_{R,\alpha;p}$, we replace $x_{w;p-1}$ by $x_{C,\beta;p} = x_{G;p-1} + \beta \cdot \left(x_{G;p-1} - x_{w;p-1} \right)$, with $\beta \in (0,1)$ (we do a "contraction of the reflection").
6) If the reflection is totally unfavorable, i.e. $x_{R,\alpha;p}$ is even worse than $x_{w;p-1}$, we do a contraction of the solutions' space: each possible solution at previous step, $x_{p-1}^{(i)}$ with $i = \overline{1,k+1}$ is replaced by $x_{B;p-1} + \frac{x_{p-1}^{(i)} - x_{B;p-1}}{2}$.

We notice first that the transformation from the last step does not change the best solution (we add zero for the above fraction). The best values for α, β and γ resulting from experiments are $\alpha = 1$, $\beta = 0.5$ and $\gamma = 2$, values proposed by the authors of the algorithm (Nelder and Mead).

2. The method and applications

In the algorithm simplex we consider that each component of *x* can have any real value. In our case we have to maximize the likelihood or to minimize the chi square, and the possible solutions are given by the values of the parameters. But some parameters, as the variance in a normal distribution, or *c* in a Pareto distribution could not have any real value.
We have to manage this situation in the following way: 1) we take a feasible solution in the interior of the domain (not on the border), which will be considered in the place of the origin $0_{\square k}$ in the algorithm, 2) we take *t* small enough to have all $k+1$ initial solution feasible, 3) if a new possible solution does not belong to the domain of parameters, we consider the objective function to take a fixed smaller value in the case of maximum likelihood, respectively a fixed higher value in the case of minimum chi square, in comparison to a feasible solution. If a parameter θ_i belongs to an interval $[a,b]$ we have to consider the particular cases $\theta_i = a$ and $\theta_i = b$. Analogous, we have to consider the particular case $\theta_i = a$ if θ_i belongs to the interval $[a,\infty)$, or to the interval $(-\infty,a]$. Of course, the above particular cases are considered only if *a*, respectively *b* are possible values for θ_i. Therefore for instance $\sigma^2 = 0$ can not be considered as particular case for normal distribution.

Example 1. We generate 500 Pareto variables with parameters a=-0.5, b=4 and c=10, and 500 generalized Gamma variables with parameters a=2, b=3 and c=0. In the case of the Pareto distribution we obtain the maximum likelihood equal with -1432.74 for a=-0.305, b=5.598 and c=10, and the minimum χ^2 equal with 1.541 for a=-0.344, b=5.207 and c=9.628. The number of iterations is 48 in the case of maximum likelihood method, and 45 in the case of minimum χ^2 method. In the case of the generalized Gamma distribution we obtain the maximum likelihood equal with -465.41 for a=2.204, b=3.616 and c=0, and the minimum χ^2 equal with 10.62 for a=2.27, b=3.5 and c=0. The number of iterations is 21 in the case of maximum likelihood method, and 22 in the case of minimum χ^2 method.

Example 2. Consider 78 data sets on the Danube discharges at Oltenita, in the period 1931-2008. We will consider for these data the generalized Pareto distribution and the generalized Gamma distribution, and for each case we will apply the maximum likelihood method and minimum chi squared method through the algorithm simplex of Nelder and Mead. Finally we will compare the results to that obtained by the method of moments (see [3] for the case of Pareto distribution). In the case of the Pareto distribution we obtain the maximum likelihood equal with -765.22 for a=-0.33, b=2035.06 and c=6419.47, and the minimum χ^2 equal with 114.32 for a=-0.42, b=2035.54 and c=6420. The number of iterations is 20 in the case of maximum likelihood method, and 26 in the case of minimum χ^2 method. In the case of the generalized Gamma distribution we obtain the maximum likelihood equal with -722.51 for a=2.05, b=2035.16 and c=6419.27, and the minimum χ^2 equal with 1.66 for a=5.3, b=880.11 and c=6419. The number of iterations is 48 in the case of maximum likelihood method, and 25 in the case of minimum χ^2 method.

3. Conclusions

Even for the simulated data both models yield acceptable results, in the case of real data only the Gamma model is appropriate. In [4] we have estimated the parameters of Pareto marginal distributions using the moments' method. We have noticed that the nonlinear system has to be solved by substitution method, but finally we had to solve a nonlinear equation in *a*.

In [3] other methods to estimate the generalized Pareto distribution's parameters are presented: the probability-weighted moments (PWM) method, the maximum likelihood method and the maximum entropy method. For these methods we have neither analytical solutions.

Because both distributions families (Pareto and Gamma) contain the $\exp(1)$ distribution, we consider first the objective function value equal to that of this case if the parameters are not in the domain ($c \le \min(X_i)$, $b > 0$ for both distributions, $a < 0$ for Pareto and $a > 0$ for Gamma). To this value we subtract one in the case of maximum likelihood method, and we add one in the case of minimum chi squared method.

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ON A TYPE I TRANSFORMATION GROUPOID CROSSED PRODUCT

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Abstract: If G is a locally compact group acting on a topological space X, it is known that, since $(C_0(X), G, lt)$ is a regular dynamical system and every stability group G_x is type I, the associated crossed product $C_0(X) \times_{lt} G$ is type I. The purpose of this paper is to extend this result to the case of G being a locally compact groupoid, with a Haar system, acting on a topological space X.
Mathematics Subject Classification (2010): 46L05, 22A22, 47L65

Key words: General theory of C^*-algebras, topological groupoids, crossed product algebras

1. Introduction

According to [3, Proposition 7.29], if G is a locally compact, topological group left (right) acting on a topological space X, such that the associated dynamical system $(C_0(X), G, lt)$ is regular and every stability group G_x is type I, then the crossed product $C_0(X) \times_{lt} G$ is a type I C^*-algebra. In this paper we consider a topological, locally compact, second countable groupoid G with a Haar system of measures $\{\lambda^u\}_{u \in G^{(0)}}$ acting on a topological space X and we associate to this action a groupoid dynamical system $(\mathcal{A}, G, \alpha)$, where $\mathcal{A}$ is a bundle of C^*-algebras indexed by the unit space of G and $\alpha : G \to Iso(\mathcal{A})$ a homomorphism such that $\alpha_g : A(s(g)) \to A(r(g))$ is an isomorphism of C^*-algebras. If $(\mathcal{A}, G, \alpha)$ is a groupoid dynamical system such that G is an amenable groupoid and every stability group $G/\{u\}$, $u \in G^{(0)}$ is a type I group, we show that the associated crossed product $C^*(G, \mathcal{A})$ is a type I C^*-algebra.

2. The main results

Lemma 2.1. If G is a topological, locally compact, second countable groupoid G with a Haar system of measures $\{\lambda^u\}_{u \in G^{(0)}}$ acting on a topological space X and if we consider

$\mathcal{A} = \bigcup_{u \in G^{(0)}} C_0(p^{-1}(u))$, there exists a continuous homomorphism $lt : G \to Iso(\mathcal{A})$ such that $lt_g : C_0(p^{-1}(s(g))) \to C_0(p^{-1}(r(g)))$, $lt_g(f)(x) = f(g^{-1} \cdot x)$, for all $g \in G, x \in p^{-1}(r(g))$, $f \in C_0(p^{-1}(s(g)))$ is an isomorphism of C^*-algebras.

Proof Since $x \in p^{-1}(r(g))$, it results $p(x) = r(g) = s(g^{-1})$, $(g^{-1}, x) \in G * X$, and the element $g^{-1} \cdot x$ exists. Moreover, $p(g^{-1} \cdot x) = r(g^{-1}) = s(g)$ and it results that $g^{-1} \cdot x \in p^{-1}(s(g))$.

If $(g_1, g_2) \in G^{(2)}$, $s(g_1) = r(g_2)$, we have $lt_{g_1 g_2}(f)(x) = f((g_1 g_2)^{-1} \cdot x) =$

$= f(g_2^{-1} \cdot (g_1^{-1} \cdot x)) = lt_{g_2}(f)(g^{-1} \cdot x) = lt_{g_1}(lt_{g_2}(f))(x) = (lt_{g_1} \circ lt_{g_2})(f)(x)$.

Hence the map $lt : G \to Iso(\mathcal{A})$ is a homomorphism. Because the groupoidal action on space X is continuous and every function $f \in C_0(p^{-1}(u))$ is continuous, it is clear that the map lt is continuous.

Since

$$lt_g(f_1 \cdot f_2)(x) = (f_1 \cdot f_2)(g^{-1} \cdot x) = f_1(g^{-1} \cdot x) f_2(g^{-1} \cdot x) = lt_g(f_1)(x) lt_g(f_2(x)) = \\ = (lt_g(f_1) lt_g(f_2))(x)$$

and

$$lt_g(f)^*(x) = \overline{lt_g(f)(x)} = \overline{f(g^{-1} \cdot x)} = f^*(g^{-1} \cdot x) = lt_g(f^*)(x),$$

it follows that the map lt_g is a $*$-homomorphism from $C_0(p^{-1}(s(g)))$ to $C_0(p^{-1}(r(g)))$. Moreover, since the map $(g^{-1}, x) \to g^{-1} \cdot x$ and the the functions from $C_0(p^{-1}(s(g)))$ are continuous the map lt_g is continuous.

Similarly, we show that the inverse map of lt_g, denoted by lt_g^{-1},

$$lt_g^{-1} : C_0(p^{-1}(r(g))) \to C_0(p^{-1}(s(g))),\ lt_g^{-1}(f)(x) = f(g \cdot x)$$

has the same properties as lt_g. Moreover, for every $f \in C_0(p^{-1}(s(g)))$, we have $(lt_g \circ lt_g^{-1})(f) = f$, for every $f \in C_0(p^{-1}(r(g)))$, we have $(lt_g^{-1} \circ lt_g)(f) = f$ and it results that lt_g is an isomorphism of C^*-algebras.

Lemma 2.2. If G is a topological, locally compact, second countable groupoid G with a Haar system of measures $\{\lambda^u\}_{u \in G^{(0)}}$ acting on a topological space X, the stability group associated to the groupoidal action on X,

$$G_x = \{g \in G / g \cdot x = x, \forall (g, x) \in G * X\}$$

coincides with the stability group associated to the group action of the stability group $G/\{u\}$ on X,

$$(G/\{u\})_x = \{\gamma \in G/\{u\} / \gamma \cdot x = x, x \in p^{-1}(u)\}.$$

Proof It is clear that $(G/\{u\})_x \subseteq G_x$. If $g \in G_x$, it results $p(x) = s(g)$ and $g \cdot x = x$. Since $p(x) = p(g \cdot x)$ and $p(g \cdot x) = r(g)$, we deduce that

$$r(g) = s(g) = p(x) = u.$$

Hence $g \in (G/\{u\})_x$, and the equality of the sets G_x and $(G/\{u\})_x$ follows.

Proposition 2.3. Let *G* be a topological, locally compact, second countable groupoid *G* with a Haar system of measures $\{\lambda^u\}_{u\in G^{(0)}}$ acting on a topological space *X*. If, for every $u \in G^{(0)}$, the dynamical system $(C_0(p^{-1}(u)), G/\{u\}, {}^{lt}\!/_{G/\{u\}})$ is regular and for every $x \in X$, the stability groups G_x are type I, then for every $u \in G^{(0)}$ the crossed product $C_0(p^{-1}(u)) \times_{{}^{lt}\!/_{G/\{u\}}} G/\{u\}$ is a type I C^*-algebra.

Proof According to [3, Proposition 7.29], the regularity of the dynamical system $(C_0(p^{-1}(u)), G/\{u\}, {}^{lt}\!/_{G/\{u\}})$, the fact that for every $x \in X$, the stability groups G_x are type I and Lemma 2.2., it follows that $C_0(p^{-1}(u)) \times_{{}^{lt}\!/_{G/\{u\}}} G/\{u\}$ is a type I C^*-algebra.

Proposition 2.4. Let *G* be a topological, locally compact, second countable, amenable groupoid *G* with a Haar system of measures $\{\lambda^u\}_{u\in G^{(0)}}$ acting on a topological space *X*. If for every $u \in G^{(0)}$, the dynamical system $(C_0(p^{-1}(u)), G/\{u\}, {}^{lt}\!/_{G/\{u\}})$ is regular and for every $x \in X$ the stability groups G_x are type I, then the crossed product $C^*(G, \mathcal{A})$ is a type I C^*-algebra, where $\mathcal{A} = \bigcup_{u\in G^{(0)}} C_0(p^{-1}(u))$.

Proof According to [2, Theorem 3.6], if $(\mathcal{A}, G, \alpha)$ is a groupoid dynamical system with *G* being amenable, the representations of the crossed product $C^*(G, \mathcal{A})$ are induced by the representations of the stability groups $G/\{u\}$, $u \in G^{(0)}$.

We deduce that if for every $u \in G^{(0)}$ the stability groups are type I, the crossed product $C^*(G, \mathcal{A})$ is a type I C^*-algebra (the representations of every stability group contain the C^*-algebra of compact operators, hence the induced representations of $C^*(G, \mathcal{A})$ contain the C^*-algebra of compact operators)

From Proposition 2.3., since $(C_0(p^{-1}(u)), G/\{u\}, {}^{lt}\!/_{G/\{u\}})$ is regular and for every $x \in X$ the stability groups G_x are type I, it results that for every $u \in G^{(0)}$, $C_0(p^{-1}(u)) \times_{{}^{lt}\!/_{G/\{u\}}} G/\{u\}$ is a type I C^*-algebra. From the above considerations relative to [2, Theorem 3.6], it follows that $C^*(G, \mathcal{A})$ is a type I C^*-algebra.

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THE GEOMETRY OF THE SPHERE WITH MATHCAD

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Abstract: In this paper we focus on the specific geometry on the surface of the sphere which is more intuitive and we present several examples in which we provide MathCad worksheets for some of the basic concepts of this geometry.
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Key words: sphere; straight line; plane; geodesic; small circle; spherical triangle; MathCad.

1. Introduction

MathCad is the powerful resource tool that will benefit students in any course they take which requires calculations to be performed. This will include almost every course taken by a student majoring in engineering, mathematics and physical sciences.

Why MathCad is easy to learn and easy to use?
Unlike other technical programs, a lot of professors from a variety of disciplines choose MathCad because it features an "user-friendly interface" that it easy-to-learn, allowing users to easily combine standard math notations, graphs and text regions in a single complex document. We have found that MathCad can fundamentally change the way we teach mathematics. MathCad allows us to cover more complex and interesting problems because its numerical and symbolic algebraic manipulation, and programming and optimization capabilities are outstanding. A software tool like MathCad is particularly important because students need to understand the symbolism that is used to make the abstract concepts of geometry (point, lines, planes and surfaces, etc.) observable and measurable. For example, to produce circles or spheres, one might choose parametric trigonometric functions or rational B-splines, but not ordinary B-splines. MathCad easily shows the difference.

In this work we present examples of how it is being used as a teaching and learning tool.

2. Sphere. Some background notations

In this section we want to recall the basic notions about straight line, plane and sphere and we are interested here in the geometry of an ordinary sphere.

In three-dimensional Euclidean geometry, a *sphere* is the set of all points lying the same distance (the *radius*) from a given point (the *center* or c*entre*). In analytic geometry, a sphere with centre $C(a, b, c)$ and radius R is the locus of all points (x, y, z) such that:

$$(x-a)^2+(y-b)^2+(z-c)^2=R^2 \text{ or}$$

$$x^2+y^2+z^2-2ax-2by-2cz+q=0, \text{ where } R=\sqrt{a^2+b^2+c^2-q}\,.$$

A sphere may be defined parametrically in terms of (θ,φ):

$$\begin{cases} x(\theta,\varphi) = a + R\cdot\cos\varphi\sin\theta \\ y(\theta,\varphi) = b + R\cdot\sin\varphi\sin\theta\,, \\ z(\theta,\varphi) = c + R\cdot\cos\theta \end{cases} 0\le\varphi\le 2\pi,\ 0\le\theta\le\pi\,.$$

2.1. Lines through a sphere

If we take an arbitrary straight line and a sphere several things can happen. First, the line can not intersect the sphere at all and this case is not interesting. Secondly, the line can intersect the sphere at one point in which case it is called *tangent*. The only other thing that can happen is that the line can intersect the sphere in precisely two points, the entry and exit points. Methods for distinguishing these cases and determining the Cartesian coordinates for the points in the latter cases are useful in a number of circumstances. A straight line that passes through the centre of a sphere has two intersection points, these are called *antipodal points*.

From the algebrical point of view, to find where a straight line meets a sphere means to calculate the solutions of the system formed by the equations of straight line and sphere: $\begin{cases} \dfrac{x-x_0}{l}=\dfrac{y-y_0}{m}=\dfrac{z-z_0}{n} \\ (x-a)^2+(y-b)^2+(z-c)^2=R^2 \end{cases}$. Using the parametric equations of the straight line: $x=t\,l+x_0$, $y=t\,m+y_0$, $z=t\,n+z_0$, $t\in\mathbf{R}$ and substituting into the equation of the sphere we find a quadratic equation for the real parameter t. After we solve the resulting equation we replace the value(s) found in the straight line equations to find the Cartesian coordinates of the intersection points (if there are).

2.2. Planes through a sphere

In analytic geometry, we have three possible situations of intersection between a sphere and a plane: no intersection at all (plane misses sphere), only one point (tangential contact) in which case the plane is called the *tangent plane* and an infinite number of points lying in a circle.

To find out whether the plane and sphere meet or not, we compare the distance d from the centre of the sphere to the plane with the radius R of the sphere. We will see immediately which of the above cases hold. If $d>R$, we have not intersection, if $d=R$ we have a single point of intersection and if $d<R$, the intersection is a circle.

If the plane and the sphere intersect at one point $M_0(x_0,\,y_0,\,z_0)$, then the equation of the tangent plane to the sphere at this point is:

$$(x-a)(x_0-a)+(y-b)(y_0-b)+(z-c)(z_0-c)-R^2=0 \text{ or}$$
$$x\,x_0+y\,y_0+z\,z_0-a(x+x_0)-b(y+y_0)-c(z+z_0)+q=0.$$

2.3. Geodesics on a sphere

It is easy to see that the circle of intersection between a sphere and a plane will be the largest when the plane passes through the center of the sphere and it is called *great circle*. Great circles become more important when we realize that the shortest path between two points on the sphere is along the segment of the great circle joining them.

On any surface the curves that minimize the distance between points are called *geodesics*.

In plane geometry, the basic concepts are points and lines. On the sphere we have points, but there are no straight lines, at least not in the usual sense. Thus straight lines are the geodesics on the plane, and great circles are the curves on the sphere with the same property. However, since the great circles are geodesics on the sphere, just as lines are in the plane, we should consider the great circles as replacements for lines.

Suppose we have two distinct points A and B on a sphere. Together with C, the center of the sphere, we have three points in space, and there are two possibilities. If A and B are not antipodal points, then A, B and C determine a unique plane, and consequently there is a unique great circle that contains A and B. The two points divide this great circle into two arcs of differing length. The shorter arc is the curve of shortest distance (on the sphere) between the two points. If A and B are antipodal points, then A, B and C lie on the same line in space and there are infinitely many great circles containing A and B.

The vector equation of a great circle on a sphere with centre $C(a, b, c)$ and radius R passing through the two not antipodal points $A(x_A, y_A, z_A)$ and $B(x_B, y_B, z_B)$ is:

$$\overrightarrow{g(t)} = C + \cos(t)\cdot\overrightarrow{CA} + \sin(t)\cdot\vec{u}\text{, where } \vec{u} = \frac{R}{\left\|\vec{w}\right\|}\,\vec{w} \text{ and } \vec{w} = \overrightarrow{CB} - \frac{\overrightarrow{CA}\cdot\overrightarrow{CB}}{R^2}\,\overrightarrow{CA}.$$

2.4. Small circles on a sphere

A *small circle* of a sphere is the circle constructed by a plane crossing the sphere not in its center. A small circle always has smaller diameter than the sphere itself, and a segment on its circumference does not represent the shortest path between two points on a spherical surface.

If a sphere and a plane meet in a small circle with centre $O(x_O, y_O, z_O)$ and radius r, then its vector equation is:

$$\overrightarrow{s(t)} = O + r\cos(t)\cdot\vec{u} + r\sin(t)\cdot\vec{v}\text{, where } \vec{u} = \frac{\overrightarrow{OM}}{r},\ \vec{v} = \overrightarrow{N}\times\vec{u},$$

$\overrightarrow{N}$ is a unit normal vector for the plane and $M(x_M, y_M, z_M)$ is a point of the circle.

2.5. Spherical triangles on a sphere

The spherical triangle is the spherical analog of the planar triangle, and it is sometimes called an Euler triangle. A *spherical triangle* is a figure formed on the surface of a sphere by three great circular arcs intersecting pairwise in three vertices. On any sphere, if three connecting arcs are drawn, two triangles are created. If each triangle takes up one hemisphere, then they are equal in size, but in general there will be one larger and one smaller.

3. An MathCad algorithm for plotting the concepts of spherical geometry

Example 3.1. A MathCad worksheet for plotting the intersection of straight line and sphere, respectively the tangent plane to sphere

The sphere equation is: $f(x,y,z) = 0$, where $f(x,y,z) := x^2 + y^2 + z^2 - 4\cdot x + 2\cdot y - 6\cdot z + 8$

The centre and the radius of the sphere are: $a := 2\ \ b := -1\ \ c := 3\ \ q := 8\ \ R := \sqrt{a^2 + b^2 + c^2 - q} \rightarrow \sqrt{6}$

The canonical equations of the straight line are: $\frac{x}{-1} = \frac{y-1}{1} = \frac{z+1}{-2}$, hence

$M_0 := \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ is a point on the line and $v := \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix}$ is the director vector of the straight line

Using the function *Find* we solve the system formed by the above equations of the straight line and the sphere and we found the Cartesian coordinates of intersection points:

$A := \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ and $B := \begin{pmatrix} 3 \\ -2 \\ 5 \end{pmatrix}$

To draw the sphere and the straight line we must use the parametric equations of sphere (used in geodesy) and to define the vector equation of the straight line denoted by *k*:

$x(\theta,\varphi) := a + R\cdot\cos(\theta)\cdot\cos(\varphi)$

$y(\theta,\varphi) := b + R\cdot\sin(\theta)\cdot\cos(\varphi)$ and $k(t) := M_0 + v\cdot t$ $\quad k(t) \rightarrow \begin{pmatrix} -t \\ 1+t \\ 1-2\cdot t \end{pmatrix}$

$z(\theta,\varphi) := c + R\cdot\sin(\varphi)$

The equations of the tangent planes to the sphere at the points A and B are:

$PT1(x,y) := -\frac{1}{2}\cdot x + \frac{1}{2}\cdot y + \frac{3}{2}$ and $PT2(x,y) := -\frac{1}{2}\cdot x + \frac{1}{2}\cdot y + \frac{15}{2}$

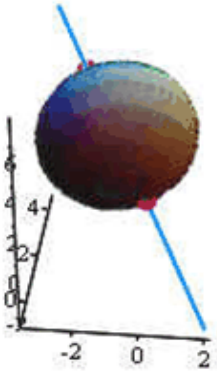

Fig. 3.1. Intersection of straight line and sphere

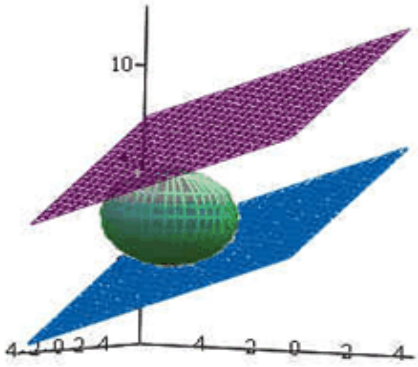

Fig. 3.2. Tangent planes to sphere

Example 3.2. A MathCad worksheet for plotting the geodesics and small circles on sphere
We consider the centre C and the radius R of a sphere, two not antipodal points A and B (different from the ones of Example 3.1) and a plane with the normal vector denoted by n:

$$C := \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \quad R := \sqrt{6} \quad A := \begin{pmatrix} 4 \\ -2 \\ 4 \end{pmatrix} \quad B := \begin{pmatrix} 3 \\ -2 \\ 5 \end{pmatrix} \quad CA := A - C \quad CB := B - C \quad n := \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

To vector equation of the great circle passing through A and B is:

$$g(t) := C + \cos(t) \cdot CA + \sin(t) \cdot u \text{, where } w := CB - \frac{CA \cdot CB}{R^2} \cdot CA \qquad u := \frac{R}{|w|} \cdot w$$

The center O and the radius r of the small circle and an arbitrary point M on this circle are:

$$O := \begin{pmatrix} 1.5 \\ -2 \\ 2.5 \end{pmatrix} \quad r := \frac{3}{\sqrt{2}} \quad M := \begin{pmatrix} 0 \\ -2 \\ 4 \end{pmatrix} \quad OM := M - O$$

To vector equation of the small circle is:

$$s(t) := O + r \cdot \cos(t) \cdot \frac{OM}{r} + r \cdot \sin(t) \cdot \left(\frac{n}{|n|} \times \frac{OM}{r} \right)$$

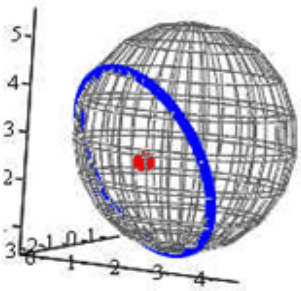

Fig. 3.3. Small circle

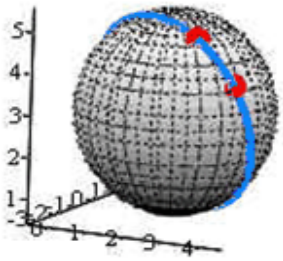

Fig. 3.4. Great circle

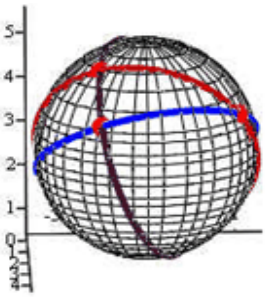

Fig. 3.5. Spherical Triangle

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ON RESTRICTIONS AND QUOTIENTS OF SPECTRAL OPERATORS AND SPECTRAL SYSTEMS

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Abstract: This paper is dedicated to the study of restrictions and quotients of spectral operators, respectively of spectral systems on Banach spaces.
It is shown that the restriction and the quotient of a spectral operator (system) with respect to an invariant subspace to the operator (system) are spectral operators (systems) if and only if the subspace is also invariant to the spectral measure of the operator (system) and therefore the restriction to an invariant subspace is a spectral operator (system) if and only if the quotient is a spectral operator (system) too. Furthermore, we study the case of spectral operators (systems) with the spectrum totally disconnected.
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Key words: spectral; spectral measure; restriction and quotient of an operator (system); totally disconnected set.

1. Introduction

This work contains certain results of the theory of spectral operators and spectral systems of commuting operators. The restrictions and the quotients of spectral operators (respectively, spectral systems) with respect to invariant subspaces are studied here and it is shown that they are also spectral operators (respectively, spectral systems). In what follows, we recall several notations and definitions from the specialized literature, which will be further needed.

Let $\mathbf{B}(X)$ denote the Banach algebra of all linear bounded operators acting on a given complex Banach space X and let $\mathcal{P}_X$ be the set of all projectors on X. If $Y \subset X$ is a linear closed subspace invariant to an operator $T \in \mathbf{B}(X)$ (respectively, to a commuting operator system $a = (a_1, a_2, ..., a_n) \subset \mathbf{B}(X)$), then $T|Y$ is the restriction of T to Y (respectively, $a|Y = (a_1|Y, a_2|Y, ..., a_n|Y)$ is the restriction system of a to Y) and $\dot{T}$ is the quotient operator induced by T (respectively, $\dot{a} = (\dot{a}_1, \dot{a}_2, ..., \dot{a}_n)$ is the quotient system induced by a) on the quotient space $\dot{X} = X/Y$. For $T \in \mathbf{B}(X)$, we also denote by $\rho(T)$ the resolvent set of T (in X) and by $\sigma(T) = \mathbf{C} \setminus \rho(T)$ the spectrum of T (in X); for a system of commuting operators $a = (a_1, a_2, ..., a_n) \subset \mathbf{B}(X)$, we denote by $\sigma(a, X)$ the spectrum of a (in X).

Recall that a closed linear subspace Y of X is a *spectral maximal space* of $T \in \mathbf{B}(X)$ (respectively, of $a = (a_1, a_2, ..., a_n) \subset \mathbf{B}(X)$) if Y is invariant to T (respectively, to a) and

for any other closed subspace $Z \subset X$, also invariant to T (respectively, to a), such that $\sigma(T|Z) \subset \sigma(T|Y)$ (respectively, $\sigma(a,Z) \subset \sigma(a,Y)$), we have $Z \subset Y$ (see [2] and [7]).

A subset of the complex plane $\mathbf{C}$ or of the space $\mathbf{C}^n$ is said to be *of dimension* 0 or *totally disconnected* if the connected component of each point is the set consisting of the point itself.
In Section 2, we present several results obtained by Dowson in [3] and [4], which characterize the restriction and the quotient of a spectral operator with respect to an invariant subspace. Furthermore, Dowson has shown that the restriction $T|Y$ and the quotient $\dot{T}$ of a spectral operator T are spectral operators if $\sigma(T|Y)$ is totally disconnected. In addition, we show that these assertions occur if the intersection $\sigma(T|Y) \cap \sigma(\dot{T})$ is totally disconnected.

In Section 3, we extend the results obtained by Dowson for a single spectral operator to spectral systems of operators.

2. Restrictions and quotients of spectral operators

Definition 2.1. For the complex plane $\mathbf{C}$, let $\mathcal{B}(\mathbf{C})$ be the family of the Borelian sets of $\mathbf{C}$. A mapping $E: \mathcal{B}(\mathbf{C}) \to \mathcal{P}_X$ is called a *spectral measure* if the following conditions are established:

1) $E(\mathbf{C}) = I_X$ and $E(\varnothing) = 0_X$
2) $E(B_1 \cap B_2) = E(B_1)E(B_2)$, $B_1, B_2 \in \mathcal{B}(\mathbf{C})$
3) $E\left(\bigcup_{m=1}^{\infty} B_m\right)x = \sum_{m=1}^{\infty} E(B_m)x$, $B_m \in \mathcal{B}(\mathbf{C})$, $B_m \cap B_p = \varnothing$, $m \neq p$, $x \in X$
4) $\sup_{B \in \mathcal{B}(\mathbf{C})} \|E(B)\| < \infty$.

An operator $T \in \mathbf{B}(X)$ is called *spectral* if there is a spectral measure E such that:

5) $TE(B) = E(B)T$, $B \in \mathcal{B}(\mathbf{C})$
6) $\sigma(T|E(B)X) \subset \overline{B}$, $B \in \mathcal{B}(\mathbf{C})$.

The spectral measure E verifying 5) and 6) is uniquely determined by T and it is called *the spectral measure* of T ([6]).

Lemma 2.2. Let $T \in \mathbf{B}(X)$ be a spectral operator and let $A \subset \mathbf{C}$ be a Borelian set. Then the restriction $S = T|E(A)X$ is a spectral operator, with the spectral measure E_S given by the relation $E_S(B) = E(A \cap B)$, for any $B \subset \mathbf{C}$ Borelian, where E is the spectral measure of T.

Theorem 2.3. ([3], [4]) Let $T \in \mathbf{B}(X)$ be a spectral operator with its spectral measure E, let $Y \subset X$ be a closed linear subspace invariant to T and let $\dot{T}$ be the quotient operator induced by T on the quotient space $\dot{X} = X/Y$. Then the following assertions are equivalent:

1. Y is also invariant to the spectral measure E
2. $T|Y$ is a spectral operator with the spectral measure $E|Y$
3. $\dot{T}$ is a spectral operator with the spectral measure $\dot{E}$ induced by E on $\dot{X}$.

Theorem 2.4. ([3], [4]) Let $T \in \mathbf{B}(X)$ be a spectral operator whose spectrum $\sigma(T)$ is totally disconnected (i.e. $\dim \sigma(T) = 0$). Then $T|Y$ and $\dot{T}$ are spectral operators, for any closed linear subspace Y of X invariant to T.

Theorem 2.5. ([3]) Let $T \in \mathbf{B}(X)$ be spectral and let Y be a closed linear subspace of X invariant to T such that $\sigma(T|Y)$ is totally disconnected. Then $T|Y$ and $\dot{T}$ are spectral.

Theorem 2.6. ([4]) Let X be a reflexive Banach space and let $T \in \mathbf{B}(X)$ be a spectral operator. If Y is a closed linear subspace of X invariant to T such that $\sigma(\dot{T})$ is totally disconnected, then $T|Y$ and $\dot{T}$ are spectral.

Theorem 2.7. Let $T \in \mathbf{B}(X)$ be spectral and let $Y \subset X$ be a closed linear subspace invariant to T such that $\sigma(T|Y) \bigcap \sigma(\dot{T})$ is totally disconnected. Then $T|Y$ and $\dot{T}$ are spectral.

3. Restrictions and quotients of spectral systems

Definition 3.1. For the space $\mathbf{C}^n$ of all elements $z = (z_1, z_2, ..., z_n)$, with $z_i \in \mathbf{C}$, $i = 1, 2, ..., n$, let $\mathcal{B}(\mathbf{C}^n)$ be the family of the Borelian sets of $\mathbf{C}^n$. An application $E: \mathcal{B}(\mathbf{C}^n) \to \mathcal{P}_X$ is called a (C^n, X) *type spectral measure* if it verifies the conditions:

1) $E(\mathbf{C}^n) = I_X$ and $E(\varnothing) = 0_X$

2) $E(B_1 \bigcap B_2) = E(B_1)E(B_2)$, $B_1, B_2 \in \mathcal{B}(\mathbf{C}^n)$

3) $E\left(\bigcup_{m=1}^{\infty} B_m\right)x = \sum_{m=1}^{\infty} E(B_m)x$, $B_m \in \mathcal{B}(\mathbf{C}^n)$, $B_m \bigcap B_p = \varnothing$, $m \neq p$, $x \in X$.

A commuting operator system $a = (a_1, a_2, ..., a_n) \subset \mathbf{B}(X)$ is called *spectral* if there is a (C^n, X) type spectral measure E such that:

4) $a_i E(B) = E(B) a_i$, $B \in \mathcal{B}(\mathbf{C}^n)$, $i = 1, 2, ..., n$

5) $\sigma(a, E(B)X) \subset \overline{B}$, $B \in \mathcal{B}(\mathbf{C}^n)$.

The spectral measure E verifying the conditions 4) and 5) is uniquely determined by a and it is called *the* (C^n, X) *type spectral measure* of a ([2]).

Lemma 3.2. ([8]) Let $a = (a_1, a_2, ..., a_n) \subset \mathbf{B}(X)$ be a spectral system and let E be its (C^n, X) type spectral measure. Then each operator a_i, $i = 1, 2, ..., n$, is spectral having the spectral measure E_i defined by the equality $E_i(B) = E(\pi_i^{-1}(B))$, where $B \in \mathcal{B}(\mathbf{C})$ and π_i is the corresponding projection.

Lemma 3.3. ([8]) Let $a = (a_1, a_2, ..., a_n) \subset \mathbf{B}(X)$ be a spectral system with the (C^n, X) type spectral measure E and let $A \subset \mathbf{C}^n$ be a Borelian set. Then the restriction $b = a|E(A)X$ is a spectral system, with the spectral measure E_b given by the relation $E_b(B) = E(A \bigcap B)$, for any $B \subset \mathbf{C}^n$ Borelian.

Theorem 3.4. Let $a=(a_1,a_2,...,a_n)\subset \mathbf{B}(X)$ be a spectral system with its $\left(C^n,X\right)$ type spectral measure E and let Y be a closed linear subspace of X invariant to a. The following three assertions are equivalent:

1. Y is also invariant to the spectral measure E
2. the restriction system $a|Y=(a_1|Y,a_2|Y,...,a_n|Y)$ is spectral having the spectral measure $E|Y$
3. the quotient system $\dot{a}=(\dot{a}_1,\dot{a}_2,...,\dot{a}_n)$, induced by a on the quotient space $\dot{X}=X/Y$, is spectral having the spectral measure $\dot{E}$ induced by E on $\dot{X}$.

Corollary 3.5. If $a=(a_1,a_2,...,a_n)\subset \mathbf{B}(X)$ is a spectral system and Y is a spectral maximal space of a, then the restriction $a|Y$ and the quotient $\dot{a}$ are spectral systems.

Proposition 3.6. ([7]) Let $a=(a_1,a_2,...,a_n)\subset \mathbf{B}(X)$ be an operator system with the spectrum $\sigma(a,X)$ totally disconnected (i.e. $\dim\sigma(a,X)=0$) and let Y be a closed linear subspace of X invariant to a. Then $\sigma(a,Y)\subset\sigma(a,X)$.

Theorem 3.7. Let $a=(a_1,a_2,...,a_n)\subset \mathbf{B}(X)$ be a spectral system with the spectrum $\sigma(a,X)$ totally disconnected. Then the restriction $a|Y$ and the quotient $\dot{a}$ are spectral systems, for any closed linear subspace Y of X invariant to a.

Corollary 3.8. Let $a=(a_1,a_2,...,a_n)\subset \mathbf{B}(X)$ be a spectral system and let Y be a closed linear subspace of X invariant to a such that $\dim\sigma(a,Y)=0$. Then the restriction $a|Y$ and the quotient $\dot{a}$ are spectral systems.

Theorem 3.9. Let $a=(a_1,a_2,...,a_n)\subset \mathbf{B}(X)$ be a spectral system and let Y be a closed linear subspace of X invariant to a such that $\dim\left(\sigma(a,Y)\cap\sigma\left(\dot{a},\dot{X}\right)\right)=0$. Then the restriction $a|Y$ and the quotient $\dot{a}$ are spectral.

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