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CONTENT

IHARA ZETA FUNCTIONS OF GRAPHS ASSOCIATED TO NON-CONGRUENT NUMBERS

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Abstract: For a number of the form

$$n=p_1p_2\ldots p_t,\ t\geq 3$$

we associate, following [1], a graph G(n). We compute the Ihara zeta function of G(n).
Mathematics Subject Classification (2010): 68R10, 14G10, 11M99
Key words: congruent numbers, Ihara zeta function, finite digraphs.

1. Introduction

Zeta Functions and L-function of an elliptic curve.

We recall basic facts from [2].
For an elliptic curve E over $\mathbf{Q}$ we denote Δ its discriminant. For each prime p we denote Ep the reduction of E modulo p, defined over $\mathbf{F}p$. Let $\#\, Ep(\mathbf{F}p)$ be the number of projective solutions. Put

$$a_p = p+1-\#\, Ep(\mathbf{F}p)$$

for all primes p. The local L-factor corresponding to the prime p is defined to be the formal power series

$$L_p(u) = 1/(1-a_pu+pu^2) \text{ if } p \text{ does not divide } \Delta$$

$$L_p(u) = 1/(1-a_pu^2) \text{ if } p \text{ divides } \Delta$$

The L-function of E is the product

$$\mathbf{L(s,E)} = \prod_p L_p(p^{-s})$$

The local zeta function of E is

$$Z(u,E_p) = \exp\left(\sum_{n=1}^{\infty} \frac{\#\,\mathbf{Ep(F_{p^n})u^n}}{n}\right)$$

It is known that $E(\mathbf{Q})$ is an abelian group
The weak version of the Birch Swinnerton-Dyer conjecture is:

$$\text{Rank}(E(\mathbf{Q}))=\text{ord}_{s=1}\, L(s,E(\mathbf{Q}))$$

Congruent Numbers.
A positive integer n is said to be a congruent number if it is the area of a right triangle with sides positive integers. For example n=6 is a congruent number since it is the area of the right triangle with sides (3,4,5). Finding all the congruent numbers is perhaps one of the oldest problems in mathematics. Assuming the weak Birch Swinnerton-Dyer to be true, J. Tunnell gave in 1983 an algorithmic criterion to decide whether an arbitrarily fixed positive integer n is congruent or not. We send the reader to [4] for more details.

There are other criteria for the congruence problem but they give an answer for particular positive integers n; they are not as general as Tunnell's criterion. For example in [1] Feng proved the non-congruence for infinitely many positive numbers n using graphs. In [1] Feng proved also that the Birch Swinnerton-Dyer conjecture is true for infinitely many elliptic curves, given by the equation

$$E_n:\ y^2 = x(x-n)(x+n)$$

with n noncongruent numbers of a particular form:
$n=p_1p_2\ldots p_t,\ t\geq 1,\ p_1\equiv 3(mod\ 8),\ pi\equiv 1(mod\ 8),\ i\geq 2$

or

$n=p_1p_2\ldots p_t,\ t\geq 1,\ p_1\equiv 5(mod\ 8),\ pi\equiv 1(mod\ 8),\ i\geq 2$
The following basic but very important result (proved in [4]) has been used in [1]:

n is a noncongruent number iff rank($E_n(\mathbf{Q})$)=0

We find interesting the fact that by studying one graph one can conclude the non-congruence of infinitely many n's.

Ihara Zeta Functions of Finite Oriented (Directed) Connected Graphs.
We follow [4],[5].
An oriented (directed) graph is denoted X=(V,E) with E a subset of the Cartesian product $V\times V$. Let e=(a,b) be an edge; we denote o(e)=a (the origin of e) and t(e)=b (the terminus of e) and $\bar{e}$ =(b,a). A *prime cycle* in X is an equivalent class of a sequence of edges such that each vertex of one of these edges is exactly once target for one edge and exactly once terminus for one edge. The length of a prime cycle p is defined to be the number of edges defining the cycle p and it is denoted by |p|. The Ihara Zeta Function of X is

$$Z(u)=\prod_p (1-u^{|p|})^{-1}$$

where the product is taken on all prime cycles p.

2. Ihara Zeta Functions of G(n)

For a number of the form

$$n=p_1p_2\ldots p_t,\ t\geq 3$$

we associate, following [1], a graph G(n). We compute the Ihara zeta function of G(n).
We define a directed graph G(n) as follows: the vertices of G(n) are all prime factors of n and for $p_i\neq p_j$ there exists an edge (p_i, p_j) in G(n) iff the Legendre symbol $(\frac{p_j}{p_i})=-1$. We put $(\frac{m}{2})=-1$ for all odd m.

Examples.

1)

$n = 105 = 3 \cdot 5 \cdot 7$

We have:

$(\frac{3}{5}) = -1 = (\frac{5}{3})$, $(\frac{3}{7})$=-1, $(\frac{7}{3})$=1, $(\frac{7}{5})$=-1=$(\frac{5}{7})$.

The edges are (3,5),(5,3),(3,7),(5,7),(7,5). The prime cycles are: [(3,5),(5,3)],[(5,7),(7,5)], [(3,7),(7,5),(5,3)]. The Ihara Zeta Function is

$$Z(u)=1/((1-u^2)(1-u^2)(1-u^3))$$

2)

$n = 1155 = 3 \cdot 5 \cdot 7 \cdot 11$

We have:

$(\frac{3}{11})$=1, $(\frac{11}{3})$=-1, $(\frac{5}{11})$=1= $(\frac{11}{5})$, $(\frac{7}{11})$-=1, $(\frac{11}{7})$=1.

The edges are (3,5),(5,3),(3,7),(5,7),(7,5), (11,3)(7,11). The prime cycles are: [(3,5),(5,3)],[(5,7),(7,5)], [(3,7),(7,5),(5,3)],[(7,11),(11,3),(3,7)] [(3,5),(5,7),(7,11),(11,3)]. The Ihara Zeta Function is

$$Z(u)=1/((1-u^2)(1-u^2)(1-u^3)\ (1-u^3)\ (1-u^4))$$

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CONTRACTIVE SEQUENCES IN METRIC SPACES

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Abstract: In this paper we define the notions of contractive and semi-contractive sequences in metric spaces and we show that every semi-contractive sequence is a Cauchy sequence. Also we show that any contractive sequence is semi-contractive and we give a nice proof for Rakotch fixed point theorem.
Mathematics Subject Classification (2010): 46B25
Key words: Contractive sequences, fixed point, metric spaces.

1. Contractions on $\mathbb{R}_+$ and associated marginal sequences

In the sequel we consider the functions $\varphi:(0,\infty)\to(0,1)$ which are decreasing. Such kind of functions will be called *functional contraction* or simply *contraction*.

Lemma 1.1 If φ is a contraction on $(0,\infty)$ and f (resp. f_x) is the function on $(0,\infty)$ given by

$$f(t)=t(1-\varphi(t))$$

$$(\text{resp. } f_x(y)=\sup\{t\in(0,\infty)/\ f(t)\le y\})$$

then we have

1). f is strictly increasing on $(0,\infty)$ and

$$\lim_{t\to 0} f(t)=0,\quad \lim_{t\to\infty} f(t)=+\infty$$

2). f_x is increasing and continuous on $(0,\infty)$ and moreover

$$\lim_{y\to 0} f_x(y)=0,\quad \lim_{y\to\infty} f_x(y)=\infty$$

3). $f_x(f(t))=t$ for all elements $t\in(0,\infty)$.

Proof. The function f is the product of two strictly positive functions, one of them being just strictly increasing and therefore the function f has all properties stated at the point 1).

The function f being strictly increasing we deduce the following equalities

$$f_x(y)=sup\{t\,|\,f(t)<y\}=inf\{t\,|\,f(t)>y\}=inf\{t\,|\,f(t)\ge y\}$$

and therefore the function $f_x:(0,\infty)\to(0,\infty)$ is increasing (not necessary strictly increasing) and continuous on $(0,\infty)$. From the above equalities we deduce also that $f_x(f(t))\le t\le f_x(f(t))$ i.e.

$$f_x(f(t))=t\quad \forall t\in(0,\infty)$$

Using the last relation and taking into account that the function f_x is increasing on $(0,\infty)$ we get

$$\lim_{y\to\infty} f_x(y)=\sup_{y\in(0,\infty)} f_x(y)\ge\sup_{t\in(0,\infty)} f_x(f(t))\ge\sup_{t\in(0,\infty)} t=\infty$$

$$\lim_{y\to 0} f_x(y) = \inf_{y\in(0,\infty)} f_x(y) \le \inf_{t\in(0,\infty)} f_x\left(f(t)\right) = \inf_{t\in(0,\infty)} t = 0$$

Remark 1.2 If the function f_x is such that

$$f_x(y') < f_x(y_0) < f_x(y'') \quad \text{if} \quad y' < y_0 < y''$$

then we have $f\left(f_x(y_0)\right) = y_0$.

Particularly if the function f_x is strictly increasing on $(0,\infty)$ then we have also

$$f\left(f_x(y)\right) = y \quad \forall y \in (0,\infty)$$

In the sequel we associate to the functional contraction φ two interesting other functions α and ω defined on $(0,\infty)$ by

$$\alpha(t) = t\varphi(t), \quad \omega(t) = t\cdot\max\left(\frac{1}{2}, \varphi\left(\frac{t}{2}\right)\right).$$

Obviously these functions are strictly positive, they are not monoton functions but we have

$$\alpha(t) \le \omega(t) < t \quad \forall t \in (0,\infty)$$

For any natural number $n \ge 1$ let us denote by α^n, respectively ω^n, the n- times iteration of the function α, respectively ω.

Theorem 1.3

a).The functions α and ω are subadditive i.e. $\alpha(t_1+t_2) \le \alpha(t_1)+\alpha(t_2)$, $\omega(t_1+t_2) \le \omega(t_1)+\omega(t_2) \quad \forall t_1, t_2 \in (0,\infty)$;

b). $\alpha(t') \le \omega(t'') \quad \forall t', t'' \in (0,\infty), t' \le t''$;

c). The sequence $\left(\omega^n\right)_n$ decreases to 0 on $(0,\infty)$;

d). The sequence of functions $\left(\alpha^n\right)_n$ is uniformely convergent to 0 on any bounded interval of $(0,\infty)$;

e). *If for any $\varepsilon \in (0,\infty)$ we denote $\delta_\varepsilon = \varepsilon - \omega(\varepsilon)$ then for any $t \in (0,\varepsilon)$ we have $\delta_\varepsilon + \alpha(t) < \varepsilon$.*

Proof. The assertion a) follows just from the definition of the functions α and ω, using the monotonicity of the function φ and the following inequalities

$$\varphi(t_1+t_2) \le \varphi(t_1), \varphi(t_1+t_2) \le \varphi(t_2), \varphi\left(\frac{t_1+t_2}{2}\right) \le \varphi\left(\frac{t_1}{2}\right), \varphi\left(\frac{t_1+t_2}{2}\right) \le \varphi\left(\frac{t_2}{2}\right)$$

b) Let $t', t'' \in (0,\infty), \quad t' \le t''$. If $t' \le \frac{t''}{2}$ then we have

$$\alpha(t') = t'\varphi(t') < t' \le \frac{t''}{2} \le \omega(t'')$$

If $t' > \frac{t''}{2}$ then we have $\varphi(t') \le \varphi\left(\frac{t''}{2}\right)$ and therefore $\alpha(t') = t'\varphi(t') \le t''\varphi\left(\frac{t''}{2}\right) \le \omega(t'')$.

Hence in both cases we get $\alpha(t') \le \omega(t'')$.

c) and d). Since $\omega(t) < t, \quad \alpha(t) < t$ for any $t \in (0,\infty)$, we deduce that the sequence of functions $\left(\omega^n\right)_n$ and $\left(\alpha^n\right)_n$ are decreasing and we have inductively

$$\alpha(t') \le \omega(t''), \quad \alpha^2(t') \le \omega^2(t'') \ldots \alpha^n(t') \le \omega^n(t'')$$

for any $t',t''\in(0,\infty),\quad t'\le t''$.

We show now that for any $r\in(0,\infty)$ we have $\lim_{n\to\infty}\omega^n(r)=0$.

In the contrary case we have

$$\beta:=\lim_{n\to\infty}\omega^n(r)=\inf\left\{\omega^n(r)\middle|n\in\mathbb{N}\right\}>0$$

and therefore

$$\beta\le\omega^{n+1}(r)=\omega^n(r)\cdot\left(\max\left(\frac{1}{2},\varphi\left(\frac{\omega^n(r)}{2}\right)\right)\right)\le\omega^n(r)\cdot\max\left(\frac{1}{2},\varphi\left(\frac{\beta}{2}\right)\right)$$

Passing to the limit when n tends to ∞ we obtain the following contradictory relation

$$\beta\le\lim_{n\to\infty}\omega^n(r)\cdot\max\left(\frac{1}{2},\varphi\left(\frac{\beta}{2}\right)\right)=\beta\max\left(\frac{1}{2},\varphi\left(\frac{\beta}{2}\right)\right)<\beta$$

Hence for any $r>0$ the sequence $\left(\omega^n(r)\right)_n$ decreases to 0.

Taking now $r\in(0,\infty)$ and $t\in(0,\infty),\quad t\le r$, from the relation $0\le\alpha^n(t)\le\omega^n(r)$ we deduce that the sequence of functions $\left(\alpha^n\right)_n$ is uniformely decreasing to 0 on the interval $(0,r]$.

e) Let $\varepsilon\in(0,\infty)$ and let δ_ε be the element of $(0,\infty)$ given by $\delta_\varepsilon=\varepsilon-\omega(\varepsilon)$. Using the assertion b) we have

$$t\in(0,\varepsilon)\Rightarrow\alpha(t)\le\omega(\varepsilon),\quad\delta_\varepsilon+\alpha(t)\le\delta_\varepsilon+\omega(\varepsilon)=\varepsilon$$

2. Contractive and semi-contractive sequences

In the sequel the function $\varphi:(0,\infty)\to(0,1)$ will be a functional contraction.

Definition. A sequence $(x_n)_n$ of $\mathbb{R}$ will be called a φ- *contractive sequence* (or simply *contractive*) if for any $n,m\in\mathbb{N}$ we have

$$|x_{n+1}-x_{m+1}|\le|x_n-x_m|\varphi(|x_n-x_m|)$$

The sequence $(x_n)_n$ of $\mathbb{R}$ will be called *semi-contractive* if there exists a sequence $\left(x_n'\right)_n$ in $\mathbb{R}$ (associated with the sequence $(x_n)_n$) such that

$$|x_n'-x_m'|\le|x_n-x_m|\varphi(|x_n-x_m|),$$

for any $n,m\in\mathbb{N}$ and such that

$$\lim_{n\to\infty}\left(x_n'-x_n\right)=0$$

Theorem 2.1

a). *Any semi-contractive sequence is convergent.*

b). *Any contractive sequence is semi-contractive.*

Proof. a) Let $(x_n)_n$ be a semi-contractive sequence in $\mathbb{R}$ and let $\left(x_n'\right)_n$ be the associated sequence. For any $n,m\in\mathbb{N}$ we have the following relations

$|x_n'-x_m'|\le|x_n-x_m|\varphi(|x_n-x_m|)$,

$|x_n-x_m|\le|x_n-x_n'|+|x_n'-x_m'|+|x_m'-x_m|$,

$|x_n-x_m|\le|x_n-x_n'|+|x_n-x_m|\varphi(|x_n-x_m|)+|x_m'-x_m|$,

$|x_n-x_m|(1-\varphi(|x_n-x_m|))\le|x_n-x_n'|+|x_m'-x_m|$.

We consider the increasing function $f_x:(0,\infty)\to(0,\infty)$ associated with the function φ (see **Lemma 1.1**) and we apply it to the last of the above inequalities:

$$|x_n - x_m| = f_x\left(|x_n - x_m|\left(1-\varphi\left(|x_n - x_m|\right)\right)\right) \le f_x\left(|x_n - x_n^{'}| + |x_m - x_m^{'}|\right).$$

Using Lemma 1.1, 2 and using the fact that $\lim_{n,m\to\infty}\left(|x_n - x_n^{'}| + |x_m - x_m^{'}|\right) = 0$ we deduce that the sequence $(x_n)_n$ is a Cauchy sequence.

b) If $(x_n)_n$ is a contractive sequence we have

$$|x_{n+1} - x_{n+2}| \le |x_n - x_{n+1}|\varphi\left(|x_n - x_{n+1}|\right) \le |x_n - x_{n+1}|$$

Let α be the limit of the decreasing sequence $\left(|x_n - x_{n+1}|\right)_n$. From the previsions inequalities we get $\varphi\left(|x_n - x_{n+1}|\right)_n \le \varphi(\alpha)$ and

$$\alpha = \inf_n |x_{n+1} - x_{n+2}| \le \alpha\varphi(\alpha) \le \alpha$$

Since for any $t\in(0,\infty)$ we have $\varphi(t)<1$ we deduce $\alpha = 0$ i.e $\lim_{n\to\infty}|x_n - x_{n+1}| = 0$. We finish the proof taking as associated sequence with $(x_n)_n$, the sequence $(x_n^{'})_n$ given by $x_n^{'} = x_{n+1}$.

Remark 2.2 We can define the notions of *contractive* (resp. *semi-contractive*) *sequence* in a *metric space* (X,d) just replacing $|x-y|$, when $x,y\in\mathbb{R}$, by $d(x,y)$, when $x,y\in X$. In this way we have the following

Theorem 2.3

a). *Any contractive sequence in a metric space is a semi-contractive sequence;*

b). If (X,d) *is complete, then any semi-contractive sequence is convergent.*

3. Applications

It is well know the famous Banach fixed point theorem namely: If (X,d) is complete metric space and $T:X\to X$ is an α - Lipschitz map, $\alpha\in\mathbb{R}$, $0\le\alpha<1$ i.e. $d(Tx,Ty)\le\alpha d(x,y)$ then there exists $x_0\in X$ such that $Tx_0 = x_0$. The point x_0 which is called a *fixed point of* T is uniquely determined.

This result was really improved by Rakotch (1962) who replaced the above constant $\alpha\in(0,1)$ by a contractive mapping.

We give bellow a proof of Rakotch's theorem using our result (Theorem 2.2) in this context

Rakotch theorem *If* (X,d) *is a complete metric space and* $\varphi:(0,\infty)\to(0,1)$ *is a functional contraction then any* φ*-contraction* $T:X\to X$ *(i.e. we have* $d(Tx,Ty)\le d(x,y)\cdot\varphi\left(d(x,y)\right)$, $\forall x,y\in X$ *) has a unique fixed point* $x_0\in X$.

Proof. We consider an arbitrary point $z\in X$ and we take the sequence $(x_n)_n$ in X inductively defined by

$$x_0 = z,\quad x_{n+1} = Tx_n \quad \forall n\in\mathbb{N}.$$

We constate immediately that the sequence $(x_n)_n$ is contractive because we have

$$d(x_{n+1}, x_{m+1}) = d(Tx_n, Tx_m) \le d(x_n, x_m)\cdot\varphi\left(d(x_n, x_m)\right)$$

Hence the sequence $(x_n)_n$ is convergent to an element $x_0\in X$. Obviously the map T is continuous and therefore $\lim_{n\to\infty} Tx_n = Tx_0$.

But $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x_{n+1} = x_0$ and therefore $Tx_0 = x_0$.

If $x_0^{'} \in X$ is such that $Tx_0^{'} = x_0^{'}$ we have $d\left(x_0, x_0^{'}\right) = d\left(Tx_0, Tx_0^{'}\right) \le d\left(x_0, x_0^{'}\right) \cdot \varphi\left(d\left(x_0, x_0^{'}\right)\right)$,

$d\left(x_0, x_0^{'}\right)\left(1-\varphi\left(d\left(x_0, x_0^{'}\right)\right)\right) = 0$.

If $x_0^{'} \neq x_0$ we have $d\left(x_0, x_0^{'}\right) > 0, \quad 1-\varphi\left(d\left(x_0, x_0^{'}\right)\right) > 0$ and we contradict the above equality.

Hence $x_0^{'} = x_0$.

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ON SOME EXAMPLES OF SPECTRAL GROUPS OF AUTOMORPHISMS IN QUANTUM LOGICS

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Abstract: Hilbert space theory has long been established as the appropriate mathematical framework for the study of quantum mechanics [1]. The bivalent propositions concerning a quantum system form an orthomodular lattice, the so-called logic associated to the quantum system, isomorphic to the partially ordered set of projection operators on the Hilbert space associated to the system [2,3].
In previous papers [4,5], we aimed to find out to what extent some of the fundamental physical facts concerning quantum systems can be described in the more general framework of orthomodular lattices (also known as *quantum logics*), without the support of Hilbert space specific tools. To this end, we introduced the notion of *spectral automorphism* and attempted to create in orthomodular lattices an analogue of the Hilbert space spectral theory [4]. In [5], the theory of spectral automorphisms was further developed and the notion of *spectral family of automorphisms* was introduced. A *Stone-type theorem* (in the sense of the Stone theorem concerning strongly continuous one-parameter unitary groups) was proved for spectral groups of automorphisms fulfilling certain conditions.
In this note, we give and discuss a number of previously missing examples of spectral groups of automorphisms satisfying the conditions of the above mentioned theorem, thus providing more substance to this result.
Mathematics Subject Classification (2010): 81P10, 03G12, 06C15.
Key words:quantum logic, orthomodular lattice, spectral automorphism, Stone theorem, automorphisms group.

1. Introduction

Hilbert space theory has long been established as the appropriate mathematical framework for the study of quantum mechanics [1]. Since the quantum phenomena do not conform to the classical Boolean logic, it was also necessary to find the appropriate type of logic to describe them. It turned out that the bivalent propositions concerning a quantum system form an orthomodular lattice (OML), the so-called logic associated to the quantum system, isomorphic to the partially ordered set of projection operators on the Hilbert space associated to the system [2,3].
In previous papers [4,5], we aimed to find out to what extent some of the fundamental physical facts concerning quantum systems can be described in the more general framework of orthomodular lattices (also known as quantum logics), without the support of Hilbert space specific tools. To this end, we introduced the notion of spectral automorphism and attempted to create in orthomodular lattices an analogue of the Hilbert space spectral theory [4]. In [5], the theory of spectral automorphisms was further developed and the notion of spectral family of automorphisms was introduced. An analogue, in the framework of orthomodular lattices, of the Stone theorem concerning strongly continuous one-parameter unitary groups was proved for spectral groups of automorphisms fulfilling certain conditions.

In this note, we give and discuss a number of previously missing examples of spectral groups of automorphisms satisfying the conditions of the above mentioned theorem, thus providing more substance to this result.
In the next section, we recall a few basic facts about orthomodular lattices. Then, in the third section of the paper, we briefly summarise previously obtained results on spectral automorphisms and spectral families of automorphisms, which are necessary for our discussion. Finally, in the last section, we investigate the properties of the spectral groups of automorphisms fulfilling the conditions of the above-mentioned Stone-type theorem, thus elaborating on the requirements that have to be met by such spectral groups. This allows us to find new (types of) examples of such spectral groups and make a few remarks on how more examples can be found – or should be looked for.

2. Basics on orthomodular lattices

Let us recall now some of the important facts about orthomodular lattices. For a detailed treatment of the subject, we refer to [6].

Definition 2.1 An *orthomodular lattice* is a structure $(L,\leq,',\mathbf{0},\mathbf{1})$ with $\mathbf{0}\neq\mathbf{1}$ such that:

(a) $(L,\leq)$ is a bounded lattice, with $\mathbf{0}$ as the least element and $\mathbf{1}$ as the greatest element;

(b) the unary operation $'$ is an orthocomplementation, i.e., the following hold for every $a,b\in L$: $(i)\ (a')'=a$; $(ii)\ a\leq b\Rightarrow b'\leq a'$; $(iii)\ a\vee a'=\mathbf{0}$;

(c) $a\leq b$ implies $b=a\vee(a'\wedge b)$ for every a,b in L (*orthomodular law*).

In the following, L will denote an orthomodular lattice. Elements $a,b\in L$ are *orthogonal*, denoted $a\perp b$, if $a\leq b'$ (i.e., $b\leq a'$). Minimal non-zero elements of L are called *atoms.* L is *atomic* if its every element dominates an atom and *atomistic* if every element of L is the least upper bound of the set of atoms it dominates. Every atomic OML is atomistic. An OML L is *complete* if the supremum of arbitrary subsets (and therefore the infimum as well) exists in L.

Definition 2.2 Let L be an OML. A mapping $\varphi: L\to L$ is an *automorphism* if and only if it satisfies the following conditions:

(a) $\varphi(\mathbf{1})=\mathbf{1}$;

(b) $a\perp b$ implies $\varphi(a)\perp\varphi(b)$ for all $a,b\in L$;

(c) $a\leq b$ if and only if $\varphi(a)\leq\varphi(b)$ for all $a,b\in L$;

(d) φ is surjective.

We shall make use, in the sequel, of the following important result [3] :

Theorem 2.3 Every automorphism of an atomic, complete OML L is uniquely determined by its restriction to the set of atoms (denoted $\Omega(L)$), which is a bijective map that preserves orthogonality both ways.

The above result obviously means that every bijective map $\varphi:\Omega(L)\to\Omega(L)$ that preserves orthogonality both ways can be extended in an unique way to an automorphism of L. Thus, it will prove to be a useful tool for building examples of automorphisms later on.

Let us now define the important and physically meaningful relation of compatibility.

Definition 2.4 Elements a,b of the OML L are *compatible* if $a=(a\wedge b)\vee(a\wedge b')$. In this case we write $a\leftrightarrow b$.

Compatible pairs represent simultaneously verifiable events, hence their importance in quantum theory.

It is not difficult to see that, in orthomodular lattices, the compatibility relation is symmetric. It is also evident from the definition that $a\leftrightarrow b$ if and only if $a\leftrightarrow b'$ —a fact that we shall make use of in what follows. An OML in which every pair of elements is compatible is a Boolean algebra. For a subset M of L and an element $a\in L$, we will write $a\leftrightarrow M$ when

$a \leftrightarrow x$ for all $x \in M$. The *center* of an OML L is the set $C(L) = \{x \in L \mid x \leftrightarrow L\}$. The center of an OML is a Boolean subalgebra of L. A *block* of an OML L is a maximal set of pairwise compatible elements, i.e., a maximal Boolean subalgebra of L. Every element of L is part of (at least) a block of L, hence L is the set-theoretical union of its blocks. On the other hand, it is clear that the set-theoretical intersection of all blocks of L is its center $C(L)$.

3. Previous results on spectral automorphisms, spectral families and a Stone-type theorem

As mentioned earlier, a (perhaps tentative) analogue in abstract OMLs of the Hilbert space spectral theory was elaborated in previous articles [4,5]. Let us recall now the essential elements of this construction, that are necessary for discussing our main results here.

Definition 3.1 Let L be an OML and $\varphi : L \to L$ be an automorphism. The automorphism φ is *spectral* if there is a Boolean subalgebra $B \subset L$ such that $\varphi(a) = a$ if and only if $a \leftrightarrow B$.

Such a Boolean subalgebra B is called a *spectral algebra* of φ. The greatest spectral algebra of φ is the *spectrum* of the spectral automorphism φ, denoted by σ_φ. We shall also denote by L_φ the set of φ-invariant elements of L. It is a straightforward verification that, for every automorphism $\varphi : L \to L$, L_φ is a subortholattice of L.

The following result allow us to find the spectrum of a spectral automorphism:

Proposition 3.2 Let L be an OML and $\varphi : L \to L$ be a spectral automorphism of L. Then $\sigma_\varphi = C(L_\varphi)$.

Proposition 3.3 Let L be an OML. Then:

(a) the identity $id : L \to L$ is a spectral automorphism, with $\sigma_{id} = C(L)$;

(b) if $\varphi : L \to L$ is a spectral automorphism, then φ^{-1} is also a spectral automorphism and $\sigma_{\varphi^{-1}} = \sigma_\varphi$.

Definition 3.4 Let L be an OML and Φ be a family of automorphisms of L. The family of automorphism Φ is *spectral* if there is a Boolean subalgebra $B \subset L$ such that $\varphi(a) = a$ for every $\varphi \in \Phi$ if and only if $a \leftrightarrow B$.

A Boolean subalgebra B with the above property is called a *spectral algebra* of Φ. The greatest spectral algebra of Φ is the *spectrum* of the spectral family Φ, denoted by σ_Φ. Let us denote by L_Φ the set of Φ-invariant elements of L. Obviously, $L_\Phi = \bigcap_{\varphi \in \Phi} L_\varphi$ and it is a subortholattice of L.

It should be noted that a family of spectral automorphisms is *not* necessarily a spectral family of automorphisms.

The following results concerning spectral families of automorphisms hold [5]:

Proposition 3.5 If Φ is a spectral family of automorphisms of an OML L, then $\sigma_\Phi = C(L_\Phi)$.

Theorem 3.6 Let Φ be a family of spectral automorphisms of an OML L. Then, Φ is a spectral family of automorphisms of L if and only if $\sigma_\varphi \leftrightarrow \sigma_\psi$ for every $\varphi, \psi \in \Phi$.

The next theorem [5,Theorem 6.8] was intended as an analogue, in the framework of quantum logics, of the Stone theorem concerning strongly continuous one parameter groups of unitary operators.

Theorem 3.7 (*A Stone-type theorem*) Let L be an OML and Φ be a family of spectral automorphisms of L. If Φ is an Abelian group and $\varphi(L_\psi) = L_{\varphi\psi}$ for every $\varphi, \psi \in \Phi - \{id, \varphi^{-1}\}$, then :

(a) $L_\varphi = L_\psi$ for every $\varphi, \psi \in \Phi - \{id\}$;

(b) $\sigma_\varphi = \sigma_\psi$ for every $\varphi, \psi \in \Phi - \{id\}$;

(c) Φ is a spectral family of automorphisms.

Let us now use an example from [5] to illustrate some of the facts presented so far. In order to represent the OMLs in this example and in other examples that will follow, we will use the technique of Greechie diagrams [6]. This type of diagrams consist of a set of points and a set of lines. The points represent the atoms of the OML, while the lines correspond to its blocks. In the examples that follows, when defining an automorphism on a finite atomic OML, we will only specify its action on atoms, since an automorphism of such an OML is uniquely determined by its restriction to the set of atoms, according to Theorem 2.3.

Example 3.8 [5, Example 6.9] Let L be the OML described by the Greechie diagram in Fig.1. It is the union of three blocks: the first determined by atoms $\{a, b, c\}$, the second by atoms $\{c, d, e\}$ and the last determined by atoms $\{e, f, g, h, i\}$. Let φ be an automorphism of L that leaves atoms *a,b,c,d,e* invariant, while performing a circular permutation on atoms *f,g,h,i*. We will prove that φ is spectral. Clearly, L_φ is the union of the first two blocks of L, determined by atoms $\{a, b, c\}$ and $\{c, d, e\}$, respectively. It follows that the center $C(L_\varphi)$ is the Boolean algebra $\{\mathbf{0}, c, c', \mathbf{1}\}$. Thus, for an element $x \in L$, the following are equivalent: $x \in L_\varphi$, $x \leftrightarrow c$, $x \leftrightarrow C(L_\varphi)$, and therefore, φ is spectral and $\sigma_\varphi = C(L_\varphi) = \{\mathbf{0}, c, c', \mathbf{1}\}$. Let now Φ be the cyclic group generated by φ, i.e., $\Phi = \{id, \varphi, \varphi^{\mathbf{2}}, \varphi^{\mathbf{3}}\}$ (since $\varphi^{\mathbf{4}} = id$). Clearly Φ is an Abelian group. Moreover, $L_\varphi = L_{\varphi^2} = L_{\varphi^3} = L_\Phi$, therefore $C(L_\varphi) = C(L_{\varphi^2}) = C(L_{\varphi^3}) = $ $= C(L_\Phi)$ and it follows that all the automorphisms of Φ are spectral, and have the same spectrum. It is now easy to see that the group Φ verifies the hypothesis of Theorem 3.7, and also its conclusion.

4. Main results

The Stone-type theorem stated above may be useful, like any mathematical theorem, only if it does not operate in void, i.e., if there are such mathematical objects to which the theorem applies. Surely, the larger the set of mathematical objects fulfilling the hypothesis of the theorem, the more relevant it is. The previous Example 3.8 shows that indeed there exist nontrivial groups of automorphisms of an OML, to which Theorem 3.7 may be applied. However, this example is, in a sense that will be made precise shortly, somewhat too simple.

We discuss, in what follows, the properties of the spectral groups of automorphisms fulfilling the conditions of Theorem 3.7. We will then be able to find new examples of spectral groups to which the theorem applies and make a few remarks on how such examples can be found.

Firstly, let us estabilish the following result:

Theorem 4.1 Let L be an OML, let Φ be a family of automorphisms of L and let $G = \langle \Phi \rangle$ be the group of automorphisms of L generated by Φ. The following are equivalent:

(a) Φ is a spectral family of automorphisms of L;

(b) G is a spectral group of automorphisms of L.

Proof: We have $L_\Phi = \bigcap_{\varphi \in \Phi} L_\varphi$ and $L_G = \bigcap_{\phi \in G} L_\phi$. Since $\Phi \subset G$, it follows that $L_G \subset L_\Phi$. Let us prove the inverse inclusion. To this end, let us notice that, for every $\varphi, \psi \in \Phi$ we have:

(1) $L_{\varphi^{-1}} = L_\varphi$, since $\varphi(x) = x$ if and only if $x = \varphi^{-1}(x)$, for all $x \in L$,

(2) $L_\varphi \cap L_\psi \subset L_{\varphi\psi}$, since $\varphi(x) = x$ and $\psi(x) = x$ imply $\varphi(\psi(x)) = x$, for all $x \in L$.

Because every element $\phi \in G$ is a product of elements of Φ and of inverses of such elements, by applying the above observations (1)–(2) repeatedly, if necessary, we find that $L_\phi \supset \bigcap_{\varphi\in\Phi} L_\varphi$, for all $\phi \in G$. It follows that $\bigcap_{\phi\in G} L_\phi \supset \bigcap_{\varphi\in\Phi} L_\varphi$, i.e., $L_G \supset L_\Phi$, hence $L_G = L_\Phi$.

We are now able to derive the conclusion of the theorem by noticing that the following is a sequence of equivalent assertions: Φ is a spectral family of automorphisms of L, $x \in L_\Phi$ if and ony if $x \leftrightarrow C(L_\Phi)$, $x \in L_G$ if and ony if $x \leftrightarrow C(L_G)$, G is a spectral family of automorphisms of L ■

Corollary 4.2 Let L be an OML, let φ be an automorphism of L and let $G = \langle\varphi\rangle$. If φ is a spectral automorphism, then G is a spectral group of automorphisms of L.

Proof: The result is a straightforward consequence of Theorem 4.1 ■

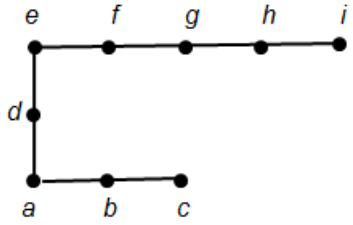

Fig. 1 Greechie diagram of the OML used in Example 3.8

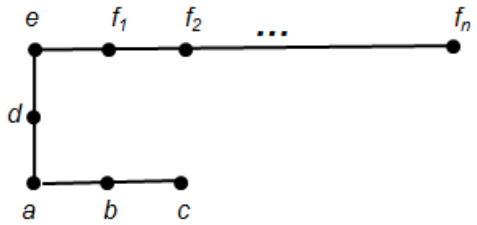

Fig. 2 Greechie diagram of the OML *L(n)* used in Examples 4.3 and 4.4

Let us now have a closer look to Example 3.8. The family Φ considered there is the cyclic group $\langle\varphi\rangle$. It is then obvious that the set of invariant elements is the same for all the nontrivial elements of Φ. Since φ is spectral, in view of Corollary 4.2, the family Φ has to be spectral. We conclude that a more interesting example to illustrate Theorem 3.7 should refer to a *noncyclic* (but Abelian) group of spectral automorphisms.

Let us turn again to the OML from Example 3.8 and generalise it to a lattice $L(n)$ corresponding to the Greechie diagram in Fig.2. We will try to find noncyclic groups Φ of automorphisms of *L(n)* fulfilling the conditions of Theorem 3.7. Obviously, such a group will also satisfy the conclusions of the theorem, e.g., the set of invariant elements must be the same for all the nontrivial automorphisms of the group. It is not difficult to see that, by taking an example that satisfies $L_\varphi = L_\psi$ for every $\varphi, \psi \in \Phi - \{id\}$, the condition $\varphi(L_\psi) = L_{\varphi\psi}$ from the hypothesis of the Theorem 3.7 is also verified for every $\varphi, \psi \in \Phi - \{id, \varphi^{-1}\}$.

Let us assume that the first two blocks of *L(n)*, determined by atoms $\{a,b,c\}$ and $\{c,d,e\}$ respectively, are invariant under all elements of the group (otherwise, if only one block is invariant, we have $L(n)_\Phi = C(L(n)_\Phi) = \sigma_\Phi$ which leads to a simplistic example). We will prove that none of the atoms f_i, *i=1..n* may be invariant under the automorphisms of the group Φ. Indeed, let us assume to the contrary, that one of these atoms f_i is invariant under the elements of Φ. Then $C(L(n)_\Phi) = \{\mathbf{0},\mathbf{1}\} = \sigma_\Phi$, which in turn implies, by the spectrality of the automorphisms in Φ, that $\Phi = \{id\}$, since the identity is the only spetral automorphism with trivial spectrum $\{\mathbf{0},\mathbf{1}\}$.

We conclude that our task can be summarized as follows: *find an Abelian noncyclic group of automorphisms of L(n) that leave atoms a,b,c,d,e invariant and permute (without leaving*

invariant) all the atoms f_i, *i=1..n.* Obviously, this corresponds to finding Abelian noncyclic subgroups of the symmetric group S_n such that no element of the subgroup has fixed points.

The simplest example of such a subgroup can be found for *n=4*, namely the Klein group. Using the cycle notations, it can be written :{*id*, (*12*)(*34*), (*13*)(*24*), (*14*)(*23*)}. For *n=6*, the only such subgroups of S_6 are the ones of the same type as the following: {*id*, (*0,1,2*)(*3,5,4*), (*0,2,1*)(*3,4,5*), (*0,3*)(*1,5*)(*2,4*), (*0,4,1,3,2,5*), (*0,5,2,3,1,4*)}. Thus, we can produce the corresponding two examples of spectral noncyclic Abelian groups of automorphisms that fulfill the conditions of Theorem 3.7:

Example 4.3 Let us consider the lattice *L(n)* for *n=4*. Of course, it is the same as the lattice in Example 3.8, except for the notation of the atoms. We take $\Phi = \{id, \varphi_1, \varphi_2, \varphi_3\}$ such that the blocks determined by atoms $\{a,b,c\}$, and $\{c,d,e\}$ are invariant under φ_i, *i=1..3* and $\varphi_i(f_j) = f_{\sigma_i(j)}$ for all *i=1..3* and all *j=1..4*, where $\sigma_1 = (12)(34)$, $\sigma_2 = (13)(24)$, $\sigma_3 = (14)(23)$ are elements of S_4 written in cycle notation.

Example 4.4 Similarly, let us consider the lattice *L(n)* for *n=6* and $\Phi = \{id, \phi_1, \phi_2, \phi_3, \phi_4, \phi_5\}$ such that the blocks determined by atoms $\{a,b,c\}$, and $\{c,d,e\}$ are invariant under ϕ_i, *i=1..5* and $\phi_i(f_j) = f_{\sigma_i(j)}$ for all *i=1..5* and all *j=1..6*, where $\sigma_1 = (0,1,2)(3,5,4)$, $\sigma_2 = (0,2,1)(3,4,5)$, $\sigma_3 = (0,3)(1,5)(2,4)$, $\sigma_4 = (0,4,1,3,2,5)$, $\sigma_5 = (0,5,2,3,1,4)$ are elements of S_6 written in cycle notation.

The verification of the fact that the families of automorphisms in Examples 4.3 and 4.4 satisfy the hypothesis of Theorem 3.7 is similar to the proof for Example 3.8, and therefore we omit it.

Our intention here was only to show that proper examples of groups of automorphisms to which Theorem 3.7 apply exist and, more important, to illustrate the technique that can be used to find them. It is clear that, by using the same technique explained here and investigating the subgroups of the symmetric group S_n, one can find more such examples of groups of automorphisms of the lattice *L(n)* that fulfill the conditions of Theorem 3.7. Surely, in trying to find groups of automorphisms to which Theorem 3.7 apply, one is not restricted to the lattice *L(n)*. It is, for example, possible to add more blocks to it that will have non-invariant elements. This would considerably enlarge the number of possibilities of finding the desired examples.

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CONSUMER MODELS AND THE COMMON INFLUENCE OF INCREASING VAT AND DECREASING WEDGES

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Abstract: In this paper we study the common impact of increasing VAT and decreasing incomes in consumer models. The considered models are linear ones (see [3], [4] and [8]). It is in fact the extension of the study [2], where there was performed the study of the impact of only one of the mentioned government decision (increasing VAT). We have already noticed that applying the simple three rule is not appropriate. But the problems that arise come from the common impact. It is possible that if it is applied only the decreasing of the wedges (25%), the incomes from selling products decreases by the ratio β, if we apply only increasing VAT the income decreases by the ratio α (we have obtained in [2] $\alpha = 4.01653$ ignoring the dependence of quantity on wedges), but if there are applied both the income decreases by the ratio $\gamma \neq 1-(1-\alpha)(1-\beta)$.

This is the general case, and the explanation of such phenomenon comes from analogous reasons as in [2]: the total income is the sold quantity multiplied by the price, hence we have not linearity. Another explanation comes from the least squares method: in the obtained linear system for estimating the three parameters (intercept, coefficient of prices and coefficient of wedges) both variables influence the result.
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Key words: Linear regression, consumer models, interests.

1. Introduction

Consider n points in R^{k+1}, $X^{(1)},...,X^{(n)}$, where $X^{(i)} = \left(X_1^{(i)}, X_2^{(i)}, ..., X_k^{(i)}, Y_i\right)$. The regression hyper-plane has the equation (see [7,3])

$$H: Y = A_0 + \sum_{i=1}^{k} A_i X_i \text{ such that} \tag{1}$$

$$\sum_{i=1}^{n} u_i^2 \text{ is minimum,} \tag{1'}$$

where the residues u_i have the formula

$$u_i = Y_i - A_0 - \sum_{j=1}^{k} A_j X_j^{(i)}. \tag{1''}$$

For the computation of A_i from (1) we have to solve the system (see [7,4])

$$\sum_{j=0}^{k} \overline{X_i \cdot X_j} \cdot A_j = \overline{X_i \cdot Y},\ i = \overline{0,k}, \tag{2}$$

where $\overline{X_0 \cdot X_i} = \overline{X_i}$ and $\overline{X_0^2} = 1$.

If we apply the linear regression to a consumer model, the dependent variable is the sold quantity. One of the explanatory variables is the unit price, and there are some other explanatory variables for the multi-linear model, as income and advertising. As we expect, the coefficient of price is negative.

Next we will present some notion about the theory of interest in order to point out the errors that appear when we replace the multiplicative model by the additive one.

The interest for a financial operation represents the sum added periodically to the initial sum S_0 during the operation time T, until its end. The interest can be simple or compound (see [6]).

Denoting by S_t the amount at the moment t from the initial moment to the maturity, T, and by R the interest rate (the ratio of the initial sum added periodically), we obtain

$$S_t = S_0(1 + R \cdot t). \tag{3}$$

The above interest rate can be expressed even as a number between 0 and 1 (the ratio between the periodically added sum and the initial one), even as percentage: $100 \times R = p\%$. If we want to compute the initial capital in terms of the final capital, the duration of the operation and the interest rate, we obtain

$$S_0 = \frac{S_T}{1 + R \cdot T}. \tag{3'}$$

In the case of compound interest the maturity is divided into n periods, T_1, T_2,..., T_n. The final capital is in this case (see [6])

$$S_T = S_0 \cdot \prod_{i=1}^{n} (1 + R_i T_i), \tag{4}$$

where S_0 is the initial capital, and R_i is the interest rate on the period T_i.

Usually T is divided into equal time periods, the common length of these time periods becoming time unit $(T_i = 1)$. If the interest rate is constant, R, over all the duration T, the formula (4) becomes

$$S_T = S_0(1 + R)^T. \tag{4'}$$

Remark 1. *Sometimes the maturity,* T*, is not supposed to be integer, considering* $T = n + T_{n+1}$ *with* $T_{n+1} \in (0,1)$. *In this case the above value* S_t *is the trading solution, and the rational solution is* $S_T = S_0(1 + R)^n \cdot (1 + R \cdot T_{n+1})$.

In the case of deposits at given term, it is used also the solution with lost interest: $S_T = S_0(1 + R)^n$.

The initial capital is computed using (4) and (4'), as in the case of simple interest. We obtain the general formula

$$S_0 = \frac{S_T}{\prod_{i=1}^{n} (1 + R_i T_i)}, \tag{5}$$

and in the particular case $T_i = 1$ and $R_i = R$

$$S_0 = \frac{S_T}{(1 + R)^T}. \tag{5'}$$

In [2] we have pointed out some errors that can appear in consumer models. First one is that if VAT increases from 19% to 24% even the prices do not increase by 5%: the increasement of prices is only 4.20168%. Another error is to believe that the income of selling company increase by this percentage. In the application we have found that in fact the

income decreases by 4.01653%.

2. The common impact

The above decrease is explained because the sold quantity Y is expressed as linear regression on prices X [3,4]

$$Y = b_0 + b_1 X, \tag{6}$$

where b_1 is negative. The explanation of the decrease of income is that the total income of the selling company is the sum of the product between X_i and Y_i, hence modifications of X_i yields to modification of Y_i.

But in [3,4] in consumer models the sold quantity Y can be expressed as multi-linear regression on wedge X_1 and the price X_2:

$$Y = a_0 + a_1 X_1 + a_2 X_2, \tag{7}$$

where a_1 is positive a_2 is negative.

The wedges, by the coefficient a_1 from (7) takes a part of the influence of the prices, but their presence modify the intercept, and by this modification they change the linear part of the model (the total income is the sum of prices multiplied by quantities, hence the linear part is the intercept multiplied by the sum of prices).

The above differences can be noticed from expressing the income Z in terms of prices in (6), namely

$$Z = b_0 X + b_1 X^2, \tag{8}$$

which is a parabolic cylinder, respectively expressing the income in terms of both variables (wedges and prices)

$$Z = a_0 \cdot X_2 + a_1 \cdot X_1 \cdot X_2 + a_2 \cdot X_2^2, \tag{9}$$

which is a hyperbolic paraboloid, because the matrix of the conic given by the plane section $Z = z_0$ is

$$A = \begin{pmatrix} 0 & \frac{a_1}{2} & 0 \\ \frac{a_1}{2} & a_2 & \frac{a_0}{2} \\ 0 & \frac{a_0}{2} & -z_0 \end{pmatrix}. \tag{10}$$

3. Application

Consider the following consumer model, where X_1 is the wedge, X_2 is the unit price and Y the sold quantity (see [3,4]).

Table 1: The consumer model of sold quantity in terms of wedge and price

X_1	3	2	0.8	2.5	2	1.4	2.5	2.5	3	1.4	1	1.2	1.6
X_2	1.3	2.8	1.5	0.2	1.8	4	1.8	2	0.5	2.8	3.2	2.5	1.3
Y	2	0.5	1.5	3	1	0.2	2.1	1.8	3	0.7	0.5	1	1.4
X_1	0.5	2.8	3.2	2.5	1.3	2.2	3.5	1.1	0.001	0.2	2	1.2	
X_2	2.2	3.5	1.1	0.1	0.2	2	1.2	3	3	0.6	3.2	0.3	
Y	1.2	0.8	2.3	3.5	3.8	1.8	2.6	0.8	1.2	4.2	0.8	2.5	

The total income maintaining the above data is 52.6975, and the linear regression is

$$Y = 1.98335 + 0.44061 X_1 - 0.6386 X_2.$$

We consider four transformations and we study the impact of both variables separately (maintaining the values of the other explanatory variable constant), respectively

simultaneously (modifying both variables simultaneously). First transformation starts from initial data, the wedges decrease by 25% and the prices increase by 4.20168%.

The second transformation starts from new data obtained by the above simultaneous transformation, the wedges increase by 5% and the prices decrease by 12.09677% (in fact the last value is from decreasing VAT from 24% to 9%: $\left(\left(1-\frac{109}{124}\right)\cdot 100 = 12.09677\right)$.

The last two transformations start from the data obtained after the second transformation. For the third transformation the wedges increase by 5% and the prices increase by 2.75%, and for the fourth transformation the wedges decrease by 2.5% and the prices decrease by 5%. The results are presented in the following table.

Table 2: The modifications on incomes after the four transformations

Transformation	Wedges only		Prices only		Wedges and prices		$100\left(\left(1+\frac{\alpha}{100}\right)\left(1+\frac{\beta}{100}\right)-1\right)$
	Value	Percentage	Value	Percentage	Value	Percentage	
First transformation	43.6469	-17.17462	51.63935	-2.00798	42.20847	-19.90422	-18.83774
Second transformation	43.6231	3.35153	45.73207	8.34808	46.97558	11.29419	11.9794
Third transformation	48.28126	2.77949	46.49553	-1.02191	47.83711	1.83402	1.72918
Fourth transformation	46.32273	-1.38975	47.6054	1.34075	46.9852	0.02049	-0.06763

4. Conclusions

In our example if we decrease/ increase wedge maintaining constant price we have an increasement (of course, if the income decrease we consider an increasement by a negative value) of $\alpha\%$, and analogously we have an increasement of $\beta\%$ if we modify only the prices. But if we modify both variables we have an increasement of $\gamma\%$, where $\gamma \neq 100\left(\left(1+\frac{\alpha}{100}\right)\left(1+\frac{\beta}{100}\right)-1\right)$. Some explanations of this phenomenon are the non-linearity and the fact that in the obtained linear system for estimating the three parameters (intercept, coefficient of prices and coefficient of wedges) both variables influence the result.

Another phenomenon that appears in our example, analogous to the resonance when people cross a bridge, is that in the above relation γ is greater if both measures decrease the sold quantity, namely decreasing wedges and increasing VAT. For instance in the case of decreasing wedges by 25% and increasing VAT (and prices) by 4.20168%, we have $\alpha = 17.17462$ and $\beta = 2.00798$, and $100\left(1-\left(1+\frac{\alpha}{100}\right)\left(1+\frac{\beta}{100}\right)\right) = 18.83774$. But if we apply the modifications for both variable, the decrease of income is 19.90422%.

We notice that the income is not linear in wedges and prices, but according (9) it is linear on price, price times wedges and the square of price. An open problem is to estimate the linear regression of incomes Z from the above formula in terms of the above mentioned explanatory variables. We can also test the significance of the intercept, wich does not appear in (9).

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SOLVING THE BILEVEL LINEAR PROGRAMMING PROBLEM BY THE MONTE CARLO METHOD

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Abstract: In this paper we propose to use the "Monte Carlo" method for solving bilevel linear programming (linear bi-level programming problem – BLP problem). In the BLP problem, each decision maker tries to optimize its own objective function without considering the objective of the other party, but the decision of each party affects the objective value of the other party as the decision space. The existing methods for solving BLP problem can be grouped into four categories: a) methods based on vertices enumeration; b) methods based on Kuhn-Tuck conditions; c) the fuzzy approach; d) metaheuristics methods. Starting from Gnedenko's theorem, this paper uses the "Monte Carlo" method for determining the approximate solution of BLP problem. The numerical example presents the performance of the proposed approach**.**
Mathematics Subject Classification (2010): 90C05, 90C29
Key words: vertices enumeration method, decision makers, random search method, uniform random numbers, relative error

1. Bi-level linear programming problem formulation

The bi-level programming problem has its origins in the work of Stackelberg „The Theory of the Market Economy (Oxford University Press", 1952). The BLP arises in paper of Bracken and McGill "Mathematical Programs with Optimization Problems in the Constraints" (Operations Research Vol. 21 No. 1, 1973) [1]. Although the origins of BLP problem are in the game theory, BLP problem has applications in many technical fields and to set an example in hydrology to efficient use of water resources [3].
Many refereces can be found in the report of Vicente and Calamai, Bilevel and Multilevel Programming:A Bibliography Review [8].
The general formulation of a bilevel programming problem is as follows [2] (upper level):

$$\begin{cases} \min\limits_{x \in X, y} F(x, y) \\ s.t.\ G(x, y) \le 0 \end{cases} \tag{1}$$

where y can be solved from (lower level)

$$\begin{cases} \min\limits_{y} f(x, y) \\ s.t.\ g(x, y) \le 0 \end{cases} \tag{2}$$

where $x \in R^{n_1}$, $y \in R^{n_2}$. The variables of problem (1) , $x \in R^{n_1}$ are considered upper level and $y \in R^{n_2}$ are the lower level variables. The upper level decision maker controls over vector x, and the lower level decision maker controls over vector y.
In the following we consider the particular case when the objectives functions and the constraintes are liniar functions, obaining the BLP problem.
In the BLP problem, each decision maker tries to optimize its own objective function without considering the objective of the other party, but the decision of each party affects the

objective value of the other party as well as the decision space. The liniar BLP problem is an optimization model formulated as follows:

$$\begin{cases} \min\limits_{x} F(x,y) = c_1 x + d_1 y \\ \min\limits_{y} f(x,y) = c_2 x + d_2 y \\ s.t. \qquad A_1 x + A_2 y \le b, \\ \qquad\qquad x,\ y \ge 0 \end{cases} \tag{3}$$

where: $F(x,y)$ is the objective function of the leader and $f(x,y)$ is the objective function of the follower. Also, $x \in R^{n_1}$ is a vector controlled by leader and $y \in R^{n_2}$ is a vector by follower, $A_1 \in \mathcal{M}_{m,n_1}(R)$ is a $(m \times n_1)$ real matrix and $A_2 \in \mathcal{M}_{m,n_2}(R)$ is a $(m \times n_2)$ real matrix. Here $x \ge 0$ means $x_i \ge 0$, $i = \overline{1,n_1}$.

The leader decision is priority and the leader gets feedback from the follower. There are many methods to solve this kind of problems; the existing methods can be grouped into the following categories [4]:

(a) Methods based on vertices enumeration: the optimal solution should be in vertex points belonging in feasible space determined by constraints [2],

(b) Methods Based on Kuhn-Tucker conditions: the BLP problem becoming a one-level problem replacing the second level problem with complementary constraints [9]

(c) The fuzzy approach: the objective functions of the leader or the follower or both objective functions are considered like a memberships function [6], and

(d) Methods based on meta heuristics: to solve the BLP problem can be used genetic algorithm [4,5], algorithm based on simulated annealing etc.

Our proposed approach can be included in last category of methods to solve BLP problem.

2. The proposed method: optimization using random search

In this section we present methods of optimization based on *random search.* The problem is the following,

$$\min_{x \in D} f(x),$$

where $D \subset R^k$ is a $k-$dimensional set. There is a vast variety of professional literature based on properties of $f(x)$ or of D. Here we are interested in determining the point $x^* \in D$ such as $\min\limits_{x \in D} f(x) = f(x^*) = f^*$, i.e. x^* is a *global* minimum point. The global minimum point is selected from several local minimums.

The idea of *random search* is the following [7]:

Generate a large number N of random points $X_1, X_2, ..., X_N$, uniformly distributed in D (supposed to be a bounded set). Then calculate $f(X_i)$, $1 \le i \le n$ and take $f^*_{(N)} = f\left(X^*_{(N)}\right) = \min\limits_{1 \le i \le N} f(X_i)$. A theorem of Gnedenko (1943) says that in some conditions (i.e. f is a continuous function), we have $\lim\limits_{N \to \infty} f^*_{(N)} = f^*$, $\lim\limits_{N \to \infty} X^*_{(N)} = x^*$. If the optimum solution $x^* \in T^* \subset D$, (T^* is a capture set containing the solution), then it is known that for a given risk ε, $0 < \varepsilon < 1$ there is a p such as [7]:

$$P\left(X_N \in T^*; \left\| X_N - x^* \right\| \ge \varepsilon\right) = p,$$

it is necessary to use

$$N > \left[\frac{\log \varepsilon}{\log(1-p)} + 1 \right] = N^* \tag{4}$$

the previous probability being calculated for the assumed distribution of X. As p is not known, one can use $p > 1-\varepsilon$. The following algorithm allows the approximation of $x^* \in D$.

The RandSeach algorithm

Step 0. Input: $N -$ number of points uniformly distributed on D used for determining optimum value of leader function

$M -$ number of points extracted from the N used for determining optimum value of follower function X^*

$K -$ number of points uniformly distributed on the segment $\left[X^*, Z^* \right]$

Initialize: $i \leftarrow 1$; $j \leftarrow 0$;

Step 1. While $i \le N$ execute

generate $x \sim > \mathbf{U}(D)$

calculate $C_{i,1} = F(x)$

for $j = \overline{2, k+1}$ $C_{i,j} = x_{j-1}$

endfor

endwhile

Step 2. Sort C after the first column in C

for $j = \overline{2, k+1}$ $x^*_{1,j-1} = C_{1,j}$

retain M lines of C in E

for $i = \overline{1, M}$ calculated

$B_{i,1} = f\left(C_{i,2}, C_{i,3}, ..., C_{i,k}\right)$

for $j = \overline{2, k+1}$ $B_{i,j} = x_{j-1}$

endfor

endfor

Sort B after the first column in B

for $j = \overline{2, k+1}$ $y^*_{1,j-1} = B_{1,j}$

for $i = \overline{1, K}$

Generate $x \sim > \mathcal{U}\left(\left[x^*, y^* \right]\right)$

Calculate $F(x)$, $f(x)$.

$S_{i,1} = F(x)$

$S_{i,k+2} = f(x)$

for $j = \overline{2, k+1}$ $x^{**}_{i,j-1} = S_{i,j}$

Sort S after the first column in S;

Step 3. Delivre $F_{opt} = F(x^{**})$, $f_{opt} = f(x^{**})$, obtained in ; x^{**}

Stop!

Remark: The matrix S have the first line and first column the optimum value for leader objective function, the first line and last column the optimum value for follower objective

function and the interior columns of the first line, x^{**}, are the coordinates of approximate optimal point!

3. Application

We will customize the algorithm proposed in the previous section for the case of k=3 and we will compare the solution obtained with the help of the polyhedron vertex enumeration method with the approximate solution obtained with the proposed method. At the end of the section we will determine the relative error and we will make the necessary considerations.
Consider the following problem:

$$\min_{(x,y)} F(x,y,z) = 3x + y + z$$

$$\min_{z} f(x,y,z) = 2x - y - 7z$$

restrictions

$$\begin{cases} 2x - y + 2z \geq 3 \\ x - y + z \leq 1 \\ x - y + 2z \leq 2 \\ y - z \leq 1 \end{cases}$$

These planes define an ABCD tetrahedron with the following vertices:

$$A(1,1,1), B(2,1,0)\ C(1,3,2)\ , D(2,2,1),$$

3.1. Enumeration method

Firstly we use the enumeration vertices method to find exact solution.

$$F(A) = F(1,1,1) = 3\cdot 1 + 1 + 1 = 5\ , F(B) = F(2,1,0) = 3\cdot 2 + 1 + 0 = 7\ ,$$

$$F(C) = F(1,3,2) = 3\cdot 1 + 3 + 2 = 8\ , F(D) = F(2,2,1) = 3\cdot 2 + 2 + 1 = 9$$

$\min_{t\in\{A,B,C,D\}} \{F(t)\}$ obtained in the point A.

We have:

$$m = \max\left\{\frac{3}{2} - x + \frac{y}{2}, y - 1, 0\right\} \leq z \leq \min\left\{1 - x + y, 1 - \frac{x}{2} + \frac{y}{2}\right\} = M$$

For that $[m, M] \neq \Phi$ it needs that $m \leq M$, which is equivalent with the next inequalities

$$\frac{3}{2} - x + \frac{y}{2} < 1 - x + y \Rightarrow y \geq 1 \quad -x + \frac{y}{2} < 1 - x + y \Rightarrow y \geq 1 \tag{5}$$

$$\frac{3}{2} - x + \frac{y}{2} < 1 - \frac{x}{2} + \frac{y}{2} \Rightarrow x \geq 1 \tag{6}$$

$$y - 1 \leq 1 - x + y \Rightarrow x \leq 2 \tag{7}$$

$$y - 1 \leq 1 - \frac{x}{2} + \frac{y}{2} \Rightarrow x + y \leq 4 \tag{8}$$

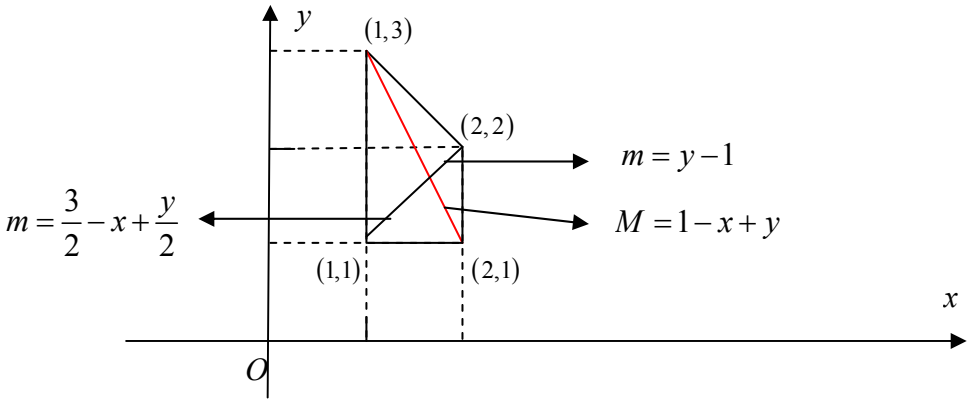
$$0 \leq 1 - x + y \Rightarrow x - y \leq 1 \tag{9}$$

$$0 \leq 1 + \frac{x}{2} + \frac{y}{2} \Rightarrow x - y \leq 2 \tag{10}$$

The (10) relation is superfluous because it results from the relation (9). The (9) relation is superfluous because it results from the (5) and (7) relations. We can note that in the domain described by the above inequalities we obtain from the (5) relation, y-1 ≥ 0, so we can discard the zero from the description of m

$$1 - x + y \leq 1 - \frac{x}{2} + \frac{y}{2} \Rightarrow y \leq x$$

$$\frac{3}{2}-x+\frac{y}{2}\geq y-1,\ \frac{5}{2}\geq x+\frac{y}{2},\ 5\geq 2x+y$$



We have $z^* = M$ because f is decreasing in the variable z.

$z^* = M = 1-x+y$ in the triangle with the vertices $(1,1),\ (2,1),\ (2,2)$

$$F(x,y,z^*) = 3x+y+1-x+y = 2x+2y+1,$$

$(1,1)\to 2\cdot 1+2\cdot 1+1=5,\ (2,1)\to 2\cdot 2+2\cdot 1+1=7,\ (2,2)\to 2\cdot 2+2\cdot 2+1=9$

$$z^* = M = 1-\frac{x}{2}+\frac{y}{2} \qquad (1,1),\ (1,3),\ (2,2)$$

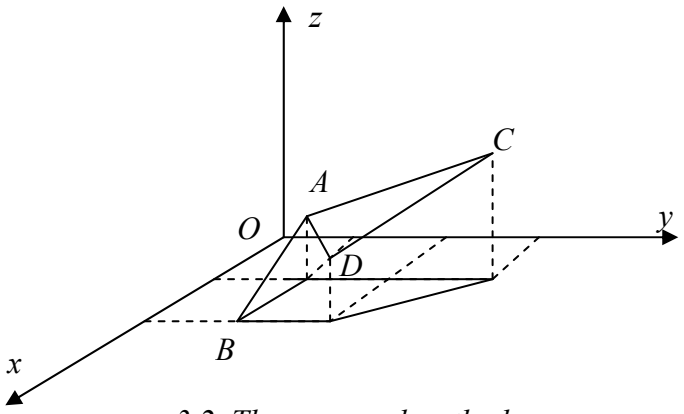
$$F(x,y,z^*) = 3x+y+1-\frac{x}{2}+\frac{y}{2} = \frac{5}{2}x+\frac{3}{2}y+1$$

$(1,1)\to \frac{5}{2}\cdot 1+\frac{3}{2}\cdot 1+1=5,\ (1,3)\to \frac{5}{2}\cdot 1+\frac{3}{2}\cdot 3+1=8,\ (2,2)\to \frac{5}{2}\cdot 2+\frac{3}{2}\cdot 2+1=9$

The optimum is obtained for $(x,y)=(1,1)$.

When $(x,y)=(1,1),\quad M=1-x+y=1,\quad M=\frac{3}{2}-\frac{x}{2}+\frac{y}{2}=1$ When $(x,y)=(1,1)$, we have

$$m=\max\left\{\frac{3}{2}-1+\frac{1}{2},1-1\right\}=1 \text{ and } M=\min\left\{1-1+1,1-\frac{1}{2}+\frac{1}{2}\right\}=1\Rightarrow z^*=1 \qquad z=1, z^*=1,$$



3.2. The proposed method

Although the minimum number of points given by equation (4) $\varepsilon = 10^{-5}$ and $p = 10^{-3}$ is $N = 11507$, we use this example the following values $N = 10^6$;.

Using the RandSeach algorithm for the previous numerical example we consider:

i) $N = 10^6;\ M = \frac{N}{10};\ K = \frac{M}{10};$

ii) the leader's function to optimize $F(x, y, z) := 3x + y + z$;

iii) the follower's function to optimize $f(x, y, z) := 2x - y - 7z$.

We obtain the approximate solution of this example

$$\min F := 5.039,\ \min f := -6.052,\ x = 1.004,\ y = 1.012,\ z = 1.005$$

By comparing the solution obtained by enumeration vertices method with the approximate solution given by proposed method we find the following relative error

$$\text{err} := \sqrt{\frac{\sum_{i=2}^{4} |S_{1,i} - S_{i-1}|^2}{\sum_{i=1}^{3} |S_i|^2}} \qquad \text{err} := 7.739 \times 10^{-3}$$

which leads us to affirm that the proposed method is "good"!

4. Conclusions

From the list of references can be seen that the BLP problem is still actual. The simplicity of the proposed approach would recommend it for practical applications. The only problem would be the finding an efficient algorithm for the numerical simulation of uniformly distributed points in the *D*.

In the considered numeric example, we can be observed that the approximate solution is sufficiently "good", the relative error being of the 10^{-3} order.

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SOME RESULTS IN THE EQUIVARIANT ALGEBRAIC TOPOLOGY

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Abstract: For a fixed compact Lie group G, we may study G – spaces using the methods of G – equivariant algebraic topology. Many classical techniques and results of algebraic topology generalise to a "global" equivariant world, although the calculations are significantly more difficult in this case (see [3]).
A very useful result for calculations in equivariant topological K – theory is the "Segal's finiteness theorem". In this paper we consider a generalization of the Segal's result to G – cohomology theories defined on a suitable category of G – spaces.
Mathematics Subject Classification (2010): 55N15, 55T99.
Key words: G -cohomology theory, equivariant K – theory, the generalized Atiyah – Hirzebruch spectral sequence.

1. Introduction

Let G be a compact Lie group and X be a compact G – space.

Definition. The G - space X is called *locally G – contractible* if each $x \in X$ has arbitrarily G_x - stable neighbourhood which is G_x - contractible in himself to x (e.g. each orbit xG has arbitrarily small G – neighbourhood of which it is a G – deformation retract).

One application in equivariant topological K – theory is the following useful finiteness theorem du to Segal ([8]).

Theorem 1. If X is a locally G – contractible compact G – space such that the orbit space X/G has finite covering dimension, then K_G^* (X) is a finite R (G) – module (one denotes by R(G) the representation ring of G)

Scheme of proof. We use the spectral sequence of Atiyah and Hirzebruch:

$$H^p(X/G\ ; \mathcal{K}_G^q) \Rightarrow \quad K_G^*\ (X)$$

where $\mathcal{K}_G^q$ is the sheaf on X/G, whose the stalk at the orbit xG = G / G_x is R(G_x) if q is even, and $\mathcal{K}_G^q = 0$ if q is odd.

Because the orbit space X/G has finite dimension (in the sense of Lebesgue), the spectral sequence of Atiyah and Hirzebruch is convergent, and so is suffices to show that $H^*(X/G\ ;\ \mathcal{K}_G^*)$ is finite over R (G).

On the other hand, X is locally G–contractible and this implies that each orbit $xG \in X/G$ has arbitrarily neighbourhood V such that $\mathcal{K}_G^q(V) \cong R(G_x)$, which is finite over R (G); one considers a covering $\mathcal{V}$ of X/G with neighbourhoods V as above and because the covering $\mathcal{V}$ is acyclic we obtain

$$H^*(\mathcal{V}\ ; \mathcal{K}_G^*) \cong H^*(X/G, \mathcal{K}_G^*)$$

The orbit space X/G having finite dimension it follows that $H^*(\mathcal{V};\mathcal{K}_G^*)$ is a direct sum of R(G_x) and quotient modules (R(G) is noetherian - see Segal [9]); this implies that $H^*(X/G,\mathcal{K}_G^*))$ is finite over R (G).

Remarks. 1. For example, a differentiable manifold X on which the compact Lie group G acts smoothly is locally G – contractible and the hypothesis that the orbit space X/G has finite dimension is satisfied in this case because X/G is then a finite union of open manifolds (Palais, 1960).

2. For the entire proof see [4]; some steps of the above proof will be used in section 3 for showing the main result of this note.

Now, let C^2 be the category of CW – pairs (X, A) and maps (X, A) → (Y, B) and we define a functor R : $C^2 \to C^2$ by setting R (X, A) = (A, Ø).

Definition. A *generalized cohomology theory* (see [6], [5]) on C^2 is a family $\{h^n, \delta^n\}, n \in \mathbb{Z}$ of contravariant functors $h^n : C^2 \to$ Ab and natural transformations $\delta^n : h^n \circ R \to h^{n+1}$ satisfying the following axioms:

(Homotopy axiom) If (X, A), (Y, B) ∈ C^2 and the maps f, g : (X, A) → (Y, B) are homotopic, then the induced homomorphisms

$$h^n(f), h^n(g) : h^n(Y,B) \to h^n(X,A)$$

coincide for every n.

(Exactness axiom) For every pair (X, A) ∈ C^2, the sequence

$$\ldots \to h^{n-1}(A,\varnothing) \xrightarrow{\delta^{n-1}} h^n(X,A) \xrightarrow{h^n(j)} h^n(X,\varnothing) \xrightarrow{h^n(i)} h^n(A,\varnothing)\ldots$$

is exact. Here i : (A, Ø) → (X, Ø) and j : (X, Ø) → (X, A) are the inclusions.

(Collapse axiom) For every pair (X, A) ∈ C^2, the collapse c : (X, A) → (X/A, {pt}) induces an isomorphism

$$h^n(c) : h^n(X/A,\{pt\}) \to h^n(X,A).$$

For the category C^* of pointed CW – spaces (X, x_0), a *reduced cohomology theory* $\{\widetilde{h^n}, s^n\}$ can be introduced and connected with the generalized ones as above. In this case $\widetilde{h^n}$: $C^* \to$ Ab is a contravariant functor and $s^n : \widetilde{h^n} \to \widetilde{h^{n+1}}$ S is a natural equivalence, where one denotes by S the suspension functor (see [6]).

The groups $h^n(\{pt\},\varnothing) = \widetilde{h^n}(S^0, *)$ are called the *coefficient groups* of the generalized cohomology theory $\{h^n, \delta^n\}$.

However, at present the mathematicians call these objects just cohomology theories and I will use this terminology in the rest of my note.

2. The generalized Borel construction. Complete families

There are many examples of such G – cohomology theories known; we shall show how to construct a large class of examples (certainly not all known examples …).

For any family S of closed subgroups of G and any G- space X, we define a topological category β (X; S) as follows

$$\text{Object space of } \beta(X;S) = \coprod_{H \in S} X^H$$

$$\text{Morphism space of } \beta(X;S) = \coprod_{(H,K)\in S\times S} Hom_G\left(\frac{G}{H},\frac{G}{K}\right)\times X^K.$$

The structural maps of $\beta(X;S)$ are defined as follows: a G – map $f: G/H \to G/K$ is represented by a coset aK for $a \in G$ (because $Hom_G(G/H, G/K)$) is naturally homeomorphic to $\left(\frac{G}{K}\right)^H$) satisfying $a^{-1}Ha \subseteq K$ and is given by $f(gH) = gaK$.

Clearly G acts on $Ob\,\beta(X;S)$ and on $Mor\,\beta(X;S)$ by the left action of G on the factors G/H, and all structural maps are equivariant with respect to this action. Thus, we have an induced G – action on the geometric realization $B(X;S)$ of $\beta(X;S)$; here we take the geometric realization $B(X;S)$ without degeneracies as in [7].

If (X, A) is a G – pair, then $B(A;S)$ is a closed subspace of $B(X;S)$, also $B(\_\,;S)$ preserves disjoint unions, and if $g: (X, A) \to (Y, C)$ is a relative G – homeomorphism, then

$$B(g;S) : (B(X;S), B(A;S)) \to (B(Y;S), B(C;S))$$

is a relative homeomorphism.

Thus, if h^* is a cohomology theory (non-equivariant) defined on the category of pairs C^2, we can define a *G - cohomology theory* on G – pairs, by

$$h^*_G(X, A; S) = h^*(B(X;S), B(A;S)).$$

One says that $h^*_G(\_\,;S)$ is the G - cohomology theory obtained from h^* by the *generalized Borel construction with respect to family* S (see [10], [9]).

Given a G - cohomology theory h^*_G one defines for each closed subgroup H of G and for an H - pair (X, A), the *associated H – cohomology theory* h^*_H, by the formula:

$$h^*_H(X, A) = h^*_G(G\times_H X, G\times_H X)$$

Here $G\times_H X$ is the orbit space of $G\times X$ under the action of H, given by $h(g,x) = (gh^{-1}, hx)$ and is a G-space via the action of G on the first factor.

Remark. Observe that, if $1 \in S$ and $h^*_G(\_\,;S)$ is defined from a cohomology theory h^* using the generalized Borel construction, then the associated cohomology theory to $h^*_G(\_\,;S)$ is canonically isomorphic to h^*.

Now we introduce some notions from [10] :

Definition 1. Let S be a family of closed subgroups of the compact Lie group G; if S is closed with respect to conjugation and finite intersection, we call S a *complete family* of subgroups.

Definition 2. A G – map $f: X \to Y$ is a *weak- (G, S) – equivalence* if for each $H \in S$, the morphism $f^H : X^H \to Y^H$ is a weak homotopy equivalence; the G – space X is called a *weak- (G, S) –CW – complex* if the fixed point set X^H is a CW –complex for each $H \in S$.

Definition 3. A G – map $f: X \to Y$ will be called a *strong- (G, S) – equivalence* if f is an H – equivariant homotopy equivalence for each $H \in S$.

Remarks 1. Clearly, if $G \in S$, then a strong- (G, S) – equivalence is just a G – homotopy equivalence; on the other hand, if $S = \{1\}$, then f is a strong- (G, S) – equivalence if f is a G –map which is a (non-equivariant) homotopy equivalence.

2. In general, G – homotopy equivalence implies strong- (G, S) – equivalence implies weak- (G, S) – equivalence.

Using the above definitions, precisely 2.2 and 2.3, one can define the S – complete theories as follows:

Definition 4. A G - cohomology theory h_G^* will be called *strongly - S - complete* if $f^* : h_G^*(Y) \to h_G^*(X)$ is an isomorphism whenever $f : X \to Y$ is a weak- (G, S) – equivalence.

Definition 5. A G - cohomology theory h_G^* will be called *weakly-S - complete* if $f^* : h_G^*(Y) \to h_G^*(X)$ is an isomorphism whenever $f : X \to Y$ is a strong - (G, S) – equivalence.

Example. Suppose G is connected and let T be a maximal torus of G. Then the equivariant K – theory K_G^* is weakly - (T) - complete on compact G – spaces.

Remarks. 1. Clearly, strong- S- completeness implies weak - S- completeness.

2. The equivariant K – theory K_G^* is not weakly - {1}- complete. This is shown in ([2]), where an example of a G-map $f : X \to Y$ is constructed with f a homotopy equivalence, but which does not induce an isomorphism on K_G^* .

3. An equivariant generalization of the Segal's finiteness theorem

If X is a G-space then $G \times_H X$ (see section 2) is naturally G- homeomorphic to $G/H \times X$ (where G acts on both factors) ; hence the projection $G/H \times X \to X$ induces a natural transformation

$$c_H : h_G^*(X) \to h_H^*(X) .$$

If W_G (H) is the Weyl group of H in G, then W_G (H) = Hom_G (G/H, G/H) and so acts on G/H; further, this action is equivariant with respect to the left action of G on G/H and so W_G (H) acts as a group of automorphisms of h_H^* restricted to G – spaces. By continuity, this action factors through an action of the homotopy group of W_G (H); that means that c_H maps h_G^* (X) into the invariants of h_H^* (X).

Remark. As a particular case of the above, take H = 1. Then we obtain the associated cohomology theory h^* defined on spaces, together with a natural transformation $c_H : h_G^*(X) \to h_H^*(X)$ on G – spaces. Further, G acts as a group of automorphisms of h^* restricted to G – spaces.

Definition 1. A *(G, S) – CW – complex* is a G- space X together with an increasing filtration by closed, G – invariant subspaces:

$$X_0 \subseteq X_1 \subseteq \ldots \subseteq X_n \subseteq \ldots \subseteq X$$

with $X = \bigcup_n X_n$ and such that X_0 is the disjoint union of (G, S) – 0–cells and X_n is obtain from X_{n-1} by attaching (G, S) – n–cells.

Definition 2 A *(G, S)-n-cell* is a G-space of the form $D^n \times G/H$, where G acts trivially on D^n and $H \in S$.

Remark. By the (G, S) – Whitehead theorem, every G - cohomology theory is strongly-S- complete when restricted to (G, S) – CW – complexes.

Proposition 1. (see [10]). Let $\theta : h_G^* \to h_H^*$ be a natural transformation between G - cohomology theories such that the induced morphisms

$$\theta_H : h_H^*(pt) \to k_H^*(pt)$$

are isomorphisms for each H ∈ S. Then

$$\theta : h_G^*(X) \to k_G^*(X))$$

is an isomorphism for any (G, S) – CW – complex X.

Now we present the main result of this note :

Theorem. Let X be a compact G – space satisfying the hypotheses of Segal's finiteness theorem (see theorem 1.1.); for h_G^* a multiplicative G- cohomology theory, we suppose that h_G^* is weakly-S- complete and also that S is a complete family.

Then h_G^* (X) is a finite $\overline{h_G}$ – module, where one denotes by $\overline{h_G}$ the graded ring h_G^* (point).

Sketch of proof. First observe that, using the proposition 3.1, it suffices to show that h_H^* (X) is a finite $\overline{h_H}$ – module, for each H ∈ S.

On the other hand, having the above natural transformation $c_H : h_G^*(X) \to h_H^*(X)$ we obtain, after a little manipulation, that there is a subset $A \subseteq h_G^*(X)$ such that c_H maps A bijectively into a basis for h_H^* (X) as a finite $\overline{h_H}$ – module, for each H ∈ S.

Now, let μ_H be the composite

$$(A)\ \overline{h_G} \otimes_{\overline{h_G}} \overline{h_G}(Y) \to h_G^*(X) \otimes_{\overline{h_G}} \overline{h_G}(Y) \to h_G^*(X \times Y)$$

here, for a ring R, one denotes by (A) R the free right R-module on basis A.

Then μ is a natural transformation between the G-cohomology theories $h_G^*(X \times \_)$ and $(A)\overline{h_G} \otimes_{\overline{h_G}} \overline{h_G}(\_)$; these theories are clearly weakly-S-complete if h_G^* is.

Further, the composite $\mu_{G/H}$

$$(A)\ \overline{h_G} \otimes_{\overline{h_G}} \overline{h_H} \xrightarrow{\cong} (c_H(A))\overline{h_H} \otimes_{\overline{h_H}} \overline{h_H} \xrightarrow{\cong} h_H^*(X) \otimes_{\overline{h_H}} \overline{h_H} \xrightarrow{\cong} h_H^*(X)$$

is an isomorphism for each H ∈ S.

Also we use the generalized spectral sequence of Atiyah and Hirzebruch: Segal associated (see [7] and [8]) to any G- cohomology theory h_G^* and any G –map f : X → Y, with Y a trivial G – space, a spectral sequence

$$E_2^{p,q} = H^p(\,Y\,;\ \mathcal{K}_G^q\ (f)\,) \Rightarrow\ \ h_G^*\ (X)$$

where $\mathcal{K}_G^q$ (f) is the sheaf on Y associated to the presheaf U → h_G^* (f^{-1} (U)).

Using the projections G/H → {point}, where H ∈ S, the stalks of the above sheaf are h_H^* (point) if h_G^* is "suitably continuous".

Remark. This proof is a combination between the techniques from [10] and my proof of the the Segal's finiteness theorem (see 1.1).

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CLOSURE SUBLINEAR OPERATORS

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Abstract: In this paper we show that the technique used by the author in [3] and [4] for the construction of the Dedekind completion of *C(X)*, the Riesz space (vector lattice) of all real-valued continuous functions on a compact topological space *X*, can be used in the case of any Riesz space *F*. This technique is based on the properties of some closure sublinear or supralinear operators defined on *F*.
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1. Introduction

The necessity to study the properties of some closure sublinear operators arises in connection with the construction of the Dedekind completion of a Riesz space (vector lattice). We recall ([9], p. 196) that if *G* is a Riesz space, then a Dedekind complete Riesz space $G^\wedge$ is called a *Dedekind completion* of the Riesz space *G* if the following conditions hold:

(i) There exists an one-to-one Riesz homomorphism $\phi : G \to G^\wedge$ of *G* into $G^\wedge$.

(ii) If we identify the Riesz subspace $\phi(G)$ of $G^\wedge$ with *G*, then we have:

$$\sup\{g : g \in G,\, g \le \hat{f}\} = \hat{f} = \inf\{g : g \in G,\, g \ge \hat{f}\},\ \ \forall \hat{f} \in G^\wedge.$$

Let *X* be a (Hausdorff) compact topological space and let *G* be *C(X)*, the Riesz space of all real-valued continuous functions on *X*. The construction of $C(X)^\wedge$ is an old problem, which does not have yet a satisfactory widely accepted solution. All the monographs dedicated to the Riesz spaces mention the famous theorem of H. Nakano from 1941: *C(X) is Dedekind complete if and only if the compact space X is extremely disconnected (that is, the closure of every open set is open)* ([9], Theorem 43.11). This result is nice but not useful to be used in applications in numerical analysis and partial differential equations, where the functions must be defined on an usual compact space *X*. Since the supremum and the infimum of a family of real-valued continuous functions on *X* are semicontinuous functions, lower or upper, respectively, it is obvious that the construction of the Dedekind completion of *C(X)* must be made using semicontinuous functions. For this purpose Dilworth introduced in 1950 [4] the notion of normal (upper or lower) semicontinuous function.

In order to recall the definition of a normal semicontinuous function we need first to define some operators on $B(X)$, the Dedekind complete Riesz space of all real-valued bounded functions on *X*. For every *f* in $B(X)$, $I(f)$ is the lower limit function of *f* and $S(f)$ is the upper limit function of *f*, that is,

$$I(f):\ X \to \mathbb{R},\ \ I(f)(x) = \sup_{V \in N_x} \inf_{y \in V} f(y),\ \ x \in X,$$

$$S(f):\ X \to \mathbb{R},\ \ S(f)(x) = \inf_{V \in N_x} \sup_{y \in V} f(y),\ \ x \in X,$$

where N_x denotes the set of all neighborhoods of the point $x \in X$. The operators

$I, S : B(X) \to B(X)$ are called the *Baire operators*, lower and upper, respectively, in honor of R. Baire who used these operators for the first time in his book [2].

C(X) is always a Riesz subspace of *B(X)*, but not a Riesz ideal. When *X* is a compact topological space, for every continuous function $f \in C(X)$ there exists a constant $M > 0$ such that $|f| \leq M$. This shows that the Riesz ideal generated by *C(X)* in *B(X)* is the whole *B(X)*. This equality ensures that the operators $L, U : B(X) \to B(X)$ defined below are well defined.

$$L(f) = \sup\{g : g \in C(X),\, g \leq f\}, \quad U(f) = \inf\{g : g \in C(X),\, g \geq f\}.$$

In general, for every $f \in B(X)$, the following inequalities hold,

$$L(f) \leq I(f) \leq f \leq S(f) \leq U(f),$$

but when *X* is compact, or only completely regular, we have ([5], Lemma 4.1),

$$I(f) = L(f) \text{ and } S(f) = U(f).$$

The properties of the operators *L* and *U* have been studied in details by Kaplan in [7] and [8]. In Kaplan's papers these operators are defined on the second dual of $C(X)$, but their properties depend only on the order between functions and not of the structure of these functions. So the properties of *L* and *U* also hold in our settings. We will call these operators the *Kaplan operators*.

It is worth to note that the Baire operators are defined by using the topological structure of *X*, while the Kaplan operators are defined by using the order structure on *C(X)*. Since the Dedekind completion of *C(X)* is an order construction and not a topological one, it is absolutely natural to used the Kaplan operators and not the Baire operators in the procedure of construction of $C(X)^{\wedge}$.

The operators $U, L : B(X) \to B(X)$ are examples of closure sublinear operators and dual closure supralinear operators, respectively (see Section 2 for definitions). In connection with these operators we consider a new one $\delta : B(X) \to B(X)$, defined by $\delta(f) = U(f) - L(f)$. The value $\delta(f)$ is the oscillation or the saltus of the function *f*, and $\delta(f) = 0$ if and only if *f* is a continuous function on *X*.

With the aid of the operators *U* and *L* we can characterize some types of known bounded discontinuous functions or can introduce some new ones. For example, a function $f \in B(X)$ is lower semicontinuous on *X* if and only if $L(f) = f$ and is upper semicontinuous on *X* if and only if $U(f) = f$. In 1950 Dilworth introduced the notion of normal semicontinuous function [5]. More precisely, $f \in B(X)$ is normal lower semicontinuous if $LU(f) = f$ and normal upper semicontinuous if $UL(f) = f$. Using these types of functions Dilworth proved that, for a compact topological space *X*, the Dedekind completion of the lattice $C(X)$ is isomorphic with the lattice of all real-valued normal upper semicontinuous functions on *X* ([5], Theorem 4.1). Let us remark that Dilworth's result is only about the *lattice* structure of $C(X)$ and not about the Riesz structure of this space. Much more later, in 2014, the author showed how can be defined a Riesz structure on the set of all real-valued normal upper semicontinuous functions on *X* ([3], Theorem 4.2).

In this paper we show how some results obtained in the construction of Dedekind completion of $C(X)$ can be proved for any Riesz space *F*. For this aim, in Section 2, we consider a closure sublinear operator *U* on *F* and its associated dual closure supralinear operator *L*, their difference $\delta = U - L$, and present some of their properties. Section 3 contains the study of the order structure of the sets of fixed points of the operators *U*, *L*, *UL* and *LU*. Theorem 1 is the abstract version of Theorem 4.2 proved by the author in [4]. In Sections 4 we introduce two types of elements of *F*, rare and dc-continuous. Using these

elements we construct a quotient Riesz space, which is Dedekind complete when F is Dedekind complete (Theorem 2). Section 5 contains the characterizations of the rare and dc-continuous elements of F in the particular case when F is $B(X)$. Section 6 presents an abstract form of the notion of Hausdorff continuous interval-valued function used by Anguelov to construct $C(X)^{\wedge}$ [1].

The unexplained terminology can be found [9]. The proofs of the results presented in this paper will be published elsewhere.

2. Closure sublinear operators

Let F be a Riesz space (vector lattice). A map $\Phi : F \to F$ is called: (1) *increasing* if $f_1 \le f_2 \Rightarrow \Phi(f_1) \le \Phi(f_2)$; (2) *idempotent* if $\Phi(\Phi(f)) = \Phi(f)$; (3) *extensive* if $f \le \Phi(f)$; (4) *intensive* if $\Phi(f) \le f$; (5) *positive homogeneous* if $\Phi(\lambda f) = \lambda \Phi(f)$ for all real numbers $\lambda \ge 0$; (6) *sub-additive* if $\Phi(f_1 + f_2) \le \Phi(f_1) + \Phi(f_2)$; and (7) *supra-additive* if $\Phi(f_1) + \Phi(f_2) \le \Phi(f_1 + f_2)$.

The map $\Phi : F \to F$ is called a *closure* operator if it is extensive, increasing and idempotent. If $\Phi : F \to F$ is intensive, increasing and idempotent then Φ is called an *interior* operator or a *dual closure* operator.

A map $U : F \to F$ is called a *closure sublinear* operator if U is a *closure* operator which is *sublinear* and *commutes with the finite supremums*, that is, for all $f_1, f_2 \in F$ we have $U(f_1 \vee f_2) = U(f_1) \vee U(f_2)$.

If U is a closure sublinear operator, we defined a new operator $L : F \to F$ by putting $L(f) = -U(-f)$. The operator L is a *dual closure* operator which is *supralinear* and *commutes with the finite infimums*, that is, $L(f_1 \wedge f_2) = L(f_1) \wedge L(f_2)$, for all $f_1, f_2 \in F$.

This pair of operators $U, L : F \to F$ has a lot of good properties for computation. Here we mention a few.

(a) $L(f) \le f \le U(f)$.

(b) $L(f_1 + f_2) \le L(f_1) + U(f_2) \le U(f_1 + f_2)$.

(c) $L(f_1) - U(f_2) \le L(f_1 - f_2) \le \begin{matrix} U(f_1) - U(f_2) \\ L(f_1) - L(f_2) \end{matrix} \le U(f_1 - f_2) \le U(f_1) - L(f_2)$.

(d) LU and UL are idempotent operators.

(e) $L \le LUL \le LU \le ULU \le U$, where LU can be replaced with UL.

(f) $L(f_1 \wedge f_2) \le L(f_1) \wedge U(f_2) \le U(f_1 \wedge f_2)$.

(g) $f_1 \wedge f_2 = 0 \Rightarrow L(f_1) \wedge U(f_2) = 0$.

(h) $L(f_1 \vee f_2) \le L(f_1) \vee U(f_2) \le U(f_1 \vee f_2)$.

In connection of this pair of operators we can define a new one, the *delta* operator,

$$\delta(f) = U(f) - L(f).$$

This operator $\delta : F \to F$ also have good properties. Here are some of them.

(a) $\delta(f) \ge 0$ and $\delta(f) = 0 \Leftrightarrow L(f) = f = U(f)$.

(b) For every real λ, $\delta(\lambda f) = |\lambda| \delta(f)$. In particular, $\delta(-f) = \delta(f)$.

(c) $\delta(f_1 \vee f_2) \le \delta(f_1) \vee \delta(f_2)$.

(d) $\delta(f_1) - \delta(f_2) \le \begin{matrix} \delta(f_1 + f_2) \\ \delta(f_1 - f_2) \end{matrix} \le \delta(f_1) + \delta(f_2)$.

(e) $\delta(f_1 + f_2) \le \delta(f_1 \vee f_2) + \delta(f_1 \wedge f_2) \le \delta(f_1) + \delta(f_2)$.

(f) $\delta(f)=\delta(f^+)+\delta(f^-),\ \ \delta(|f|)\le\delta(f)\le 2U(|f|)$.

For the proofs of all these properties of the operators U, L and δ see [7] and [8].

3. Some subsets of elements of the Riesz space F and their structures

Using the operators L and U we define some subsets of the Riesz space F, which are fixed points of the operators L, U, LU and UL.

$L_{sc}=\{f\in F: f=L(f)\}$. An element $f\in L_{sc}$ is called a *lsc-element* of F.

$U_{sc}=\{f\in F: f=U(f)\}$. An element $f\in U_{sc}$ is called an *usc-element* of F.

$NL_{sc}=\{f\in L_{sc}: f=LU(f)\}$. An element $f\in NL_{sc}$ is called a *normal lsc-element* of F.

$NU_{sc}=\{f\in U_{sc}: f=UL(f)\}$. An element $f\in NU_{sc}$ is called a *normal usc-element* of F.

A subset W of a linear space is called a *wedge* if W is closed to addition and scalar multiplication with positive scalars.

The set L_{sc} is a wedge with the property that $f_1, f_2\in L_{sc}\Rightarrow f_1\vee f_2,\ f_1\wedge f_2\in L_{sc}$, and for every subset $\{f_\gamma\}$ of L_{sc} for which there exist $\underset{F}{\wedge} f_\gamma$ and $\underset{F}{\vee} f_\gamma$ in F we have:

$$\underset{L_{sc}}{\wedge} f_\gamma = L(\underset{F}{\wedge} f_\gamma) \text{ and } \underset{L_{sc}}{\vee} f_\gamma = \underset{F}{\vee} f_\gamma .$$

Similarly, U_{sc} is a wedge with the property $f_1, f_2\in U_{sc}\Rightarrow f_1\vee f_2,\ f_1\wedge f_2\in U_{sc}$, and for every subset $\{f_\gamma\}$ of U_{sc} for which there exist $\underset{F}{\wedge} f_\gamma$ and $\underset{F}{\vee} f_\gamma$ in F we have:

$$\underset{U_{sc}}{\wedge} f_\gamma = \underset{F}{\wedge} f_\gamma \text{ and } \underset{U_{sc}}{\vee} f_\gamma = U(\underset{F}{\vee} f_\gamma).$$

Using the wedges L_{sc} and U_{sc} we define two *Riesz subspaces* of F. One is $C=L_{sc}\cap U_{sc}$ and the other is S_c, the linear subspace of F generated by $L_{sc}\cup U_{sc}$. An element f of C is called a *c-element* of F and the following characterization holds:

$$f\in C \ \Leftrightarrow\ \delta(f)=0 .$$

An element of S_c is called a *sc-elements* of F and S_c has the following description:

$$S_c = L_{sc}-L_{sc}=U_{sc}-U_{sc}=L_{sc}+U_{sc} .$$

The sets NL_{sc} and NU_{sc} become Riesz spaces if we define the algebraic and the lattice operations using the operators LU and UL, respectively. More precisely, the theorem from below shows how NL_{sc} can be organized as a Riesz space.

Theorem 1. *The set NL_{sc} endowed with the addition and the scalar multiplication*

$$f\underline{\oplus}g=LU(f+g),\ \ \lambda\underline{\otimes}f=LU(\lambda f)=\begin{cases}\lambda f, & \lambda\ge 0,\\ \lambda U(f) & \lambda<0,\end{cases}$$

and the lattice operations $f\underset{NL_{sc}}{\vee} g=LU(f\vee g),\ \ f\underset{NL_{sc}}{\wedge} g=f\wedge g,$ is a Riesz space.

If F is a Dedekind complete Riesz space then the Riesz space NL_{sc} is also Dedekind complete and the supremum and the infimum of any nonempty order bounded subset $\{f_\gamma\}_{\gamma\in\Gamma}$ in NL_{sc} are computed with the following formulae: $\underset{NL_{sc}}{\vee} f_\gamma = LU(\underset{\gamma}{\vee} f_\gamma)$, $\underset{NL_{sc}}{\wedge} f_\gamma = L(\underset{\gamma}{\wedge} f_\gamma)$.

Theorem 1 is the abstract form of the Theorem 4.2 proved by the author in [3]. A similar result with Theorem 1 can be proved for NU_{sc} (see [3], Theorem 5.1).

4. Rare and dc-continuous elements of the Riesz space F

An element $f \in F$ is called *rare* if $LU(|f|) = 0$. The set of all rare elements of F, denoted by Ra, is a *Riesz ideal* of F ([8], p. 348). For a rare element $f \in F$ we have the following useful equivalences: $f \in Ra \Leftrightarrow LU(f) = 0 = UL(f)$, and

$$f \in Ra \Leftrightarrow L(|f|) = 0 \text{ and } L(\delta(f)) = 0. \tag{4.1}$$

Now we introduce a new set of elements of F denoted by

$$D_c = \{f \in F : L(\delta(f)) = 0\}.$$

An element $f \in D_c$ is called a *dc-continuous* element of F. Since $U(\delta(f)) = \delta(f)$ we have $LU(\delta(f)) = L(\delta(f)) = 0$, and so, $f \in D_c \Leftrightarrow \delta(f) \in Ra$. D_c is a *Riesz subspace* of F and $C \subset S_c \subset D_c$. More precisely, the following equalities hold:

$$D_c = S_c + Ra = U_{sc} + Ra = L_{sc} + Ra = NU_{sc} \oplus Ra = NL_{sc} \oplus Ra.$$

Since $Ra \subset D_c$ and Ra is a Riesz ideal of D_c, we can consider on D_c the equivalence relation, $f \approx g \Leftrightarrow f - g \in Ra$, and then the quotient Riesz space D_c / Ra. The equivalence class of $f \in D_c$ is denoted by $\hat{f}$. For the order relation on D_c / Ra we have:

$$\hat{f} \le \hat{g} \Leftrightarrow (f-g)^+ \in Ra \Leftrightarrow LU(f-g) \le 0 \Leftrightarrow LU(f) \le LU(g) \Leftrightarrow UL(f) \le UL(g).$$

Theorem 2. *If the Riesz space F is Dedekind complete, then the quotient space D_c / Ra is Dedekind complete.*

5. Rare and dc-continuous elements of $B(F)$

In order to understand the importance of Theorem 2 we give his interpretation when the Riesz space F is $B(X)$. First we remark that a function $f \in B(X)$ is a dc-element of $B(X)$, that is $L(\delta(f)) = 0$, if and only if f is *densely continuous* (or *pointwise discontinuous)* on X, that is, the set C_f of points of continuity of the function f is dense in X. We denote by $C_d(X)$ the set of all real-valued densely continuous functions on X. The equivalence

$$f \in C_d(X) \Leftrightarrow L(\delta(f)) = 0$$

was remarked since 1912 by Ford [6], shortly after the publication of Baire's book [2].

Now we can describe what it means that a function $f \in B(X)$ is a rare element of $B(X)$. By using the equivalence (4.1) we obtain that $f \in B(X)$ is rare if and only if f is densely continuous on X and $f(x) = 0$ for all $x \in C_f$.

With these explanations we can reformulate Theorem 2 in the form from below, which is nothing else then Theorem 5 of [4] in the case when the topological space X is compact.

Theorem 3. *Let X be a compact topological space. Then the quotient Riesz space $C_d(X) / Ra$ is Dedekind complete.*

6. Regular pairs

A pair of elements $(\underline{f}, \overline{f})$ of F is called a *regular pair* if: $\underline{f} \in L_{sc}$, $\overline{f} \in U_{sc}$, $\underline{f} \le \overline{f}$, and $U(\underline{f}) = \overline{f}$, $L(\overline{f}) = \underline{f}$. The pair $(\underline{f}, \overline{f})$ is regular if and only if $\underline{f} \in NL_{sc}$, $\overline{f} \in NU_{sc}$ and $L(\overline{f} - \underline{f}) = 0$. Every element $f \in F$ generates two regular pairs: $\underline{f} = LUL(f)$, $\overline{f} = UL(f)$, and $\underline{g} = LU(f)$, $\overline{g} = ULU(f)$, which are such that $\underline{f} \le \underline{g}$ and $\overline{f} \le \overline{g}$. If $f \in D_c$ then f generates *only one* regular pair $\underline{f} = LU(f)$, $\overline{f} = UL(f)$ and $\underline{f}, \overline{f} \in \hat{f} \in D_c / Ra$.

When $F = B(X)$, a regular pair of functions $(\underline{f}, \overline{f})$ corresponds to the *Hausdorff continuous interval-valued* function $\overline{\underline{f}} : X \to \mathbf{IR}$, $x \to [\underline{f}(x), \overline{f}(x)]$ (see [1]).

Inspired by a result of Kaplan ([8], p. 382) we can prove the following theorem.

Theorem 4. *If the element $f \in D_C$, then:*

(i) The equivalence class $\hat{f} \in D_C / Ra$ contains exactly one regular pair $(\underline{f}, \overline{f})$, which is defined by $\underline{f} = LU(f)$, $\overline{f} = UL(f)$.

(ii) $\underline{f}$ is the largest lsc-element in $\hat{f}$ and $\overline{f}$ is the smallest usc-element in $\hat{f}$.

We denote by *H* the set of all regular pairs of *F*. Since $\underline{f} \in NL_{sc}$ and $\overline{f} \in NU_{sc}$ we define on *H* the *addition* of two regular pairs $(\underline{f}, \overline{f})$ and $(\underline{g}, \overline{g})$ by putting,

$$(\underline{f}, \overline{f}) \oplus (\underline{g}, \overline{g}) = (\underline{f} \underline{\oplus} \underline{g}, \overline{f} \overline{\oplus} \overline{g}),$$

where $\underline{\oplus}$ and $\overline{\oplus}$ are the additions defined on NL_{sc} and NU_{sc}, respectively. More precisely, we have: $(\underline{f}, \overline{f}) \oplus (\underline{g}, \overline{g}) = (LU(\underline{f} + \underline{g}), UL(\overline{f} + \overline{g}))$.

The *scalar multiplication* between a real number λ and a regular pair $(\underline{f}, \overline{f})$ is defined by

$$\lambda \otimes (\underline{f}, \overline{f}) = (\lambda \otimes \underline{f}, \lambda \otimes \overline{f})) = (LU(\lambda \underline{f}), UL(\lambda \overline{f})).$$

The *lattice operations* on *H* are defined as follows:

$$(\underline{f}, \overline{f}) \vee (\underline{g}, \overline{g}) = (\underline{f} \underset{NL_{sc}}{\vee} \underline{g}, \overline{f} \underset{NU_{sc}}{\vee} \overline{g}) = (LU(\underline{f} \vee \underline{g}), \overline{f} \vee \overline{g}),$$

$$(\underline{f}, \overline{f}) \wedge (\underline{g}, \overline{g}) = (\underline{f} \underset{NL_{sc}}{\wedge} \underline{g}, \overline{f} \underset{NU_{sc}}{\wedge} \overline{g}) = (\underline{f} \vee \underline{g}, UL(\overline{f} \vee \overline{g})).$$

Theorem 5. *(i) H is a Riesz space.*

(ii) The Riesz spaces D_C / Ra, H, NL_{sc} and NU_{sc} are order isomorphic.

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FROM VECTOR SPACES TO INTERVAL-SPACES via THE TECHNIQUE OF THE AUXILIARY SUBLINEAR FUNCTIONAL

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Abstract. This paper is like a brief trip from *vector spaces* to *interval-spaces* passing through the *quasilinear spaces* (in the sense of Markov). We highlight this trip by using the so-called *technique of the auxiliary sublinear functional*. To apply this technique in interval-space setting we must overcome the difficulty created by the *inexistence of the opposite element for a nondegenerate closed interval*. Notice that different types of closed intervals are considered: in the set of real numbers, in the space-time geometry, in the Allen's algebra, in economics. We also considered intervals in an arbitrary ordered vector space.
Mathematics Subject Classification: Primary 46A22; Secondary 06F05, 46S99.
Key words: interval analysis, interval-spaces, interval-(sub)linear functional, Mazur-Orlicz type theorem.

1. Preliminaries

What is a vector space is known by any mathematician. A *real vector space* is a set E endowed with two algebraic operations, an addition and a multiplication with (real) scalars, such that:

1) $(x+y)+z=x+(y+z)$, for all $x,\ y,\ z\in E$;

2) there exists $0\in E$ with $x+0=x$, for each $x\in E$;

3) $x+y=y+x$, for all $x,\ y\in E$;

4) for each $x\in E$ there exists $-x\in E$ with $x+(-x)=0$;

5) $\alpha\cdot(\beta\cdot x)=(\alpha\beta)\cdot x$, for all $x\in E$ and $\alpha,\ \beta\in\mathbf{R}$;

6) $1\cdot x=x$, for all $x\in E$;

7) $\alpha\cdot(x+y)=\alpha\cdot x+\alpha\cdot y$, for all $x,\ y\in E$ and $\alpha\in\mathbf{R}$ (the so-called "first distributive law");

8) $(\alpha+\beta)\cdot x=\alpha\cdot x+\beta\cdot x$, for all $x\in E$ and $\alpha,\ \beta\in\mathbf{R}$ (the so-called "second distributive law").

But, perhaps, fewer mathematicians have thought that the axiom 4) from the above definition is redundant. This is an observation due to S. Markov, see [9]. More precisely, 8) and 6) imply 4). To prove this, firstly we notice that if we put in 8), $\alpha=1$ and $\beta=0$, we obtain: $x+0\cdot x\overset{6)}{=}1\cdot x+0\cdot x\overset{8)}{=}(1+0)\cdot x=1\cdot x\overset{6)}{=}x$; hence $0=0\cdot x$. Now, to prove that 8) and 6) imply 4), we put in 8), $\alpha=1$ and $\beta=-1$, obtaining $0=0\cdot x=(1+(-1))\cdot x\overset{8)}{=}1\cdot x+(-1)\cdot x\overset{6)}{=}x+(-1)\cdot x$, hence $0=x+(-1)\cdot x$, that is, the opposite to x exists and it is the element $(-1)\cdot x$.

In [9], S. Markov also remarked that a vector space can be defined by relaxing the axiom 4) by the "weak cancellation law" 4'), where :

4') $x+z=y+z$ implies $x=y$, for all $x,\ y,\ z\in E$.

Hence a vector space is a nonempty set E endowed with the algebraic operations that satisfy the axioms 1)-3), 4'), 5)-7) and 8).

Obviously the axioms 1)-4) are the group axioms while 1)-3) are the axioms of a commutative monoid. Notice that further we will use the terms of "linear space" and "vector space" as synonyms. In [9], S. Markov also considered quasilinear spaces. A *quasilinear space in the sense of Markov* is a set E endowed with an addition and a multiplication with (real) scalars such that the axioms 1)-3), 4'), 5)-7) and 8') are satisfied, where 8') is the so-called "quasidistributive law":

8') $(\alpha+\beta)\cdot x=\alpha\cdot x+\beta\cdot x$ for all $x\in E$ and $\alpha,\ \beta\in\mathbf{R}$ with $\alpha,\ \beta\geq 0$.

Following the above proof for "8) and 6) imply 4)" (the axiom 4) meaning the existence of the opposite element for $x\in E$) we remark (see [9]) that we do not have "8') and 6) imply 4)".

The notion of quasilinear space in the sense of Markov, more precisely the notion of *quasilinear system*, appeared in many papers by S. Markov starting with [10]. In [10], Markov gave a brief history of the term of "quasilinear space". Of course, a linear space is a special case of a quasilinear space in the sense of Markov. In [5] we also mention the notion of *quasilinear space in the sense of Aseev*, introduced in [3] (see also [15]).

Interval-spaces

"Interval Analysis" is a name of a book written by R.E. Moore in 1966 (see [11]), but also a branch of mathematics which, according to R.E. Moore (see [12]), "appeared from the observation that if we compute a number a and a rigorous bound b on the total error in a, as an approximation to some unknown number x, such that $|x-a|\leq b$, then no matter how we compute a and b, we certainly known that x lies in the interval $[a-b,\ a+b]$".

For an informal definition, notice that a simple search on Wikipedia gives that "*Interval arithmetic*, *interval mathematics*, *interval analysis*, or *interval computation* is a method developed by mathematicians since the 1950s and 1960s as an approach to putting bounds on rounding errors and measurement errors in mathematical computation and thus developing numerical methods that yield reliable results".

Generally, interval analysis operates with closed intervals of real numbers. Obviously the thinking about real numbers in terms of intervals does not appeared for the first time in the last century (see the contribution of Archimedes in the 3rd century BC, to frame the irrational numbers; he found, for example, that $\frac{223}{71}=\frac{1561}{71\cdot 7}<\pi<\frac{1562}{71\cdot 7}=\frac{22}{7}$, that is, $\pi\in\left[\frac{223}{71},\frac{22}{7}\right]$).

Notice that the book "Interval Analysis" by R.E. Moore (see [11]) marked the *birth of the modern interval analysis*. According to E.R. Hansen [6], Moore had the idea in Spring 1958 and a year later he published an article about Computer Arithmetic [13]. Actually, "Interval Analysis" was the outgrowth of his PhD Thesis from 1962. After the appearance of Moore's book, groups from different countries stated to investigate the theory and applications of interval analysis systematically. We cite here: **a)** the German School: U. Kulisch [8]; **b)** the Bulgarian School, illustrious represented by S. Markov, R. Anguelov and B. Sendov (for some contributions to the first author, see, for example [2]); **c)** the Spanish School, see for example, L.G. Casado and I. Garcia [4].

First were considered intervals on the *real line*: $[a]=[\underline{a},\overline{a}]$ with $\underline{a}\leq\overline{a}$ in $\mathbf{R}$. Sometimes the interval $[a]$ was denoted by $[a_L,a_R]$ or by $\overline{\underline{a}}$.

We notice also that intervals appear in the *space-time geometry*. Here the interval

$$[\underline{x},\overline{x}]=\{x\mid \underline{x}\leq x\leq\overline{x}\} \tag{1}$$

means that the event $\underline{x}$ can causally affect the given event x and the event $\overline{x}$ can causally be affected by x. (Of course, the description (1) of the interval $[\underline{x},\overline{x}]$ looks similar to the interval $[\underline{a},\overline{a}]$ of real numbers but the important difference is that the causality relation in space-time is only a partial order).

The intervals also can be considered in *economics*, where the 'order' is given by the preference of an agent for some goods.

More generally, we considered in [5] intervals in an arbitrary *ordered vector space* E. We call the set $IE=\{[x]=[\underline{x},\overline{x}]\,|\,\underline{x}\le\overline{x} \text{ in } E\}$ the *interval-set* (in short, *i-set*) associated to E. Of course $E\subset IE$ because any $a\in E$ can be written as $[a]=[a,a]$. We will call this interval a *degenerate interval*. We will call any interval $[a]\in IE\setminus E$ a *nondegenerate interval*.

The *i*-set IE becomes an *interval-space* (in short, *i-space*) if it is endowed with the following algebraic operations:

1. the (Minkovski) addition, defined by $[a]\oplus[b]=\{x+y\,|\,x\in[a] \text{ and } y\in[b]\}$, that is,

$$[a]\oplus[b]=[\underline{a}+\underline{b},\ \overline{a}+\overline{b}] \text{ if } [a]=[\underline{a},\overline{a}]\in IE \text{ and } [b]=[\underline{b},\overline{b}]\in IE;$$

2. the scalar multiplication with reals, defined by

$$\alpha\cdot[a]=\{\alpha x\,|\,x\in[\underline{a},\overline{a}]\}, \text{ that is, } \alpha\cdot[a]=\begin{cases}[\alpha\underline{a},\alpha\overline{a}], & \text{if } \alpha\in\mathbf{R},\ \alpha\ge 0\\ [\alpha\overline{a},\alpha\underline{a}], & \text{if } \alpha\in\mathbf{R},\ \alpha<0\end{cases}, \text{ where } [a]=[\underline{a},\overline{a}]\in IE.$$

Sometimes, we will denote $\alpha\cdot[a]$ by $\alpha[a]$. It is known that, endowed with these algebraic operations, IE is ***not*** a (real) *vector space*. More precisely, $(IE,\oplus)$ is a commutative monoid, with the *neutral* (or *identity*) element $\mathbf{0}$ ($\mathbf{0}=[0,0]$, sometimes also denoted by $[0]$), but it *is not a group*, because a nondegenerate closed interval has *no* inverse with respect to the addition, that is, *has no opposite*. Indeed, by way of contradiction, suppose that for the order interval $[a]=[\underline{a},\overline{a}]$ with $\underline{a}<\overline{a}$ in E, there exists an inverse $[b]=[\underline{b},\overline{b}]$. Hence $[\underline{a},\overline{a}]\oplus[\underline{b},\overline{b}]=\mathbf{0}$, that is, $\underline{a}+\underline{b}=0$ and $\overline{a}+\overline{b}=0$. Therefore, $\underline{b}=-\underline{a}$ and $\overline{b}=-\overline{a}$. But $\underline{b}\le\overline{b}$ implies that $-\underline{a}\le-\overline{a}$ or, equivalently, $\overline{a}\le\underline{a}$, which contradicts that $\underline{a}<\overline{a}$. We also notice that the scalar multiplication in IE has the following properties:

$$\alpha(\beta[a])=(\alpha\beta)[a], \text{ if } [a]\in IE \text{ and } \alpha,\ \beta\in\mathbf{R}; \tag{2}$$

$$1\cdot[a]=[a] \text{ for each } [a]\in IE; \tag{3}$$

$$\alpha([a]\oplus[b])=\alpha[a]\oplus\alpha[b], \text{ if } [a],[b]\in IE \text{ and } \alpha\in\mathbf{R}; \tag{4}$$

$$(\alpha+\beta)[a]=\alpha[a]\oplus\beta[a], \text{ if } [a]\in IE \text{ and } \alpha,\ \beta\in\mathbf{R}, \text{ with } \alpha\beta>0. \tag{5}$$

Hence, an *i*-space is **not** a linear space but it is a quasilinear space in the sense of Markov (and in the sense of Aseev, too).

We also consider the *subtraction* in IE: $[a]\ominus[b]=[a]\oplus(-[b])$, where $-[b]=(-1)\cdot[b]$. If $[a]=[\underline{a},\overline{a}]$, then $[a]\ominus[a]=[-(\overline{a}-\underline{a}),\overline{a}-\underline{a}]$, that is, it is a *symmetric interval*. If $\underline{a}<\overline{a}$ (that is, $[a]$ is a nondegenerate interval), then $[a]\ominus[a]\ne\mathbf{0}$ and again we conclude that IE is not a vector space.

We can consider the set $\mathcal{O}=\{[-b,b]\,|\,b\ge 0,\ b\in E\}$ of all symmetric intervals in IE. Then $[a]\ominus[a]\in\mathcal{O}$ for all $[a]\in IE$. Since if $[a]$ is a degenerate interval, then $[a]\ominus[a]=\mathbf{0}\in\mathcal{O}$, we

call the set $\mathcal{O}$ the *null set* of IE. We will denote by $[o]$ its generic element (that is, for each $[o]\in\mathcal{O}$, there exists $[a]\in IE$, with $[o]=[a]\ominus[a]$). Obviously $\mathcal{O}$ is closed under the algebraic operations on IE.

Notice that, to make a decision choosing from several possibilities expressed quantitatively, we need to compare the values of respective amounts. In practice, most often, we know these values with interval uncertainty. Thus we come to compare intervals. Many *order relations* on IE, with $E=\mathbf{R}$, are considered in the literature (see [7]) using the *end-points* $\underline{a}(=a_L)$, $\overline{a}(=a_R)$ of an interval $[a]=[\underline{a},\overline{a}]\,(=[a_L,\,a_R])$ or its *center* $a_C=\frac{\underline{a}+\overline{a}}{2}$ and its *radius* $a_w=\frac{\overline{a}-\underline{a}}{2}$ (the interval $[\underline{a},\overline{a}]$ can be also denoted by $<a_c,\,a_w>$).

Among the most known order relations between $[a]=[\underline{a},\overline{a}]$ and $[b]=[\underline{b},\overline{b}]\in IE$ we cite:

1) the *strong order*: $[a]\prec[b]\Leftrightarrow\overline{a}<\underline{b}$ or $[a]=[b]$ (notice that $[a]=[b]\Leftrightarrow\underline{a}=\underline{b}$ and $\overline{a}=\overline{b}$);

2) the *containment order*: $[a]\le[b]\Leftrightarrow\underline{b}\le\underline{a}$ and $\overline{a}\le\overline{b}$ (that is, $[a]\subseteq[b]$);

3) the *weak order*: $[a]\le[b]\Leftrightarrow\underline{a}\le\underline{b}$ and $\overline{a}\le\overline{b}$ (used, for example, in [5]).

The *pioneer* of the study of order relations on IE, for $E=\mathbf{R}$, was R.E. Moore [14]. Many different relations (including order relations) between the intervals belonging to IE, with $E=\mathbf{R}$, are defined in *Allen's algebra*. This algebra is a calculus for temporal reasoning, introduced by James F. Allen in 1983 (see [1]). There are 13 basic relations capturing all possible relations between intervals. The first relation $[a]<[b]$ (or, equivalently $[b]>[a]$- the second relation) is interpreted as "$[a]$ takes place before $[b]$". Actually we remark that we can summarize the relations of Allen's interval algebra, considering that the following situations can occur, for two intervals (belonging to IE with $E=\mathbf{R}$): *Type I)* Non-overlapping intervals; *Type II)* Partially overlapping intervals; *Type III)* Completely overlapping intervals.

2. A parallel examination between vector space setting and interval-space setting

We have seen so far which are the 'objects' in the vector space setting and interval-space setting. But what are the 'morphisms' in these settings? It is known that the morphisms in the first setting are the *linear operators*. Among these operators next we will consider only the linear functionals. We also recall what means a sublinear functional.

Definition. Let X a vector space and $L:X\to\mathbf{R}$, $S:X\to\mathbf{R}$ two maps, such that:

1) $L(x_1+x_2)=L(x_1)+L(x_2)$ and $S(x_1+x_2)\le S(x_1)+S(x_2)$ for all $x_1,\,x_2\in X$;

2) $L(\alpha x)=\alpha L(x)$ and $S(\beta x)=\beta S(x)$ for all $x\in X$ and $\alpha\in\mathbf{R}$ and $\beta\in\mathbf{R}$, $\beta\ge 0$.

We say that L is a *linear functional* and S is a *sublinear functional*. If $L(x)\le S(x)$ for all $x\in X$, we say that $L\le S$ on X.

Definition. (see [5, Definitions 3 and 5]) Let IE an i-space and $l:IE\to\mathbf{R}$, $s:IE\to\mathbf{R}$ two maps such that:

1) $l([x_1]\oplus[x_2])=l([x_1])+l([x_2])$ and $s([x_1]\oplus[x_2])\le s([x_1])+s([x_2])$ for all $[x_1]$, $[x_2]\in IE$;

2) $l(\alpha[x])=\alpha l([x])$ and $s(\beta[x])=\beta s([x])$ for all $[x]\in IE$, $\alpha\in\mathbf{R}$ and $\beta\in\mathbf{R}$, $\beta\ge 0$.

3) $s([x]\oplus[o])=s([x])$ for all $[x]\in IE$ and $[o]\in\mathcal{O}$.

We say that l is an *interval-linear* (in short, i*-linear*) *functional* and s is an *interval-sublinear* (in short, i*-sublinear*) *functional*.

In [5] we give some examples and we showed that if f is an i-linear functional, then $f([o])=0$ and, consequently $f([x]\oplus[o])=f([x])$ for all $[o]\in\mathcal{O}$ and $[x]\in IE$. So, because any i-linear functional must be an i-sublinear functional, ***the hypothesis*** 3) ***imposed to any*** i***-sublinear functional is justified***. Now we will give an example to understand how we can use this assumption. In [5] we gave a Mazur-Orlicz type theorem (an existence theorem) and, as its Corollary, a Hahn-Banach type extension theorem in the i-space setting. To prove our Mazur-Orlicz type theorem we will use a Hahn-Banach type existence theorem which in turn is demonstrated using the *pointwise order on* i*-sublinear functionals*, just like in the proof given by H. König (1982) and S. Simons (1970) for the corresponding theorem in the classic case of vector space setting. To be more precise, firstly we recall the statement of the Hahn-Banach existence theorem.

a) in the vector space setting: "*For every sublinear functional* $S:X\to\mathbf{R}$ *there exists a linear functional* $L:X\to\mathbf{R}$ *such that* $L\le S$ on X".

b) in the i-space setting: "*For every* i*-sublinear functional* $s:IE\to\mathbf{R}$ *there exists an* i*-linear functional* $l:IE\to\mathbf{R}$ *such that* $l([x])\le s([x])$ *for all* $[x]\in IE$".

Now we highlight a *step* of the *proofs* of these two statements. In this step, starting from a(n) (i-)sublinear functional we construct a new (i-)sublinear functional as a part of so-called the "***technique of the auxiliary*** **(*i*-)*sublinear functional***". (This technique, dating from the 1956's, is due to V. Pták, who used it to give a much simpler proof to an existence theorem, known as the Mazur-Orlicz Theorem.) So, for:

"**a)**": *if* $L:X\to\mathbf{R}$ *is a sublinear functional we define the auxiliary functional* $H:X\to\mathbf{R}$ *by*

$$H(x)=\inf_{\substack{u\in X\\ \alpha>0}}\left(L(x+\alpha u)-\alpha L(u)\right),\ x\in X.$$

Then: 1) H *is well-defined;* 2) H *is a sublinear functional*; 3) $H\le L$.

"**b)**": similarly, *if* $l:IE\to\mathbf{R}$ *is an* i*-sublinear functional we define* $h:IE\to\mathbf{R}$ *by*

$$h([x])=\inf_{\substack{[u]\in IE\\ \alpha>0}}\left(l([x]+\alpha[u])-\alpha l([u])\right),\ [x]\in IE.$$

Then: 1) h *is well-defined;* 2) h *is an* i*-sublinear functional;* 3) $h\le l$.

Let us prove 1) (the proofs for 2) and 3) are routine exercises).

In "**a)**": By using that L is sublinear, it follows for u, $x\in X$ and $\alpha>0$:

$$\alpha L(u)=L(\alpha u)=L(\alpha u+0)=L(\alpha u+x-x)\le L(x+\alpha u)+L(-x) \tag{6}$$

Consequently, for a fixed $x\in X$, and all $u\in X$ and $\alpha>0$, it follows that:

$$-L(-x)\le L(x+\alpha u)-\alpha L(u),$$

for all $u\in X$ and $\alpha>0$. Then the set $\{L(x+\alpha u)-\alpha L(u)\mid u\in X,\ \alpha>0\}$ is bounded from below in $\mathbf{R}$ and therefore there exists its infimum, denoted by $H(x)$. Notice that we can write $\alpha u=\alpha u+x-x$ in (6), since this equality occurs in a *vector space* which is an additive group.

In "**b)**": We can not repeat the above proof because a relation similar with (6) is not valid in an i*-space*. Indeed there is *no* additive inverse for a nondegenerate interval in any i-space. ***But we put the hypothesis*** 3) ***in the definition of an*** i***-sublinear functional***. Hence, because l is an i-sublinear functional we have:

$$\alpha l([u]) \overset{\alpha>0}{=} l(\alpha[u]) \overset{3)}{=} l(\alpha[u] \oplus [o]) \text{ for all } [o] \in \mathcal{O}.$$

Take $[o] = [x] \ominus [x] = [x] \oplus (-[x])$. Therefore

$$\alpha l([u]) = l(\alpha[u] \oplus [x] \oplus (-[x])) \leq l(\alpha[u] \oplus [x]) + l(-[x]).$$

It follows that:

$$-l(-[x]) \leq l(\alpha[u] \oplus [x]) - \alpha l([u]),$$

for all $[u] \in IE$ and $\alpha > 0$.

Then the set $\{l([x] \oplus \alpha[u]) - \alpha l([u]) \mid [u] \in IE, \alpha > 0\}$ is bounded from below in $\mathbf{R}$ and therefore there exists its infimum, denoted by $h([x])$.

For ***another example showing how we use the assumption*** 3) ***from the definition of an i-sublinear functional***, you can read the proof of our Mazur-Orlicz type theorem in [5].

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TABU SEARCH OPTIMIZATION WITH APPLICATIONS IN GROUNDWATER MANAGEMENT

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Abstract: This paper presents tabu search (TS), a metaheuristic algorithm, applied to nonlinear optimization problems in the context of groundwater management. We use TS to establish the optimal wells location in a remediation design of a polluted aquifer.
TS is a local search procedure used to identify global optimum allowing non-improving moves to the best solutions in the neighbourhood of the local optimum solution. TS uses analogy with the human memory process: a tabu list records forbids moves that would return to a solution previously visited, so that it will not be stuck in a local minimum. TS gives superior solution to the best one obtained by other alternative methods.
Mathematics Subject Classification (2010): 00A69, 90C27, 90C59, 97R40
Key words: global optimization methods, parameter estimation, Tabu Search, groundwater management

1. Purpose and Scope

A metaheuristic (MA) is a high level iterative optimization algorithm, superior to classical optimization technique, very promising for difficult engineering optimization problems. MA's are evolutionary nature-inspired stochastic algorithms that combine intelligently different concepts derived from classical heuristics, artificial intelligence, biological evolution, natural and physical sciences.
Glover [3] makes some comparison between four principal methods of artificial intelligence: artificial neural networks (ANNs), simulated annealing (SA), genetic algorithms (GAs) and tabu search (TS).
ANNs are statistical learning models inspired by biological neural networks that have been used successfully in pattern recognition problems while in more general problems of optimization results were less impressive.
SA is based on analogy between the annealing process of solids and the search for the optimum in a general system. Raising the temperature up, the atoms have a higher energy state and a high possibility to re-arrange the crystalline structure. A completely ordered crystalline structure is equivalent with the optimal solution for the problem.
GAs use concepts of "Natural Selection" and "Genetic Inheritance" (Darwin) and follow the idea of survival of the fittest: better and better solutions evolve from previous generations until a near optimal solution is obtained.
While SA and GA can be considered memoryless systems, other artificial intelligence technique try to explore more flexible memory structures [2].
Such TS emulates intelligent uses of memory in the human problem solving process. TS is a neighbourhood search method which employs "intelligent" search and flexible memory technique to avoid being trapped at local optimum.

2. Tabu search

The modern form of TS derives from Glover (1986 and 1989) and introduces memory structures in Metaheuristics [Clever algorithms]. TS is an extension of local / neighbourhood search (LS), which records and exploits the history of the search so to escape local optima and lead the search towards (hopefully) higher-quality solutions [Glover, Laguna].

LS methods are attractive because of their flexibility and because the resulting algorithms perform fast in practice. Like other metaheuristic search methods (GA, SA, ANNs, etc…) TS has the ability of finding the global optimal solution.

In order to speed up the search process moves are intelligently selected using *flexible memory structures*, *strategic restrictions* and *aspiration criteria* to explore otherwise forbidden avenues [3, 4].

Several types of memories are used in order to improve the exploration quality. One important component is the short-term memory, which is referred to as the tabu list. The purpose of the tabu list is to force the search away from solutions that are selected in recent iterations so that it will not be trapped in local minima. New solutions are rejected if they satisfy conditions given by the tabu list [4, 5].

The long-term memory record the frequencies of moves and it is used to diversify the search into new regions. Intermediate-term memory structures can be introduced to focus the search on promising areas of the search space (intensification), called aspiration criteria (AC) [2].

AC is used to determine when tabu restrictions should be overridden.

TS does not converge naturally and stopping conditions can be: a fixed number of iterations (a fixed number of iterations has elapsed since the last best solution was found) or / and a given amount of CPU time has been used.

3. Tabu search application in groundwater management

Based on the idea of reducing costs associated to an existing or a new pump and treat (P&T) system it was proven that optimization codes using metaheuristic procedure like GAs, SA or TS are expected to be cost effective at 25%-30% of those sites. Studies completed by US Environmental Protection Agency and Naval Facilities Engineering Service Center indicate that the majority of the P&T systems are not operating as designed, have unachievable or undefined goals, and have not been optimized since installation [4,6].

Pumping well optimization based on TS is used to estimate the most cost-effective pumping system design by minimizing factors such as the number of pumping wells, the total pumping rates and to find the ideal pumping well locations.

Starting with a real-life problem, MGO code search possible and allowable combinations of pumping strategies for an optimal solution. The MGO code is implemented with three global optimization methods: genetic algorithms (GA), simulated annealing (SA), and tabu search (TS).[4]

The MGO code couples groundwater flow simulation models MODFLOW with mathematical optimization techniques to address groundwater management problems.

Three-dimensional groundwater flow equation through a permeable medium has the form

$$\frac{\partial}{\partial x}(K_x \cdot \frac{\partial H}{\partial x}) + \frac{\partial}{\partial y}(K_y \cdot \frac{\partial H}{\partial y}) + \frac{\partial}{\partial z}(K_z \cdot \frac{\partial H}{\partial z}) + q_s = S_s \cdot \frac{\partial H}{\partial t} \qquad (1)$$

where $H = H(x, y, z, t)$ is hydraulic head $[L]$, $K = (K_x, K_y, K_z)$ is hydraulic conductivity $[LT^{-1}]$, q_s is the volumetric flow rate of fluid sinks/sources per unit volume of aquifer $[T^{-1}]$, S_s is the specific storage $[L^{-1}]$ and can be solved in terms of uniqueness by imposing boundary and initial conditions.

Numerical modeling of flow processes in the saturated area is performed by finite difference method which is based on discretization of partial differential equations which are included in the mathematical model (1) by approximating the partial derivatives with finite differences. This gives a linear equation where the unknown is the hydraulic head H. Groundwater flow equation (1) is solved with MODFLOW, a USGS's three-dimensional (3D) finite-difference groundwater model considered an international standard for simulating and predicting groundwater conditions.

In this paper we use pumping well optimization technology to determine the ideal pumping well locations and ideal pumping rates at these locations, in order to minimize the sum of the capital costs. Capital cost is including fixed capital costs of installation and drilling operations plus the operating and maintenance costs associated with groundwater extraction and treatment for the entire project duration.

MGO is a coupled simulation-optimization model, the simulation component updates the state variables (hydraulic head) and the optimization component determines the optimal values for all the decision variables.

The objective function J of total cost to be minimized is:

$$J = a_1 \sum_{i=1}^{N} y_i + a_2 \sum_{i=1}^{N} y_i d_i + a_3 \sum_{i=1}^{N} y_i Q_i \Delta t_i \quad (2)$$

which is subject to a set of constraints:

$$\sum_{i=1}^{N} y_i \le NW \le N \quad (2.1);$$

$$Q_{\min} \le Q_i \le Q_{\max} \quad (2.2);$$

$$H_{\min} \le H_m \le H_{\max} \quad (2.3);$$

$$H_m^{out} - H_m^{\text{int}} \ge \Delta H_{\min} \quad (2.4)$$

where

- N is the total number of parameters (wells) to be optimized
- Parameter i represent a particular well location which is associated with: Q_i the pumping rate of well, y_i is a binary variable equal with 1 if parameter i is active (the associated flow rate is not zero) or $y_i = 0$ if parameter i is inactive (the associated flow rate is zero), d_i is the depth of well bore, Δt_i is the duration of pumping.
- The coefficients: a_1 is the fixed capital cost per well, a_2 is the installation and drilling cost per unit of depth of well bore; a_3 is the pumping and/or treatment cost per unit of volume of flow.

The constraint (2.1) is stating that the total number of actual wells must not exceed a fixed number, NW out of the total candidate wells, N. According to (2.2) the flow rate must be within the specified minimum and maximum values ($Q_{\min}$ and $Q_{\max}$). The constraint (2.3) is limiting the hydraulic head at any monitoring location, H_m, within specified lower and upper bounds ($H_{\min}$ and $H_{\max}$). Finally, according to (2.4) the head difference between an "outside" and an "inside" monitoring well must be greater than a minimum value, $\Delta H_{\min}$. [5, 6]

TS was originally designed for combinatorial optimization. Zeng and Wang [4,5] use TS to allocate the optimal well locations and find optimal pumping rates in a remediation design. The basic TS optimization procedure for solving problem (2) subject to constraints (2.1)-(2.4) is summarized in Table 1.

1.Initialization	The tabu search starts with a feasible well configuration $I = \{I_j, j = \overline{1,2NW}\}$ where the coordinates of the well locations are I_j. Construct the neighbourhood set for *I* and set the tabu list empty $L = \phi$
2. Evaluation	For each well configuration in the neighbourhood set, evaluate the objective function
3. Updating	Find the best solution in the neighbourhood excluding the configurations in the tabu list. Update the tabu list, i.e., add the change between the new solution and the old solution if the tabu list is not full. Otherwise, delete the oldest element in tabu list and record the change. In addition, construct the new neighbourhood set for next iteration.
4.Checking convergence criteria	Checking convergence criteria. If the stopping criteria is met, stop. Otherwise, go to step 2.

Table1: basic TS optimization algorithm use by MGO

Based on the basic concepts of TS, new components have been added to MGO:

- *Adaptive step size*

Initially, the step size should be large for diversification purposes and gradually, the step size should decrease for intensification.

- *Dynamic short-term tabu memory*

Allow TS to have a variable length of the short-term tabu list. The user can define the initial tabu memory size NSIZE0 and the increment, INC in the tabu solution initial setting file. If there is no improvement after certain number of iterations, it is likely that the search path is in cycling. One way to break the cycle is to increment the size by INC, NSIZE=NSIZE+INC

- *Automatic cycling detection*(ACD)

ACD is designed to detect periodicity of the path and to get out of it when it occurs.

To break the cycling: start with a new solution, which can be selected randomly or deterministically, or refresh the short-term tabu memory in order to find a new better solution.

- *Long-term tabu memory*
  The frequency of visiting each value is used as the long-term tabu memory and helps the tabu search to explore new solution regions. The least frequently used values have higher probability to be selected [4].

4. Numerical Experiments

We applied tabu search methodology to minimize the sum of capital cost associated to a P&T system for an aquifer assumed to be homogenous and confined. After integrating on vertical the flow equation (1), we obtain a two-dimensional equation for the steady-state flow model of Poisson type:

$$\nabla^2 H = -\frac{Q}{T} \quad (3)$$

where $T = (T_x, T_y)$ is the aquifer transmissivity with $T_* = \int_{z1}^{z2} K_* dz \ [L^2 T^{-1}]$; $Q = \int_{z1}^{z2} q_s dz$ is the flux change by aquifer with the exterior $[LT^{-1}]$.

We consider aquifer transmissivity $T = 600 m^2 day^{-1}$. The flow domain is bounded by constant-head boundaries at the south $H = 10\,m$ and $H = 20\,m$ to north borders and no-flow boundaries on the east and west borders.

For the numerical example we are using Groundwater Vistas. The direct problem is solved with MODFLOW using finite differences with block-centred grid applied to flow equation (3). The aquifer is divided into 30×18 nodes with $\Delta x = \Delta y = 100\,m$.

For the optimization problems we are using MGO that contains tabu search option. The coefficients of the objective function (2) and of the constraints are summarized in table 2

Coefficients	Values	Units
a_1	0	$\$m^{-1}$
a_2	200	$\$m^{-1}$
a_3	0.4	$\$m^{-3}$
$H_{\min}$	0	m
$Q_{\min}$	0	m^3
$Q_{\max}$	10000	m^3
d	50	m
$\Delta H_{\min}$	0.01	m
Δt	5	years

Table 2: Cost coefficients and constraints variables

We have an initial configuration of the intended capture zone that contains twenty pumping wells (orange area from figure 1), with pumping rate of $500 m^3 day^{-1}$ (for each well) so the total cost associated to this P&T system is $7500000\$$ for 5 years.

The objective is to determine the optimal pumping rates and locations to obtain a reduction of total costs using TS. We take the case of flexible well locations with the moving well option and we show the effect of wells number to the value of total costs.

Several test runs were made with the starting number of wells increased from one to ten to see which optimal number of wells is. The maximum number of wells with nonzero pumping rates is six. The total costs are considerably higher than the optimal value of $5625000\$$ when we take more than five wells. The best value is obtained for case 2 with 4 wells when the value of costs is reduced with 50.8%

No	Well locations		Case 1 3 wells	Case 2 4 wells	Case 3 5 wells	Case 4 6 wells
	x	y				
1	15	8	2258	1613.0	0.0	0
2	15	9	0	0.0	0.0	0
3	15	10	0	0.0	0.0	0
4	15	11	0	0.0	0.0	2742
5	16	8	0	0.0	1774.0	0
6	16	9	0	0.0	0.0	0
7	16	10	0	0.0	1290.0	0
8	16	11	1613	0.0	0.0	0
9	17	8	0	0.0	0.0	0
10	17	9	0	967.7	0.0	0
11	17	10	0	0.0	0.0	0

12	17	11	0	0.0	645.2	2258
13	18	8	0	0.0	0.0	1613
14	18	9	0	0.0	0.0	0
15	18	10	0	806.5	1613.0	2419
16	18	11	0	0.0	0.0	0
17	19	8	0	0.0	0.0	2419
18	19	9	0	0.0	0.0	0
19	19	10	0	0.0	0.0	0
20	19	11	1613	1613.0	1613.0	2419
Total pumping rates			5484	5000.2	6935.2	13870
Total costs			4033320	3690146	5112696	10185100
Cost reduction			-46.22%	-50.80%	-31.83%	35.80%

Table3: Optimized pumping rate

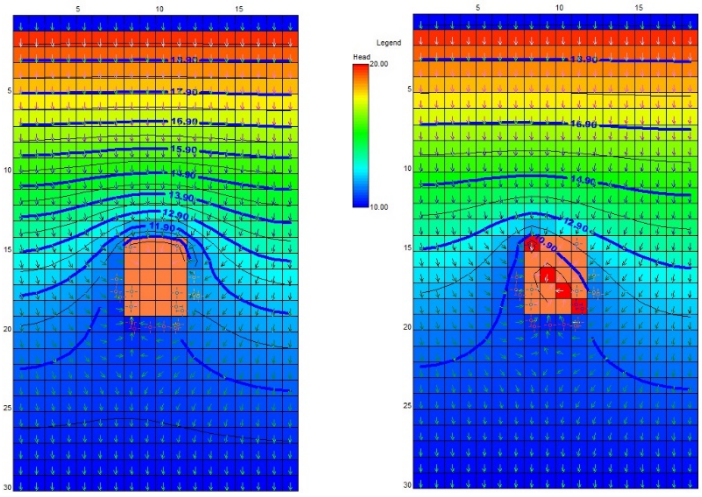

Figure1: a) The initial head distribution with 20 wells (orange area); b) The head distribution calculated for the optimal solution (case 2-4 wells). The well locations are shown as red cells

5. Conclusion

We can resume that TS is a metaheuristic algorithm for global optimization, efficiently in exploring the search space in order to find an optimal solutions using an adaptive memory.

TS can be applied for difficult problems, it obtains solutions that rival and often surpass the best solutions previously found by other approaches [3].

TS was applied with success in determination of consumption management policies. In this paper we used it to minimize the costs associated with a P&T system based on a dynamical pumping strategy in order to find optimal locations and pumping rates. The approach using MGO reduces the computational time required for simultaneous optimization of well locations and pumping rates from a few hours or more for a traditional trial-and-error modelling approach based on a forward simulation run each time a decision variable was changed to a few seconds.

Because of the good results obtained in difficult problems of optimisation, TS and other metaheuristics should be preferentially applied in practical problems of groundwater management.

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ON THE CIRCULAR EQUILIBRIUM FOR A CLASS OF NONLINEAR DYNAMICAL SYSTEMS

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Abstract: This work is motivated by a paper of Gotthans and Petržela [1] where a class of dynamical systems with circular equilibrium is investigated. The dimensionless form of this class of systems can be written as

$$x_{,t} = az\,, \qquad (1)$$

$$y_{,t} = zf_1(x,y,z)\,, \qquad (2)$$

$$z_{,t} = x^2 + y^2 - r^2 + zf_2(x,y,z)\,, \qquad (3)$$

where the parameter r became the radius of the circular equilibrium and a is a free parameter. The nonlinear functions f_1 and f_2 are quadratic polynomials în x, y, z. The fixed points of (1)-(3) form a circle and the typical property of this system is long spirally transient behavior and dissipative flow. Dynamical motion in the close neighborhood of the equilibrium circle is determined by using the cnoidal method [2]. New local features are discovered and a scenario of the hyperchaotic attractor (an attractor with a least two positive Lyapunov exponents) is described. The riddling bifurcation explains the creation of the hyperchaotic attractor [3].
Mathematics Subject Classification (2010): 70E50, 70Kxx, 70K20, 37M05
Key words: chaos, bifurcation, cnoidal method.

1. Introduction

The system of equations (1)-(3) has applications in electromagnetism [1]. The system provides a structurally stable strange attractor which fulfills the riddling bifurcation and chaos. The riddling bifurcation is the bifurcation in which one of the unstable periodic orbits embedded in a higher-dimensional chaotic attractor becomes unstable transversely to the attractor [4]. The riddling bifurcation explains the mechanism of creation of the hyperchaotic attractors [5-3]. A suggestive description of riddling and bubbling phenomena is given in [9]. When a trajectory leaves an attractor, it can be attracted by another attractor and therefore, the riddling phenomenon appears. The bubbling appears when such a trajectory comes back to the attractor, in the absence of another attractor.
The chaos-hyperchaos transition for weakly coupled continuous systems is analyzed in [10]. In such systems the coexistence of attractors located out of the invariant manifold are non-symetrical in relation to this manifold, and when the control parameteris changed, these attractors can undergo a chaos-hyperchaos transition.

2. Analysis of trajectories

In the following, we consider that the functions f_1, f_2 and the initial conditions, respectively have the form

$$f_1(x,y,z) = bx + cz^2;\ f_2(x,y,z) = dx \tag{4}$$

$$x_0 = (0,0,0)^T \tag{5}$$

The solving of the system of equations (1)-(5) is done by using the cnoidal method [2]. The general solution $\theta = \{x,y,z\}$ may be written in the terms of the *theta function* representation

$$\theta(t) = \frac{\mathrm{d}^2}{\mathrm{d}t^2}\log\Theta_n(\eta), \tag{6}$$

where $\eta = -\omega t + \phi$, ω is the frequency and ϕ is the phase. The theta function $\Theta_n(t)$ are defined as

$$\begin{aligned}
\Theta_1 &= 1 + \exp(\mathrm{i}\omega_1 t + B_{11}),\\
\Theta_2 &= 1 + \exp(\mathrm{i}\omega_1 t + B_{11}) + \exp(\mathrm{i}\omega_2 t + B_{22}) + \exp(\omega_1 + \omega_2 + B_{12}),\\
\Theta_3 &= 1 + \exp(\mathrm{i}\omega_1 t + B_{11}) + \exp(\mathrm{i}\omega_2 t + B_{22}) + \exp(\mathrm{i}\omega_3 t + B_{33}) + \exp(\omega_1 + \omega_2 + B_{12}) +\\
&+ \exp(\omega_1 + \omega_3 + B_{13}) + \exp(\omega_2 + \omega_3 + B_{23}) + \exp(\omega_1 + \omega_2 + \omega_3 + B_{12} + B_{13} + B_{23}),
\end{aligned} \tag{7}$$

and

$$\Theta_n = \sum_{M\in(-\infty,\infty)} \exp(\mathrm{i}\sum_{i=1}^{n} M_i\omega_i t + \frac{1}{2}\sum_{i<j}^{n} B_{ij}M_iM_j), \tag{8}$$

$$\exp B_{ij} = \left(\frac{\omega_i - \omega_j}{\omega_i + \omega_j}\right)^2,\ \exp B_{ii} = \omega_i^2. \tag{9}$$

with n the number of degrees of freedom for a particular solution.

Let us introduce the vectors of frequencies and constant phases

$$\omega = [\omega_1, \omega_2, ..., \omega_n],\ \phi = [\phi_1, \phi_2 ..., \phi_n],\ \eta = [\eta_1, \eta_2, ..., \eta_n]. \tag{10}$$

The vector η can be written as

$$\eta = -\omega t + \phi. \tag{11}$$

Also, we can write

$$M\eta = -\Omega t + \Phi,\ M = [M_1, M_2, ..., M_n],\ \Omega = M\omega, \Phi = M\phi. \tag{12}$$

The integer components in M are the integer indices in (8). The matrix B can be decomposed in a diagonal matrix D and an off-diagonal matrix O, that is

$$B = D + O. \tag{13}$$

THEOREM. *The general solution* $\theta(t)$ *can be written as*

$$\theta(t) = \frac{\partial^2}{\partial t^2}\log\Theta_n(\eta) = \theta_{lin}(\eta) + \theta_{nonlin}(\eta), \tag{14}$$

where θ_{lin} *represents a linear superposition of cnoidal functions*

$$\theta_{lin}(\eta) = \frac{\partial^2}{\partial t^2} \log G(\eta), \tag{15}$$

$$G(\eta) = \sum_M \exp\left(iM\eta + \frac{1}{2} M^T DM \right), \tag{16}$$

and θ_{nonlin} *represents a nonlinear interaction among the cnoidal functions*

$$\theta_{\text{int}}(\eta) = \frac{\partial^2}{\partial t^2} \log\left(1 + \frac{F\ (\eta, C)}{G\ (\eta)} \right), \tag{17}$$

$$F\ (\eta, C) = \sum_{M^\alpha} C \exp\left(iM\eta + \frac{1}{2} M^T DM \right), \tag{18}$$

$$C = \exp\left(\frac{1}{2} M^T OM \right) - 1. \tag{19}$$

The linear term (14) becomes

$$\theta_{lin}(\eta) = \sum_{m=1}^{n} \frac{\partial^2}{\partial t^2} \log G_m(M_m \eta_m) = \sum_{l=1}^{m} \alpha_l \text{cn}^2[\eta; m_l]. \tag{20}$$

The relation (20) provides the interpretation of the first term on the right-hand side of (14) as a linear superposition of cnoidal waves.
The second term θ_{nonlin} represents a nonlinear superposition or interaction among cnoidal waves. We write this term as

$$2\frac{\mathrm{d}^2}{\mathrm{d}t^2} \log\left(1 + \frac{F(t)}{G(t)} \right) \approx \frac{\beta_k \text{cn}^2(\eta, m_k)}{1 + \gamma_k \text{cn}^2(\eta, m_k)}. \tag{21}$$

Consequently, we have

$$\theta_{nonlin}(t) = \frac{\sum_{k=0}^{m} \beta_k \text{cn}^2[\eta; m_k]}{1 + \sum_{k=0}^{n} \lambda_k \text{cn}^2[\eta; m_k]}. \tag{22}$$

As a result, the cnoidal method yields to solutions consisting of a linear superposition (20) and a nonlinear superposition (22) of cnoidal functions.
The local behavior along the equilibrium circle is determined by the roots of the chatacteristic equation

$$\lambda^3 - dx\lambda^2 - 2x\left(a \pm b\sqrt{r^2 - x^2} \right)\lambda = 0, \tag{23}$$

with solutions

$$\lambda_1 = 0,\ \lambda_{2,3} = \frac{dx \pm \sqrt{d^2x^2 - 8x\left(a \pm b\sqrt{r^2 - x^2} \right)}}{2}. \tag{24}$$

The solutions $\lambda_{2,3}$ depend on the position of the equilibrium circle. From the first results we observe that we can have the saddle-type or the stable spiral equilibrium points. The typical

property of this dynamical system is long spirally transient behavior and dissipative flow given by the control parameter d .

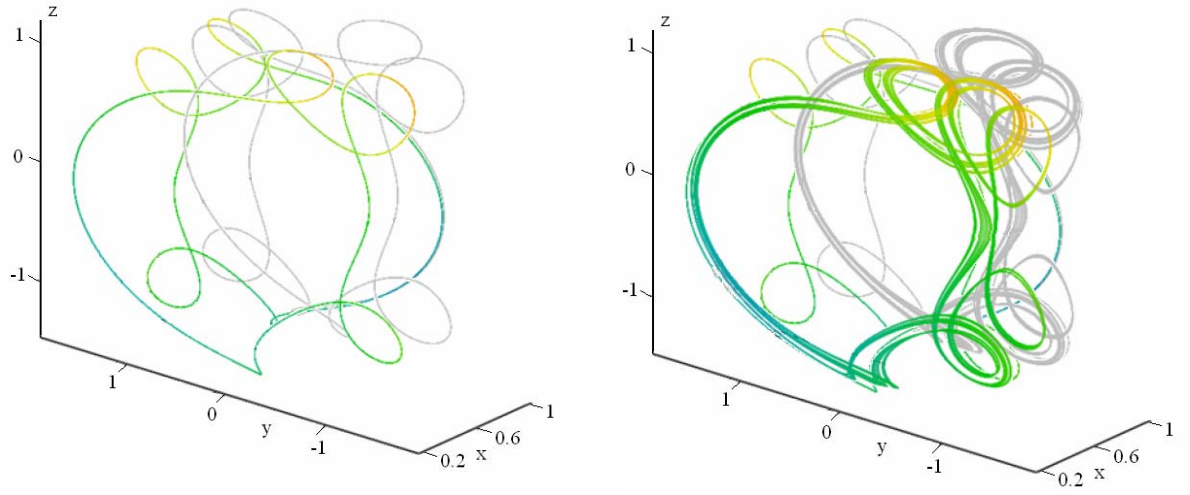

Fig.1. 3D view of the trajectories evolution for $d = -0.15$ $d = -0.12$.

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To understand the stable motion - chaos transition, we analyze the stability of trajectories. We investigate if their behavior is chaotic or not with respect to d and $t \in [0,700]$. For $t > 700$ we have no substantial differences in the behavior of trajectories. To do this we investigate the behavior of the Euclidean distance in the phase space between a parent trajectory and another trajectory obtained by a slight perturbation of the trajectory parent at $t = 0$. The numerical values of the free parameters are $a = -0.1$, $b = 3$, $c = -2.2$ and $r = 1$. The equilibrium half-circle location is in the plane $z = 0$. The control parameter d varies in the interval $[-1,0]$. Numerical results show that the solutions of the equations (1)-(4) are stable for $d \in [-1,-0.15)$. The trajectories (x, y, z) became unstable for $d \in [-0.15,0]$. Fig.1 presents the evolution of the trajectories after $t > 500$ for $d = -0.15$ (left) and $d = -0.12$ (right).

3. Chaos-hyperchaos transition

Numerical results shown us that the unstable periodic orbits $d = -0.15$ can becomes unstable with respect to $\dot{x}, \dot{y}$ or $\dot{z}$. For example, the orbits (x, y) can undergo the instability with respect to $\dot{z}$, exhibiting the riddling bifurcation. The existence of unstable orbits creates tongues anchored at these orbits, as shown in Fig. 2.

The generation of tongues of points leaving the neighborhood of the attractor after the bifurcation is explained in [9]. An infinite number of tongues can be created simultaneously, as the result of instability with respect to $\dot{x}, \dot{y}$ or $\dot{z}$. The transition between the chaos and hyperchaos is characterized by an infinite number of unstable periodic orbits which becomes unstable in the least two new variables. The bifurcation of an unstable periodic orbit is characteristic for the chaos-hyperchaos transition

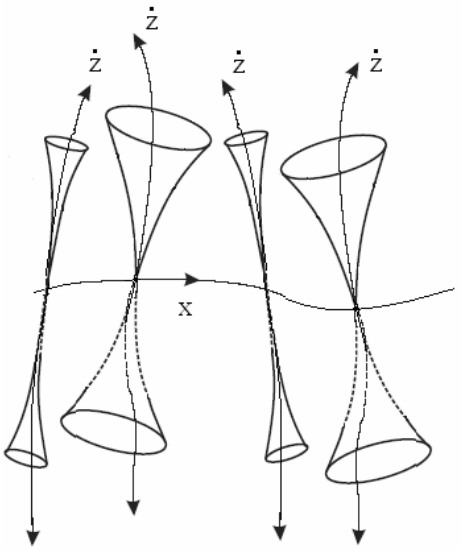

Fig. 2. Riddling bifurcations of the unstable periodic orbits with respect to x and $\dot{z}$.

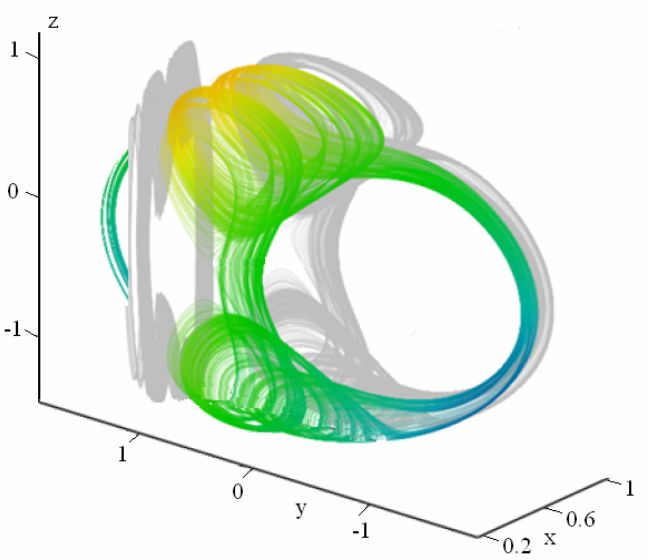

Fig.3. The transition chaos-hyperchaos of the attractor; (left) chaotic attractor for $d = -0.1$; (right) hyperchaotic attractor for $d = 0$.

4. Conclusions

In this paper a dynamical system with circular equilibrium is analyzed. A scenario of the hyperchaotic attractor formation is described via riddling bifurcations, i.e. the bifurcations in which one of the unstable periodic orbits embedded in a higher-dimensional chaotic attractor becomes unstable transversely to the attractor. When a trajectory leaves an attractor, it can be attracted by another attractor and therefore, the riddling phenomenon appears. In such systems the coexistence of attractors located out of the invariant manifold are non-symmetrical in relation to this manifold, and when the control parameter is changed, these attractors can undergo a chaos-hyperchaos transition.

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EXTREMAL POINTS IN LINEAR SPACES

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Abstract: In this article we present some results about extremal points of a nonempty convex subset of a linear space (theorem 1 and theorem 2).
Mathematics Subject Classification (2010): 46A50
Key words: Extremal Points

1. Characterization of extreme points with *n* components, $n \geq 3$

Notation: Let V be a linear space.

Definition 1. Let A be a convex subset of the linear space V, $A \neq \varnothing$. A subset K, $K \subseteq A, K \neq \varnothing$, is called *extremal subset* of A if from

$\alpha x_1 + (1-\alpha) x_2 \in K$, where $x_1, x_2 \in A$ and $\alpha \in (0,1)$,

it results that $x_1, x_2 \in K$.

If the subset K has only one element, $K = \{x_0\}$, then x_0 is called *extremal point* of the set A.

Notation: We use the notation $Ext(A)$ for the set of the extremal points of the set A.

Theorem 1. *Let V be a linear space and A a non-empty convex subset of V. If $x_0 \in A$, then the following statements are equivalent:*

a) x_0 is an extremal point of the set A;

b) Let $x_0 = \alpha x_1 + (1-\alpha) x_2$, where $x_1, x_2 \in A$ and $\alpha \in (0,1)$, then $x_1 = x_2 = x_0$;

c) Let $x_0 = \alpha_1 x_1 + \alpha_2 x_2$, where $x_1, x_2 \in A$ and $\alpha_1, \alpha_2 \in (0,1)$, $\alpha_1 + \alpha_2 = 1$, then $x_1 = x_2 = x_0$;

d) If $x_1, x_2 \in A, x_1 \neq x_2$ and $\alpha \in [0,1]$ so that $x_0 = \alpha x_1 + (1-\alpha) x_2$ then $\alpha = 0$ or $\alpha = 1$;

e) The set $A \setminus \{x_0\}$ is convex;

f) If $x_1, x_2 \in A$ and $x_0 = \frac{x_1 + x_2}{2}$, then $x_1 = x_2 = x_0$.

Proof. The statement a), b), c) are obviously equivalent to cach other $\alpha_1 = \alpha$ and $\alpha_2 = 1 - \alpha$.

$a) \Rightarrow d)$. We will use the "reductio ad absurdum"method (proof by contradiction).

Let us suppose that $\alpha \neq 0$ and $\alpha \neq 1$.

From $x_0 = \alpha x_1 + (1-\alpha) x_2$ and the fact that x_0 is an extreme point of the set A (see a)), it results that $x_1 = x_2 = x_0$, which contradicts $x_1 \neq x_2$.

This means that our supposition is false. Therefore we have $\alpha = 0$ or $\alpha = 1$.

$d) \Rightarrow a)$. Let $x_1, x_2 \in A$, $\alpha \in (0,1)$ and $x_0 = \alpha x_1 + (1-\alpha) x_2$.

If we suppose that $x_1 \neq x_2$ from b) it results that $\alpha = 0$ or $\alpha = 1$, which is in contradiction with $\alpha \in (0,1)$. Therefore we have $x_1 = x_2$.

From $x_0 = \alpha x_1 + (1-\alpha) x_2$ it results that $x_0 = x_1 = x_2$.

$a) \Rightarrow e)$. Let $x_1, x_2 \in A \setminus \{x_0\}$ and $\alpha \in [0,1]$. It is clear that $x_1 \neq x_0$ and $x_2 \neq x_0$.

It has to be proved that $\alpha x_1 + (1-\alpha) x_2 \in A \setminus \{x_0\}$, which is clear for $\alpha = 0$ and $\alpha = 1$.

If the set $A \setminus \{x_0\}$ would not be convex, it would exist $\alpha \in (0,1)$ so that $\alpha x_1 + (1-\alpha) x_2 \notin A \setminus \{x_0\}$. But A is convex, so this means that $\alpha x_1 + (1-\alpha) x_2 \in A$. Therefore we have $\alpha x_1 + (1-\alpha) x_2 = x_0$. From a) it results that $x_1 = x_2 = x_0$, which contradicts $x_1 \neq x_0$ and $x_2 \neq x_0$. This means that $A \setminus \{x_0\}$ is convex.

$e) \Rightarrow f)$. Let $x_1, x_2 \in A$ and $x_0 = \dfrac{x_1 + x_2}{2}$.

It can be noticed that if $x_1 = x_0$, then also $x_2 = x_0$.

The other way a round, if $x_2 = x_0$, then also $x_1 = x_0$.

If we suppose that $x_1 \neq x_0$ and $x_2 \neq x_0$, then $x_1, x_2 \in A \setminus \{x_0\}$. As $A \setminus \{x_0\}$ is a convex set, it results that $x_0 = \dfrac{x_1 + x_2}{2} \in A \setminus \{x_0\}$. But $x_0 = \dfrac{x_1 + x_2}{2}$. This means that $x_0 \in A \setminus \{x_0\}$, which is a contradictory result. This way we proved that $x_1 = x_2 = x_0$.

$f) \Rightarrow a)$. Let $x_1, x_2 \in A$ and $\alpha \in (0,1)$ so that $x_0 = \alpha x_1 + (1-\alpha) x_2$.

Considering $\alpha = \dfrac{1}{2}$, we obtain $x_0 = \dfrac{x_1 + x_2}{2}$ and from f) we have $x_1 = x_2 = x_0$.

If $\alpha \in \left(0, \dfrac{1}{2}\right)$, considering $x_3 = 2x_0 - x_2$ we obtain $x_3 = \lambda x_1 + (1-\lambda) x_2$, where $\lambda = 2\alpha \in (0,1)$; so we have $x_3 \in A$.

From $x_3 = 2x_0 - x_2$ we have $x_0 = \dfrac{x_2 + x_3}{2}$ and using the hypothesis from f) we obtain $x_2 = x_3 = x_0$.

From $x_0 = \alpha x_1 + (1-\alpha) x_2$ it results that $\alpha x_1 = \alpha x_2$. This is possible if and only if $x_1 = x_2$.

This way we proved that if $\alpha \in \left(0, \dfrac{1}{2}\right)$ then $x_1 = x_2 = x_0$

For $\alpha \in \left(0, \frac{1}{2}\right)$ the proof is similar. We consider $\lambda = 2\alpha - 1 \in (0,1)$, $x_3 = 2x_0 - x_1$ and we obtain $x_3 = \lambda x_1 + (1-\lambda) x_2$ so we have $x_3 \in A$. As in the case $\alpha \in \left(0, \frac{1}{2}\right)$ we can proved that $x_1 = x_2 = x_0$.

In conclusion, x_0 is an extremal point for the set A.

Theorem 2. *Let V be a linear space and A a non-empty convex subset of V. If $x_0 \in A$, then the following statements are equivalent:*

a) x_0 is an extremal point of the set A;

b) $P(n)$: If $x_1, x_2, ...x_n \in A$ and $x_0 = \frac{x_1 + x_2 + ... + x_n}{n}$, $n \geq 2$, then $x_1 = x_2 = ... = x_n = x_0$.

c) $P_1(n)$: Let $x_1, x_2, ...x_n \in A$ and $\alpha_1, \alpha_2, ...\alpha_n \in (0,1)$, $\alpha_1 + \alpha_2 + ... + \alpha_n = 1$, $\alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_n x_n = x_0$ then $x_1 = x_2 = ... = x_n = x_0$;

Proof. b). First we prove $P\left(2^m\right)$, inductively on $m \geq 1$.

If $m = 1$ obtain $P(2)$ equivalent T_1, f).

If $m = 2$ we have $P(4)$: $x_0 = \frac{1}{2}\left[\frac{x_1 + x_2}{2} + \frac{x_3 + x_4}{2}\right] \Rightarrow T_1$, f) it follows that $x_0 = \frac{x_1 + x_2}{2}$ and $x_0 = \frac{x_3 + x_4}{2} \Rightarrow x_1 = x_2 = x_0$ and $x_3 = x_4 = x_0$.

We assume by induction that $P\left(2^m\right)$ is true and we prove that $P\left(2^{m+1}\right)$ is true:

We have

$$P\left(2^{m+1}\right): x_0 = \frac{1}{2}\left[\frac{x_1 + x_2 + ... + x_{2^m}}{2^m} + \frac{x_{2^m+1} + ... + x_{2^{m+1}}}{2^m}\right].$$

By T_1, f) it follows that $x_0 = \frac{x_1 + x_2 + ... + x_{2^m}}{2^m} = \frac{x_{2^m+1} + x_{2^m+2} + ... + x_{2^{m+1}}}{2^m}$.

By the induction hypothesis we obtain

$x_1 = x_2 = ... = x_{2^m} = x_0$ and $x_{2^m+1} + x_{2^m+2} + ... + x_{2^{m+1}} = x_0$.

For $n \in \mathbb{N}$, $n \geq 3$ and $n \neq 2^k$ we consider $m \in \mathbb{N}^*$ such that $n < 2^m$ and $x_1, x_2, ...x_n \in A$, $x_{n+1} = x_{n+2} = ... = x_{2^m} = x_0$.

We have

$$x_0 = \frac{1}{2^m}\left[x_1 + x_2 + ... + x_n + \left(2^m - n\right) x_0\right].$$

By $P\left(2^m\right)$ it follows that $x_1 = x_2 = ... = x_n = x_0$.

c) First we prove $P_1\left(2^m\right)$, inductively on $m \geq 1$.

If $m=1$ obtain $P_1(2)$ equivalent T_1, c).

If $m=2$ we have $x_0=\alpha\left(\frac{\alpha_1}{\alpha}x_1+\frac{\alpha_2}{\alpha}x_2\right)+(1-\alpha)\left(\frac{\alpha_3}{1-\alpha}x_3+\frac{\alpha_4}{1-\alpha}x_4\right)$. By T_1, b) it follows that $x_0=\frac{\alpha_1}{\alpha}x_1+\frac{\alpha_2}{\alpha}x_2$ and $x_0=\frac{\alpha_3}{1-\alpha}x_3+\frac{\alpha_4}{1-\alpha}x_4\Rightarrow x_1=x_2=x_0$ and $x_3=x_4=x_0$.

We assume by induction $P_1\left(2^m\right)$ is true and we prove that $P_1\left(2^{m+1}\right)$ is true.

We have $P_1\left(2^{m+1}\right)$: $x_0=\alpha_1x_1+\alpha_2x_2+...+\alpha_{2^m}+\alpha_{2^m+1}+...+\alpha_{2^{m+1}}, \alpha_1+\alpha_2+...+\alpha_{2^{m+1}}=1$, $\alpha_k\in(0,1)$, $(\forall)k=\overline{1,m+1}$.

If $\alpha_1+\alpha_2+...+\alpha_{2^m}:=\alpha$ it follows that $\alpha_{2^m+1}+\alpha_{2^m+2}+...+\alpha_{2^{m+1}}=1-\alpha$.

If $S_1=\frac{\alpha_1}{\alpha}x_1+\frac{\alpha_2}{\alpha}x_2+...+\frac{\alpha_{2^m}}{\alpha}x_{2^m}$ and $S_2=\frac{\alpha_{2^m+1}}{1-\alpha}x_{2^m+1}+...+\frac{\alpha_{2^{m+1}}}{1-\alpha}x_{2^{m+1}}$, we have $x_0=\alpha S_1+(1-\alpha)S_2$. By T_1, b) it follows that $x_0=S_1=S_2$. By the induction hypothesis we obtain $x_1=x_2=...=x_{2^m}=x_0$ and $x_{2^m+1}=x_{2^m+2}=...=x_{2^{m+1}}=x_0$.

For $n\in\mathbb{N}$, $n\geq 3$ and $n\neq 2^k$ we consider $m\in\mathbb{N}^*$ such that $n<2^m$.

If $\alpha_1,\alpha_2,...,\alpha_n\in(0,1)$, $\alpha_1+\alpha_2+...+\alpha_n=1$ and $x_1,x_2,...,x_n\in A$, $x_{n+1}=x_{n+2}=...=x_{2^m}=x_0$.

We have $x_0=\beta_1x_1+\beta_2x_2+...+\beta_{2^m}x_{2^m}$ where,

$\beta_k=\frac{\alpha_k}{1+2^m-n}$, $(\forall)k\in\{1,2,...,n\}$ and $\beta_k=\frac{1}{1+2^m-n}$, $(\forall)k\in\{n+1,n+2,...,2^m\}$.

We have $\beta_1+\beta_2+...+\beta_{2^m}=1$, $\beta_k\in(0,1)$, $(\forall)k\in\{1,2,3,...,2^m\}$.

By $P_1\left(2^m\right)$ it follows that: $x_1=x_2=...=x_n=x_0$.

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MULTI INDEXED LAGUERRE POLYNOMIALS AND CHI SQUARE CORRELATION

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Abstract: Using the association between the moment generating function, respectively the cumulant generating function, and the generating functions for a family of orthogonal polynomials, one can study random variables involved in statistical testing, especially when sums or products of independent random variables are involved. Some well known examples in this direction are: the chi square distribution and the Laguerre polynomials, the normal distribution and the Hermite polynomials, the beta distribution and the Jacobi polynomials. In a previous work [7] we established a similar relation between the one indexed exceptional Laguerre polynomials of type 2 and a distribution obtained from the non central chi square distribution. The present work is devoted to a similar task for the multi indexed Laguerre polynomials.
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1. Introduction

The non central chi square distribution plays a central role in various statistical tests, starting with Pearson's test for goodness of fit, continuing with likelihood-ratio tests or tests for establishing when a certain form of autocorrelation is present and also for the study of the variance by means of samples taken from a normal population. When the parameters of corresponding chi square distribution $\chi_\nu^2(\delta)$ are known, namely

- the number of degrees ν, which apriori is a natural number, but was generalized to arbitrary positive numbers by means of the fractional calculus [6]and references cited there
- the non centrality parameter δ, which is a positive number

one can completely characterize the random variable $X = \chi_\nu^2(\delta)$ by computing its moments. In fact, if we consider the probability density function of X

$$f(x) = \frac{e^{-\frac{\delta+x}{2}}}{2^{\frac{\nu}{2}}} x^{\frac{\nu}{2}-1} \sum_{j=0}^{\infty} \frac{1}{2^j \Gamma(\frac{\nu}{2}+j)} (\frac{\delta}{2})^j \frac{x^j}{j!}$$

then the moment of order n is given by

$$m_n(X) = E(X^n) = \int_0^\infty x^n f(x)dx = 2^n n! L_n^{(\frac{\nu}{2}-1)}(-\frac{\delta}{2})$$

where $L_n^{(\frac{\nu}{2}-1)}(x)$ denotes the Laguerre polynomial of order n and parameter $\alpha = \frac{\nu}{2} - 1$. This formula establishes in fact a correspondence between the moments of a random variable and certain orthogonal polynomials. This correspondence was thoroughly considered in a series of papers by Karlin and his collaborators [8], where a slight modification of the probability density function was used, in order to obtain $m_n(X) = n! L_n^{(\frac{\nu}{2}-1)}(-\delta)$.

A different situation is encountered when dealing with samples from a population the parameters of which are unknown. Among different methods for estimating these parameters, we are interested in the method of the moments. By equating the theoretical moments with the statistical ones, obtained from the samples, one can get estimates for the unknown parameters with various degrees of accuracy.

Our proposal for treating the missing data situations is by means of the multi-indexed orthogonal polynomials [4],[5],[10],[11],[12] and references cited there. In order to do so, we have to consider the following additional problems:

a) the Stieltjes moment problem, which ascertains under what conditions a sequence of real numbers $\{\mu_n, n = 0,1,..\}$,eventually normalized with $\mu_0 = 1$,corresponds to the moments of a random variable [8],[13], namely that for all $n \in N$ the following determinants are positive

$$[\mu_0,..,\mu_n] = \begin{vmatrix} \mu_0 & \mu_1 & .. & \mu_n \\ \mu_1 & \mu_2 & .. & \mu_{n+1} \\ .. & .. & .. & .. \\ \mu_n & \mu_{n+1} & .. & \mu_{2n} \end{vmatrix} \text{ and } [\mu_1,..,\mu_{n+1}] = \begin{vmatrix} \mu_1 & \mu_2 & .. & \mu_{n+1} \\ \mu_2 & \mu_3 & .. & \mu_{n+2} \\ .. & .. & .. & .. \\ \mu_{n+1} & \mu_{n+2} & .. & \mu_{2n+1} \end{vmatrix}$$

b) the type of the multi-indexed orthogonal polynomials (MIOP) to be used

c) the distribution of the resulting random variable Y

For answering problem b), we relied upon the previously mentioned link between the non central chi square distribution and the Laguerre polynomials, so we choose the multi-indexed Laguerre orthogonal polynomials of first type $L_{D,n}^{(\alpha)}(z), n \in N$ [10]. A first step for answering problem a) is by finding the generating functions for the chosen polynomials, $G(t,z,\alpha,D) = \sum_{n=0}^{\infty} t^n L_{D,n}^{(\alpha)}(z)$. By considering that for some argument $\delta > 0$, $G(t,-\delta,\alpha,D)$ represents the moment generating function for the resulting random variable, we can solve the Stieltjes moment problem by means of the matrix

$$M_s(t,-\delta,\alpha,D) = \begin{bmatrix} G(t,-\delta,\alpha,D) & G'(t,-\delta,\alpha,D) & ... & G^{(s)}(t,-\delta,\alpha,D) \\ G'(t,-\delta,\alpha,D) & G''(t,-\delta,\alpha,D) & ... & G^{(s+1)}(t,-\delta,\alpha,D) \\ ... & ... & ... & ... \\ G^{(s)}(t,-\delta,\alpha,D) & G^{(s+1)}(t,-\delta,\alpha,D) & ... & G^{(2s)}(t,-\delta,\alpha,D) \end{bmatrix}$$

since the determinant of $M_s(0,-\delta,\alpha,D)$ is $[\mu_0,..,\mu_n]$.

Remark. While the zeros of the usual Laguerre polynomials lie all in the interval $[0,\infty)$, so there is no problem in taking $m_n(X) = n! L_n^{(\frac{\nu}{2}-1)}(-\delta)$, the multi-indexed Laguerre polynomials $L_{D,n}^{(\alpha)}(z)$ contain in their definition (see next section) polynomials of the form $L_j^{(\alpha)}(-z)$, $z > 0$, therefore the choice of the parameter $\delta > 0$ has to be carefully done.

2. Multi-indexed Laguerre polynomials: main properties

The solution to Sturm – Liouville problems which will encompass complete families of orthogonal polynomials having finite gaps in the sequence of their degrees (starting at some degree k or missing degrees $d_1,..,d_M$) determined the introduction and study of the exceptional orthogonal polynomials (EOP) and of the multi-indexed orthogonal polynomials (MIOP) [3],[4],[5],[10],[11],[12]. Since we are interested only in the multi-indexed Laguerre polynomials with virtual seed solutions, we shall mention here briefly the construction and the principal properties.

Consider a finite sequence of indices $D=\{d_1,..,d_M\}$ with $(C1): 1\le d_1<d_2<..<d_M$. We assume that the admissibility conditions [2], including the Krein-Adler condition, are satisfied $(C2)$: i) $\prod_{1\le j\le M}(n-d_j)\ge 0$ for any $n\in N$, and ii) $\alpha>0$. Let $L_n^{(\alpha)}(x)$ denote the Laguerre polynomial defined by [1]

$$L_n^{(\alpha)}(x)=\frac{(\alpha+1)_n}{n!}{}_1F_1(-n,\alpha+1,x)$$

where $(\alpha+1)_n=(\alpha+1)..(\alpha+n)$ is the Pochhammer symbol and the confluent hypergeometric function ${}_1F_1(-n,\alpha+1,x)=\sum_{j=0}^{n}\frac{(-n)_j}{(\alpha+1)_j}\frac{x^j}{j!}$.

Since $L_n^{(\alpha)}(0)=0$ iff $\alpha=-j$ and $j\in N$, $1\le j\le n$ [17], the condition $(C2)$ will ensure that $L_{d_j}^{(-\alpha)}(0)\neq 0$ for any $1\le j\le M$ .The following construction is adapted from [10], the type of construction being that of virtual seed solutions.

Let the functions $\phi_{d_j}(x)=e^{\frac{x^2}{2}}x^{(\frac{1}{2}+\alpha)}L_{d_j}^{\alpha}(-x^2), 1\le j\le M$, be solutions to the differential equation

$$-y''+(x^2+\frac{\alpha^2-\frac{1}{4}}{x^2})y=-2(2d_j+1+\alpha)y$$

With the notation $z=x^2$, the solutions become $\phi_{d_j}(z)=e^{\frac{z}{2}}z^{\frac{1}{2}(\frac{1}{2}+\alpha)}L_{d_j}^{\alpha}(-z)$, $1\le j\le M$. Notice that these solutions are node less and they and their inverses are not square integrable on $(0,\infty)$ [9].

Let the functions $\psi_n(x)=e^{-\frac{x^2}{2}}x^{(\frac{1}{2}+\alpha)}L_n^{\alpha}(x^2)$, $n=0,1,..$ be solutions to the differential equation

$$-y''+(x^2+\frac{\alpha^2-\frac{1}{4}}{x^2})y=2(2n+1+\alpha)y$$

With the same notation $z=x^2$, the solutions become $\psi_n(z)=e^{-\frac{z}{2}}z^{\frac{1}{2}(\frac{1}{2}+\alpha)}L_n^{\alpha}(z)$, $n=0,1,..$.

Consider the functions $W[\phi_{d_1},\phi_{d_2},..,\phi_{d_M},\psi_n](z)$, solutions to the rationally extended potential by means of the virtual states corresponding to $\{\phi_{d_j}(z),1\le j\le M\}$ [5], [10], [11], [12]. Here $W[\phi_{d_1},\phi_{d_2},..,\phi_{d_M},\psi_n](z)$ denotes the Wronskian of the functions in the brackets, with derivation relative to the variable z. Notice that the virtual states will be the *replacements* for the missing moments and the negative energies will represent the corresponding penalty weights. Our hypothesis concerning the parameter α will assure the fact that $W[\phi_{d_1},\phi_{d_2},..,\phi_{d_M}](0)\neq 0$ [17]. Moreover, $W[\phi_{d_1},\phi_{d_2},..,\phi_{d_M}](z)\neq 0$ also for any $z>0$.

Now $W[\phi_{d_1},\phi_{d_2},..,\phi_{d_M},\psi_n](z)$ can be factorized with a polynomial part, denoted $L_{D,n}^{(\alpha)}(z)$,

$$W[\phi_{d_1},\phi_{d_2},..,\phi_{d_M},\psi_n](z)=(e^{\frac{z}{2}})^{M-1}(z^{\frac{1}{2}(\alpha+\frac{1}{2})})^{M+1}L_{D,n}^{(\alpha)}(z)$$

From this it will follow that

$$L_{D,n}^{(\alpha)}(z)=(z^{\frac{1}{2}(\alpha+\frac{1}{2})})^{-(M+1)}(e^{\frac{z}{2}})^{-(M-1)}W[\phi_{d_1},..,\phi_{d_M},\psi_n](z)$$

Define the index $k=\sum_{j=1}^{M}d_j-\frac{M(M-1)}{2}$.

Although $L_{D,n}^{(\alpha)}(z)$ is generically a polynomial of degree $k+n$, the sequence $\{L_{D,n}^{(\alpha)}(z)\}_{n=0,1,..}$ forms a complete orthogonal family relative to the positive weight $\omega(z)=\frac{z^{\alpha+k}e^{-z}}{(W[\phi_{d_1},\phi_{d_2},..,\phi_{d_M}](z))^2}$ in the space $L_{\omega}^2(0,\infty)$ (see [2], [10], [12]).

We also need the fact that the associated differential operator, which is in fact the rationally extended Hamiltonian, is selfadjoint, hence its eigenfunctions are linearly independent.

3. Generating functions for type 1 multi-indexed Laguerre polynomials

In introducing the generating functions for a sequence of polynomials, we shall adapt the notation from [15], namely, if $\{f_n(z)\}_{n\in N}$ is a sequence of functions with formal power series $f(t,z)=\sum_{n=0}^{\infty}t^n f_n(z)$, then by $[t^n]f(t,z)$ will shall denote $f_n(z)$,the coefficient of the t^n term.

Since we shall use functions of two or more variables, the derivative relative to one of the variables will be subscripted with the name of the variable. Accordingly, $f_t^{(k)}(t,z)$ will stand for $\frac{d^k}{dt^k}f(t,z)$, while $f_z^{(k)}(t,z)$ will stand for $\frac{d^k}{dz^k}f(t,z)$.

Following [16] we shall use the following generating function for the Laguerre polynomials of parameter $\alpha>-1$, which clearly is our case,

$$G(t,z,\alpha)=\sum_{n=0}^{\infty}t^n L_n^{(\alpha)}(z)=(1-t)^{-(\alpha+1)}\exp(-\frac{zt}{(1-t)})$$

Notice that in the previously mentioned notation, $[t^n]((1-t)^{-(\alpha+1)}\exp(-\frac{zt}{1-t}))=L_n^{(\alpha)}(z)$. We shall assume all through this paper that $|t|<1$.

Lemma 1. The function $G(t,z,\alpha)$ has the following properties:

i) $G(t,z,\alpha)=(1-t)^s G(t,z,\alpha+k)$, for any $s\geq 1$

ii) $G(t,z,\alpha+s)=(1-t)^{M-s}G(t,z,\alpha+M)$, $1\leq s\leq M$

iii) $G_z'(t,z,\alpha)=-tG(t,z,\alpha+1)$

Proof. The statements are direct consequences of the definition of the function $G(t,z,\alpha)$.

Lemma 2. For the function $G(t,z,\alpha)$ the following relation is valid

$$[t^{n-s}](G(t,z,\alpha+s))=[t^n](t^s(1-t)^{M-s}G(t,z,\alpha+M))$$

The result is a consequence of lemma 1 and the definition of the generating function.

Let us introduce some more notations. We shall define the matrix

$$W_D^{(\alpha)}(z)=\begin{pmatrix}[(1-t)^M G(t,-z,\alpha+M)]_{t=0}^{(d_1)} & \dots & [G(t,-z,\alpha+M)]_{t=0}^{(d_1)}\\ \dots & \dots & \dots\\ [(1-t)^M G(t,-z,\alpha+M)]_{t=0}^{(d_M)} & \dots & [G(t,-z,\alpha+M)]_{t=0}^{(d_M)}\end{pmatrix}$$

Moreover, for each index $0\le j\le M$, we shall denote by $W_{D,j}^{(\alpha)}(z)$ the determinant obtained from the previous matrix by deleting the column j.

Theorem 1. The generating function for the polynomials $\{L_{D,n}^{(\alpha)}(z), n\in N\}$ is given by

$$G(t,z,D,\alpha)=\sum_{n=0}^{\infty}t^n L_{D,n}^{(\alpha)}(z)=[\sum_{j=0}^{M}(-1)^j t^j(1-t)^{M-j}W_{D,j}^{(\alpha)}(z)]G(t,z,\alpha+M)$$

By making use of the properties of the determinants and of the Laguerre polynomials, the result is a consequence of lemmas 1 and 2.

Remarks.

1) For the particular case when the index set is reduced to a single value, $D=\{k\}$, $k\in N$, $k\ge 1$, and the parameter α is shifted to $\alpha+k-\frac{3}{2}$, we obtain the generating function for type I Laguerre polynomials from [9], also recovered in [7].

2) Although the polynomials $L_{D,n}^{(\alpha)}(z)$ are generically of degree $k+n$, the definition of the generating function is given by performing a shift, namely the moments are shifted by the missing ones.

3) A similar formula can be obtained for multi-indexed type II Laguerre polynomials. In fact, a challenging result is the generating function for multi-indexed Laguerre polynomials for which the virtual seed solutions are of mixed type [9]. This and the corresponding formula for multi-indexed Jacobi polynomials are part of a forthcoming paper.

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ON BOUNDED ENTIRE REAL FUNCTIONS

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Abstract: Let $H(\mathbf{R})$ be the commutative ring of entire real functions f, where $f = \sum_{n=0}^{\infty} a_n x^n$, $a_n \in \mathbf{R}$, and $\lim_{n\to\infty} |a_n|^{\frac{1}{n}} = 0$ (see [1]). We denote by $BH(\mathbf{R}) \subset H(\mathbf{R})$ the subring of bounded entire real functions. For $f, g \in H(\mathbf{R})$, f as before and $g = \sum_{n=0}^{\infty} b_n x^n$ we consider the equivalence relation $f \sim g$ if and only if $|a_n| = |b_n|$, for every $n \in \mathbf{N}$. A necessary and sufficient condition on a function $f \in H(\mathbf{R})$ to belong to $BH(\mathbf{R})$ is proved. Examples of unbounded entire real functions which are equivalent to functions from $BH(\mathbf{R})$ are presented. Then we construct an unbounded entire real function such that all the functions $g \in H(\mathbf{R})$, $g \sim f$ are unbounded. Finally, by using results from [2], [4] and [5], some properties of the ring $BH(\mathbf{R})$ are proved..
Mathematics Subject Classification (2010): 30D20, 26A99
Key words: power series, entire function.

1. Introduction

Let $H(\mathbf{R})$ be the set of entire real functions, that is the functions $f(x)$ equal to the sum of a series $\sum_{n=0}^{\infty} a_n x^n$, $a_n \in \mathbf{R}$ which converges for every real number x. It follows easily that

$$H(\mathbf{R}) = \left\{ f(x) = \sum_{n=0}^{\infty} a_n x^n ; a_n \in \mathbf{R},\ \lim_{n\to\infty} \sqrt[n]{|a_n|} = 0 \right\}. \quad (1)$$

Then $H(\mathbf{R})$ becomes a commutative ring with respect to the addition and the multiplication of entire functions. We denote by $BH(\mathbf{R})$ the subring (of $H(\mathbf{R})$) of the bounded entire real functions.

2. Results and examples

The following result gives a necessary and sufficient condition on a function $f \in H(\mathbf{R})$ to belong to $BH(\mathbf{R})$.

Theorem 1. *Assume* $f(x) = \sum_{n=0}^{\infty} a_n x^n$ *is a entire real function. Then* $f(x) \in BH(\mathbf{R})$ *if and only if there exists a positive real number* M *and the increasing sequences* $\{x_k\}_{k\geq 1}, \{n_k\}_{k\geq 1}$ *of positive integers such that, for every* m *:*

$$(i) \quad \max_{x\in[-x_{n_m}, x_{n_m}]} \left|s_{n_m}(x)\right| > M;$$

$$(ii) \quad \left|s_{n_{m+1}}(x)\right| \leq M, \text{ for every } x \in [-x_{n_m}, x_{n_m}], \tag{2}$$

where $s_n(x) = \sum_{k=0}^{n} a_k x^k$.

Proof. Suppose $f \in BH(\mathbf{R})$ and M is a positive real number such that $\left|f(x)\right| \leq M$ for every $x \in \mathbf{R}$. We choose n_1 the smallest positive integer k such that $a_k \neq 0$ and we take the smallest positive integer x_1 such that $\max_{x\in[-x_1, x_1]} \left|s_{n_1}(x)\right| > M$. Since f is bounded it follows that we can choose the smallest positive integer $n_2 > n_1$ such that $\left|s_{n_2}(x)\right| \leq M$, for every $x \in [-x_1, x_1]$. Then we choose x_2 the smallest positive integer greater than x_1 such that $\max_{x\in[-x_2, x_2]} \left|s_{n_2}(x)\right| > M$. In this manner, by induction on m, we get (*i*) and (*ii*).

Conversely, since $f(x) = \lim_{m\to\infty} s_{n_m}(x)$, by (*ii*) it follows that $f(x) \in BH(\mathbf{R})$. □

For $f, g \in H(\mathbf{R})$, f as before, $g = \sum_{n=0}^{\infty} a_n x^n$, we consider the equivalence relation defined by : $f \sim g$ if and only if $\left|a_n\right| = \left|b_n\right|$, for every $n \in \mathbf{N}$.

It can be put the question if, for every unbounded $f \in H(\mathbf{R})$, we can find $g \in BH(\mathbf{R})$ such that $f \sim g$. For example, if $f(x) = e^x$ we can choose $g(x) = \sin x + \cos x$, and $f \sim g$. The following example shows that the answer to the above question is negative.

Example 1. Consider

$$f(x) = \sum_{k=0}^{\infty} \frac{1}{2^{k^2}} x^k \in H(\mathbf{R}) \tag{3}$$

and

$$g(x)=\sum_{k=0}^{\infty}a_k x^k,\ \text{ with } a_k\in\left\{\pm\frac{1}{2^{k^2}}\right\}. \tag{4}$$

We shall prove that every fixed g given by (4) is an unbounded entire function. Let n be an even integer greater or equal to 6. Then

$$g(2^n)=\sum_{k=0}^{\infty}(\pm\frac{1}{2^{k^2}}2^{nk})=\sum_{k=0}^{\infty}\frac{\pm1}{2^{(k-n)k}}=\sum_{k=0}^{\frac{n}{2}-2}(\pm2^{k(n-k)})\pm2^{(\frac{n}{2}-1)(\frac{n}{2}+1)}\pm2^{\frac{n}{2}\cdot\frac{n}{2}}$$

$$\pm2^{(\frac{n}{2}-1)(\frac{n}{2}+1)}\pm\sum_{k=0}^{\frac{n}{2}-2}(\pm2^{(n-k)k})\pm\frac{1}{2^{n+1}}\pm\frac{1}{2^{2(n+2)}}\pm\ldots \tag{5}$$

We denote $\sigma_{\frac{n}{2}-2}:=\sum_{k=0}^{\frac{n}{2}-2}2^{k(n-k)}$ and we shall prove that

$$\sigma_{\frac{n}{2}-2}<\frac{1}{4}2^{(\frac{n}{2}-1)(\frac{n}{2}+1)}. \tag{6}$$

At first, we shall prove, by induction on t, that

$$\sum_{k=0}^{t}2^{k(n-k)}<2^{(t+1)(n-t-1)},\ \text{for every } t\le\frac{n}{2}-3. \tag{7}$$

If $t=0,\ 1<2^{n-1}$ and (7) holds. Now, we suppose that (7) is true and we shall prove that

$$\sum_{k=0}^{t+1}2^{k(n-k)}<2^{(t+2)(n-t-2)} \tag{8}$$

Indeed,

$$\sum_{k=0}^{t+1}2^{k(n-k)}=\sum_{k=0}^{t}2^{k(n-k)}+2^{(t+1)(n-t-1)}<2^{(t+1)(n-t-1)+1}<2^{(t+2)(n-t-2)},$$

because $(t+2)(n-t-2)-(t+1)(n-t-1)-1=n-2t-4>0$ for $t<\frac{n}{2}-2.$ Hence it follows (7).

By putting $t=\frac{n}{2}-3$ in (7) we get

$$\sigma_{\frac{n}{2}-2}=2^{\left(\frac{n}{2}-2\right)\left(\frac{n}{2}-2\right)+1}=\frac{2^{\left(\frac{n}{2}-1\right)\left(\frac{n}{2}+1\right)}}{4},$$

which implies (6).

Since

$$\left|\sum_{k=0}^{\frac{n}{2}-1} \pm 2^{k(n-k)}\right| \le \sigma_{\frac{n}{2}-1} < 2\frac{\left(\frac{n}{2}-1\right)\left(\frac{n}{2}+1\right)}{4},$$

by (5) it follows that the sequence $\{g(2^n)\}_{n\ge 6}$ is bounded only when $\operatorname{sgn}(a_{\frac{n}{2}-1}) = \operatorname{sgn}(a_{\frac{n}{2}+1}) = -\operatorname{sgn}(a_{\frac{n}{2}})$. But in this case, by (5) and (6), $g(-2^n) > \frac{2^{\left(\frac{n}{2}-1\right)\left(\frac{n}{2}+1\right)}}{4}$, because $\left|a_{\frac{n}{2}-1}(-2^n)^{\frac{n}{2}-1} + a_{\frac{n}{2}}(-2^n)^{\frac{n}{2}} + a_{\frac{n}{2}+1}(-2^n)^{\frac{n}{2}+1}\right| \ge 2^{\left(\frac{n}{2}-1\right)\left(\frac{n}{2}+1\right)}$ and the sequence $\{g(-2^n)\}_{n\ge 6}$ is unbounded. Hence we obtain that $g(x)$ is unbounded.

Proposition 1. *If* $f \in BH(\mathbf{R})$, $f(x) = \sum_{n=0}^{\infty} a_n x^n$, *then the following statements are true :*

(*i*) $f = f_1 + f_2$, *where* $f_1, f_2 \in BH(\mathbf{R})$ f_1 *is an even function and* f_2 *is an odd function.*

(*ii*) *If* $g \in H(\mathbf{R})$, *then* $f \circ g \in BH(\mathbf{R})$ *and* $g \circ f \in BH(\mathbf{R})$.

(iii) If $h(x) = \sum_{n=0}^{\infty} c_n x^n$, *where* $c_n = a_n$, *for every* $n \in \mathbf{N} \setminus F$, *and* $c_n = -a_n$, *for every* $n \in F$, *with* $F \neq \{0\}$ *a finite subset of nonnegative integers, then* h *is an unbounded entire real function.*

Proof. (*i*) It is enough to take $f_1(x) = \frac{f(x) + f(-x)}{2}$ and $f_2(x) = \frac{f(x) - f(-x)}{2}$.

(*ii*) It is obvious.

(*iii*) Since $h(x) = f(x) - 2\sum_{n\in F} a_n x^n$ and every polynomial defines an unbounded entire real function, it follows the statement. □

Now, by using the results on $H(\mathbf{R})$ obtained by Helmer (see [2]), we prove similar results for $BH(\mathbf{R})$.

Theorem 2. *Assume* $f \in BH(\mathbf{R})$. *Then* f *is a unit in* $BH(\mathbf{R})$ *if and only if* $f(x) = e^{g(x)}$, *with* $g \in BH(\mathbf{R})$.

Proof. If $g \in BH(\mathbf{R})$, by Proposition 1 (*ii*) we get that $f = \pm e^g, h = \pm e^{-g} \in BH(\mathbf{R})$ and $fh = 1$.

Conversely, if $f \in BH(\mathbf{R})$ is a unit in $BH(\mathbf{R})$, by [2], Theorem 1, it follows that $f = e^g$, with $g \in H(\mathbf{R})$. Since $e^g, e^{-g} \in BH(\mathbf{R})$, we can choose $M > 0$ such that $-M \le g(x) \le M$, for every $x \in \mathbf{R}$. Hence, $g \in BH(\mathbf{R})$. □

Remark 1. Suppose $g \in H(\mathbf{R})$ is a nonnegative function. Then $f = \pm e^{-g} \in BH(\mathbf{R})$, but we can choose g such that f is not a unit in $BH(\mathbf{R})$ (for example $g(x) = x^2$). Thus, $BH(\mathbf{R})$ contains functions f which don't have zeros, and which are not units in $BH(\mathbf{R})$. We denote by $S = \{\pm e^{-g} : g \in H(\mathbf{R})$ is a nonnegative function$\}$.

Theorem 3. *S is a multiplicative system for $BH(\mathbf{R})$ and $H(\mathbf{R})$ is the ring of fractions of $BH(\mathbf{R})$ with respect to S.*

Proof. It follows easily that S is a multiplicative system. Since every element from S is a unit in $H(\mathbf{R})$, we get $S^{-1}BH(\mathbf{R}) \subset H(\mathbf{R})$.

If $f \in H(\mathbf{R})$, then $f(x) = \dfrac{f(x)e^{-f^2(x)}}{e^{-f^2(x)}} = \dfrac{h(x)}{e^{-f^2(x)}}$, where $h(x) = f(x)e^{-f^2(x)}$. Hence $|h(x)| = \dfrac{|f(x)|}{e^{f^2(x)}} \le \dfrac{|f(x)|}{1+f^2(x)} \le \dfrac{1}{2}$. Thus, $h \in BH(\mathbf{R})$ and, because $e^{-f^2} \in S$, we obtain the theorem. □

Let Z a multiset of complex numbers such that, if $z \in Z$, then also $\bar{z} \in Z$, with the same multiplicity. As in [2], we call Z *a zero-set*, if it has no finite-accumulation points.

Corollary 1. *Assume Z is a zero-set. Then, there exists $f \in BH(\mathbf{R})$, such that $Z(f)$ (the set of all zeros of f, each occurring a number of time equal to its multiplicity) is equal to Z.*

Proof. By [2], Theorem 3, there exists $g \in H(\mathbf{R})$ such that $Z(g)=Z$. Since, by Theorem 3, $g = \dfrac{f}{h}$ where $f \in BH(\mathbf{R})$ and $h \in S$, it follows the corollary. □

Corollary 2. *Let α be a complex number. Then there exists a bounded entire function f with rational coefficients such that $f(\alpha) = 0$.*

Proof. By [3], Section 4, there exists an entire function g with rational coefficients such that $g(\alpha) = 0$. If we take, as in the proof Theorem 3, $f(x) = g(x)e^{-(g(x)-g(0))^2}$, it follows the corollary. □

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ON A CLASS OF SYMMETRIC (0,1)-MATRICES

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Abstract: Let $\mathrm{Sim}_{(0,1)}(n)$ be the set of all symmetric (0,1) – matrices $X = (x_{ij})_{i,j = 1,2,...,n}$, of fixed order n, having all the entries $x_{ii} = 0$, $i = 1,2, \ldots ,n$. and the subset of $\mathrm{Sim}_{(0,1)}(n)$ defined by CyHam(n) = $\{A = P + P^{-1}$:P a permutation matrix of order n, $p_{ii} = 0, \forall i$, $P^n = I$, $P^j \neq I$, $\forall j < n \}$, where I is the identity matrix.

We prove that, for an arbitrary permutation matrix P, $A = P + P^{-1}$ belongs to the set CyHam(n) if and only if the eigenvalues of A have the form $\gamma_k = 2\cos\frac{2(k-1)\pi}{n}$, $k = 1,2,...,n$. If $A \in$ CyHam(n), we denote $F(A) = \{ X \in \mathrm{Sim}_{(0,1)}(n) : A \leq X \}$ and $\mathrm{Ham}(n) = \bigcup_{A \in \mathrm{CyHam}(n)} F(A)$. Sufficient conditions on a matrix $X \in \mathrm{Sim}_{(0,1)}(n)$ to belong to Ham(n) are given. Hence we obtain other proofs for known results in graph theory.

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Key words: (0,1)-matrix, lattice, eigenvalue, Hamiltonian cycle

1. Introduction

A (0,1) – *matrix*, called also a *logical matrix* or a *binary matrix*, is an integer matrix in which each element is 0 or 1. Let $\mathrm{Sim}_{(0,1)}(n)$ be the set of all symmetric (0,1) – matrices $X = (x_{ij})_{i,j = 1,2,...,n}$, of fixed order n having all the entries $x_{ii} = 0$, $i = 1,2, \ldots ,n$. Then $\mathrm{Sim}_{(0,1)}(n)$ is a partially ordered set, with the partial ordering, denoted by $\leq$, where for X, $Y \in \mathrm{Sim}_{(0,1)}(n)$ we have:

$$X \leq Y \quad \text{if and only if} \quad x_{ij} \leq y_{ij}, \ i,j = 1,2,...,n. \tag{1}$$

If $X, Y \in \mathrm{Sim}_{(0,1)}(n)$, then we define the meet $Z = X \wedge Y$ whose entries are :

$$z_{ij} = \min\{x_{ij}, y_{ij}\}, \quad i,j = 1,2,...,n$$

and the join $T = X \vee Y$ having the entries :

$$t_{ij} = \max\{x_{ij}, y_{ij}\}, \quad i,j = 1,2,...,n .$$

Thus $(\mathrm{Sim}_{(0,1)}(n), \leq)$ is a lattice (see, for example, [6]) . For $i, j \in \{1,2, \ldots, n\}$, $i \neq j$, we denote $S(i,j) \in \mathrm{Sim}_{(0,1)}(n)$, the matrix having 1 as their (i,j) and (j,i) entries and all other entries 0.

Let P be a permutation matrix of order n, that is a matrix characterized by the property that in each row and each column there is just one non-zero entry, which is 1. Suppose that $P^j \neq I$, for every $j < n$, where I is the identity matrix. We denote:

$\mathrm{CyHam}(n) = \{A = P + P^{-1} : P$ a permutation matrix of order n, $p_{ii} = 0, \forall i, P^n = I, P^j \neq I, \forall j < n\}$.

It follows easily that $\mathrm{CyHam}(n) \subset \mathrm{Sim}_{(0,1)}(n)$.

Example 1. If $n = 6$,

$$P_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \text{ and}$$

$$P_3 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

are permutation matrices of order six. It is easy to see that $P_1^6 = I$, and for all $j < 6$, $P_1^j \neq I$, but $P_2^3 = I$ and $P_3^5 = I$. Thus $A_1 = P_1 + P_1^{-1} \in \mathrm{CyHam}(6)$, but $A_2 = P_2 + P_2^{-1}$. $\notin \mathrm{CyHam}(6)$, $A_3 = P_3 + P_3^{-1} \notin \mathrm{CyHam}(6)$.

Remark 1. Let P be a permutation matrix of order n. Consider m the smallest positive integer such that $P^m = I$. By [1], p. 31, there is an one-to-one morphism of group $\varphi : S_n \to Gl_n(\mathbf{R})$, where S_n is the permutation group of order n and $Gl_n(\mathbf{R})$ is the multiplicative group of non-singular real matrices of order n, and $\varphi(S_n)$ is the multiplicative

group of permutation matrices. Hence $P = \varphi(\sigma)$, where $\sigma^m = e$ with e the identity permutation. If $n = 6$, from Example 1 we find $\varphi((123456)) = P_1$, $\varphi((123)(456)) = P_2$ and $\varphi((12345)) = P_3$.

2. Main results

The following result describes the eigenvalues of the matrices from CyHam(n).

Theorem 1. *Suppose P is a permutation matrix of order n. Then $A = P + P^{-1}$ belongs to* CyHam(n) *if and only if the eigenvalues of A are:*

$$\gamma_k = 2\cos\frac{2(k-1)\pi}{n}\ ,\ k = 1,2,\ldots,n. \tag{2}$$

Proof. If $A \in$ CyHam(n), we consider T the endomorphism of $\mathbf{R}^n$ defined by P. Since $P^n = I$ and $P^j \neq I$, for every $j < n$ it follows that T is a cyclic linear transformation, that is there exists a single vector $X \in \mathbf{R}^n$ such that $\{X, T(X), \ldots, T^{n-1}(X)\}$ is a basis of $\mathbf{R}^n$. By [3], Theorem 2, p. 69, it follows that the minimum polynomial $Q(\gamma)$ of P has degree n. Since $P^n = I$ it follows that $Q(\gamma)$ is a factor of $\gamma^n - 1$. Hence $Q(\gamma) = \gamma^n - 1$ and the eigenvalues of P are

$$\delta_k = \cos\frac{2(k-1)\pi}{n} + i\cdot\sin\frac{2(k-1)\pi}{n}\ ,\quad k = 1,2,\ldots,n.$$

Thus, for a fixed k, $PV_k = \delta_k \cdot V_k$, where V_k is an eigenvector associated to δ_k. This implies

$$P^{-1}V_k = P^{n-1}V_k = \delta_k^{n-1}\cdot V_k = \overline{\delta}_k \cdot V_k, \text{ where } \overline{\delta_k} = \cos\frac{2(k-1)\pi}{n} - i\cdot\sin\frac{2(k-1)\pi}{n},\ k = 1,2,\ldots,n.$$

Hence $AV_k = (\delta_k + \overline{\delta}_k)V_k$, and the eigenvalues of A are given by (2).

Conversely, suppose $A = P + P^{-1}$, where P is a permutation matrix and has the eigenvalues γ_k, given by (2). Consider m the smallest positive integer such $P^m = I$. If $m \neq n$, then by Remark 1 it follows that $P = \varphi(\sigma)$, where $\sigma = \sigma_1 \ldots \sigma_t$, with σ_j disjoint cycles of order m_j, j=1,2,...,t, such that $m_1 + m_2 + \ldots + m_t = n$. Hence

$$P = \begin{pmatrix} P_1 & 0_{m_1\times m_2} & \ldots & 0_{m_1\times m_t} \\ 0_{m_2\times m_1} & P_2 & \ldots & 0_{m_2\times m_t} \\ \ldots & \ldots & \ldots & \ldots \\ 0_{m_t\times m_1} & 0_{m_t\times m_2} & \ldots & P_t \end{pmatrix},$$

where $P_j = \varphi(\sigma_j)$, j=1,2,…, t, and $0_{m_j\times m_k}$ is a $m_j \times m_k$ matrix having all the entries equal to zero.

As in the first part of the proof it we get the eigenvalues of P_j, j = 1,2,…, t, and then those of P:

$$\tau_{j,k_j} = \cos\frac{2(k_j-1)\pi}{m_j} + i\cdot\sin\frac{2(k_j-1)\pi}{m_j},\ k_j = 1,2,...,m_j,\ j = 1,2,...,t. \tag{3}$$

If $t > 1$, then by (3) we find that P has t eigenvalues equal to 1 and A has t eigenvalues equal to 2, a contradiction because in (2) there is only one eigenvalue equal to 2. Hence it follows that $m = n$ and $A \in$ CyHam(n). □

If $A \in$ CyHam(n), we denote

$$F(A) = \{ X \in \mathrm{Sim}_{(0,1)}(n) : A \le X \} \text{ and } \mathrm{Ham}(n) = \bigcup_{A\in \mathrm{CyHam}(n)} F(A) \tag{4}$$

Then $F(A)$ is a filter (see [6]). If $X \in \mathrm{Sim}_{(0,1)}(n)$ for every i = 1,2, ... ,n we denote $L_i(X) = \sum_{j=1}^{n} x_{ij}$.

Now, we give a sufficient condition on a matrix $X \in \mathrm{Sim}_{(0,1)}(n)$ to belong to Ham*(n)*.

Theorem 2. *Suppose that* $X \in \mathrm{Sim}_{(0,1)}(n)$ *such that for every i, j* $\in \{1,2, ... ,n\}$, $i \neq j$, *with* $x_{ij} = 0$, *it follows that* $L_i(X) + L_j(X) \ge n$. Then $X \in$ Ham(n).

Proof. Suppose that $X \notin$ Ham(n). We denote U(X) = $\{Y \notin$ Ham*(n)* : $X \le Y\}$. Let X' be a maximal element of U*(X)*. Then there exists $x'_{ij} = x_{ij} = 0$, with $i \neq j$. We denote $X'' = X' \vee S(i,j)$. Since X' is a maximal element from U(X) and $X' < X''$, then exists $A = P + P^{-1} \in$ CyHam(n) such that $X'' \in F(A)$. By using Remark 1, $P = \varphi(\sigma)$, where $\sigma = (r_1, r_2, ... ,r_n)$ with $r_1 = i$ and $r_n = j$.

We prove that, for every k = 1, 2, ... , n-1,

$$x'_{i,r_{k+1}} \cdot x'_{r_k,j} = 0 \tag{5}$$

Indeed, if $x'_{i,r_{k'+1}} \cdot x'_{r_{k'},j} = 1$, for a fixed k' , we consider the cycle

$$\sigma' = (i, r_2, ... ,r_{k'}, j, r_{n-1}, r_{n-2}, ..., r_{k'+1}) \in S_n .$$

For $P' = \varphi(\sigma')$ it follows that $A' = P' + P'^{-1} \in$ CyHam(n) and $A' \le X'$. Hence we find $X' \in F(A') \subset$ Ham(n) , a contradiction which implies (5). Then , by using (5) we get

$$L_i(X') + L_j(X') = \sum_{k=1}^{n} x'_{i,k} + \sum_{k=1}^{n} x'_{k,j} = \sum_{k=1}^{n-1} x'_{i,r_{k+1}} + \sum_{k=1}^{n-1} x'_{r_k,j} \le n-1 \tag{6}$$

Since $L_i(X') + L_j(X') \geq L_i(X) + L_j(X) \geq n$, by (6) we find $n - 1 \geq n$, a contradiction which implies $X \in \text{Ham}(n)$. □

3. Applications

Let $G = (V,E)$ be a finite graph having n vertices (see [2], [4] or [5]). The number of edges incident with a vertex $x \in V$ is called the *degree* of x and it is denoted by $\deg(x)$.

A closed path through in a graph using every edge once is called an *Eulerian cycle* and the graph is called an *Eulerian graph*. The following result holds (see, for example, [4], p. 6)

Theorem 3. A *finite graph G with no isolated vertices is Eulerian if and only if it is connected and every vertex has even degree*.

G is called a *Hamiltonian graph* if there is a *Hamiltonian cycle* in *G*, that is a simple closed path that passes through each vertex exactly once. This concept seems similar to Eulerian graph but that is in reality quite different.

Based on Theorem 2 we give a proof of the following result:

Theorem 4. (Ore) Let $G = (V,E)$ *be a simple graph on n vertices such that for all* $x, y \in V$ *with* $xy \notin V$ *it follows that* $\deg(x) + \deg(y) \geq n$. *Then G is a Hamiltonian graph.*

Proof. Let X be the adjacency matrix of G. Since G is a simple graph, that is it has no loops and no two distinct edges have exactly the same pair of ends, it follows that $X \in \text{Sim}_{(0,1)}(n)$. Since the hypotheses of the Theorem 2 are fullfilled it follows that $X \in \text{Ham}(n)$. Hence there exists $A \in \text{CyHam}(n)$ and $X \in F(A)$. But $A \in \text{CyHam}(n)$ if and only if A is the adjacency matrix of a Hamiltonian cycle G'. Since $X \in F(A)$, G' can be considered as a subgraph of G and this implies the theorem. □

As usually by Theorem 4 we obtain the following result:

Corollary (Dirac) *Let* $G = (V, E)$ *be a simple graph on n vertices such that for all* $x \in V$ *it follows that* $\deg(x) \geq \frac{n}{2}$. *Then G is a Hamiltonian graph*.

Remark 2. The result of Dirac is best possible at least in the sens that it does not hold if we replace $\frac{n}{2}$ by $\frac{n-1}{2}$, but it does admits other improvements and generalizations (see [4], p. 40).

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PREDICTING THE EARTHQUAKE MAGNITUDE FROM SEISMICITY INDICATORS USING A PROBABILISTIC NEURAL NETWORK

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Abstract: The aim of the present paper is to introduce a Probabilistic Neural Network (PNN), which will be applied for solving the problem of prediction the earthquake magnitude. This problem is dealt like a classification problem, such that the magnitude range in a future time period is predicted like being one of the output classes. Our proposed model [2] is unlike the classical PNN model [1], which uses the concept of the Parzen windows classifier, by operating with a non-parametric recursive procedure. The recursive estimation of the joint probability density function (PDF) has a net advantage over the non-recursive as it can be completed with additional observations, while the non-recursive has to be completely recalculated.
Mathematics Subject Classification (2010): 68T10, 68T45
Key words: Probabilistic Neural Networks, Seismicity indicator, Earthquake magnitude prediction, Recursive estimation

1. Introduction

Artificial Neural Networks (ANNs) are used in many different application domains in order to solve various information processing problems in a lot of fields such as [1]: image and object recognition, speech recognition, robotics and computer vision, natural language and text processing, signal processing, biometrical engineering and medical diagnosis, neuroscience, optimization and nonlinear programming, construction engineering, structural engineering, transportation engineering, video and audio analysis, computer networking, control, computer security, communication analysis, satellite data and GPS, water resources engineering, reservoir engineering, air traffic control, solar activity, biology and financial forecasting.
Probabilistic Neural Networks (PNNs) represent a class of ANNs, that are crucial for solving some problems like the classification of the: underwater sonar signals radiated by ships, taxonomical and metabolic responses of the human body, text independent speaker identification in natural language processing, class prediction of leukemia and embryonic tumors, prediction of compressive strength of concrete samples.

2. Related work

The problem of prediction the earthquake magnitude could be solved using:

- back-propagation neural networks (Hung & Adeli, 1993);
- radial basis function neural networks (Adeli & Karim, 2000, Liu, Wang & Qiang, 2007, Mayorga & Arriaga, 2007);
- recurrent neural networks (Schaefer & Zimmermann, 2007, Zakeri & Pashazadeh, 2014);
- feed-forward Levenberg-Marquardt back-propagation neural networks (Panakkat & Adeli, 2007; Niksarlioglu & Kulahci, 2013);
- probabilistic neural networks (Adeli & Panakkat, 2009).

The aim of the present paper is to introduce a Probabilistic Neural Network (PNN), which will be applied for solving the problem of prediction the earthquake magnitude.
This problem is dealt like a classification problem, such that the magnitude range in a future time period is predicted like being one of the *M* output classes. The selected class, which corresponds to an input vector will maximize the joint probability density function (PDF) and yield the magnitude range of the largest earthquake in the respective time period.

3. Non-parametric methods for estimation and classification

Unlike the classical PNN model [1], which uses the concept of the Parzen windows classifier, the present PNN operates with a non-parametric recursive procedure. The recursive estimation of the probability density function (PDF) has a net advantage over the non-recursive as it can be completed with additional samples, while the non-recursive has completely to be recalculated.
As the class-conditional densities are usually unknown in practice, they must be learned [6] from the available training patterns. If the form of the class-conditional densities is unknown then we have a nonparametric decision problem. In this case, we must either estimate the density function (e.g., Parzen window approach, recursive method) or directly construct the decision boundary based on the training data (e.g., nearest neighbor rule, *k*- nearest neighbor rule). The most approaches in statistical pattern recognition implements the Bayes decision rule.
The nearest neighbor rule is the suboptimal one, i.e. it doesn't converge to Bayes optimal rule.
The minimizing condition of misclassification probability exactly corresponds to Bayes decision rule.
The Bayes decision rule, in the multiple hypothesis (there are *M* classes $\omega_1,\ldots,\omega_M$), the *a priori* probabilities being known assigns the d dimensional vector X to the class ω_j if:

$$\frac{N_j}{N}\,p(X\mid\omega_j)>\frac{N_k}{N}\,p(X\mid\omega_k),\ (\forall)\,k=\overline{1,M},\ k\neq j,$$

where:

- N_k is the number of samples from the class ω_k;
- $p(X\mid\omega_k)$ means a class- conditional density function, which can be [2] estimated recursively by $\hat{p}(X\mid\omega_k)$.

From [7] can be noticed that $\hat{p}(X\mid\omega_k)$ is an asymptotically estimator without losses, in any continuity point of $p(X\mid\omega_k)$, namely:

$$\mathrm{E}(\hat{p}(X\mid\omega_k))=p(X\mid\omega_k),$$

E being the *mean operator*.

4. PNN architecture

The architecture of the PNN is depicted in the Figure 1.
It consists of:

- ✓ an *input layer*; the number of the neurons from this layer is equal to eight, each of them representing a seismicity indicator (a mathematically computed parameter);
- ✓ a *hidden layer*, known as the *pattern layer*, whose number of neurons is equal to *n*, which represents the number of the available training input vectors;
- ✓ an *output layer*, called the *competition layer*, which has as many neurons as the number of the pre-defined input classes *M*; here are computed the PDFs corresponding to the considered classes and it is selected that class for which we achieve the maximum PDF.

The selected class will yield the magnitude range of the largest earthquake in the respective time period.

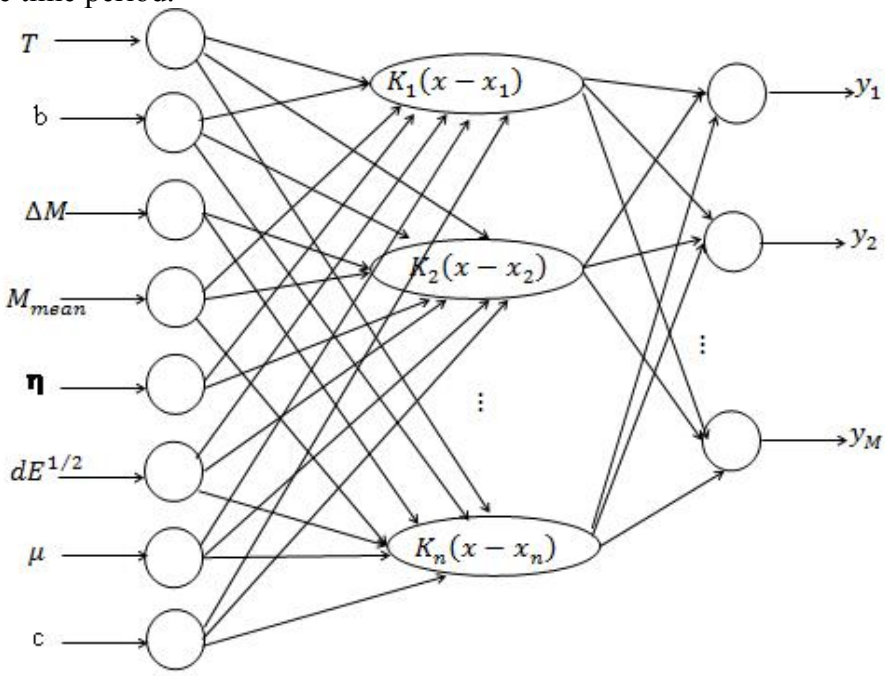

Figure 1: Architecture of the PNN.

Each component of the eight- element input vector constitutes a seismicity indicator and has the following significance [9]:

1) *T*- the time elapsed during a particular number *n* of significant seismic events before the month in question;
2) *b*- the slope of the Gutenberg- Richter inverse power law curve for the *n* events;
3) ΔM - magnitude deficit or the difference between the largest observed magnitude and the largest expected magnitude based on the Gutenberg- Richter relationship;
4) M_{mean} - the average magnitude of the last *n* events;
5) η - the mean square deviation about the regression line based on the Gutenberg- Richter inverse power law for the *n* events;
6) $dE^{1/2}$ - the rate of square root of seismic energy released during the *n* events(during time *T*);
7) μ - the mean time or period between characteristic events;
8) *c*- coefficient of variation of the mean time between characteristic events.

The eight mathematically parameters are computed for each time period and the corresponding input class, on the basis of the magnitude of the largest earthquake that occurred during that time period.

5. Experimental results

The training data set is divided into seven classes depending on the magnitude of the largest earthquake that occurred during each time period.

The Table 1 shows:

- the classes in the training dataset for the Southern California;
- the output magnitude range;
- the number of training instances available in each class.

Table 1: The classes that correspond to each output magnitude range.

Class	Output magnitude range	Number of training instances
Class1	< 4.5	10
Class2	$4.5 - 5.0$	10
Class3	$5.0 - 5.5$	10
Class4	$5.5 - 6.0$	10
Class5	$6.0 - 6.5$	10
Class6	$6.5 - 7.0$	10
Class7	$7.0 - 7.5$	10

Prediction accuracies of the PNN are evaluated using three different statistical measures [1]:

- the probability of detection (POD):

$$\mathrm{POD} = \frac{N_{pc}}{N_{pc} + N_{ni}};$$

- the false alarm ratio(FAR):

$$\mathrm{FAR} = \frac{N_{ni}}{N_{pc} + N_{pi}};$$

- the true skill score or R score:

$$\mathrm{R} = \mathrm{POD} - \mathrm{FAR},$$

where:

- N_{pc} (predicted-corrected) is the number of the time periods during which the magnitude of the largest recorded earthquake falls within the predicted magnitude range;
- N_{ni} (not predicted-incorrect) is the number of time periods during which the magnitude of the largest recorded earthquake is greater the upper limit of the predicted magnitude range;
- N_{pi} (predicted-incorrect) is the number of time periods during which the magnitude of the largest recorded earthquake is less than the lower limit of the predicted magnitude range.

The PNN approach is trained (for the 70 time periods) and tested (in the following 10 time periods) using the data for the Southern California region. The seismic data recorded in the Southern California is archived by the Southern California Earthquake Data Center (SCEC).
We received:

$$N_{pc} = 9;\ N_{ni} = 0;\ N_{pi} = 1;\ \text{hence: } \mathrm{POD} = 1,\ \mathrm{FAR} = 0.1,\ \mathrm{R} = 0.9.$$

6. Conclusion

In contrast with the other neural networks, in a PNN: no learning rule is required, no weights are assigned to links connecting the layers and no predefined convergence criteria are needed.
The drawback of the PNN in prediction problems is that it can be used only if the desired output is expressed as one of several pre-defined classes.
After each test run, the input vector is added to the training dataset; hence, in a PNN approach, the number of the traininig input vectors increases by one with each iteration thereafter, as opposed to a conventional neural network, where the addition of any new training example requires the retraining of the network.

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A COMPUTATIONAL METHOD FOR DESCRIBING THE BEHAVIOR OF THE NATURAL (MAIZE) COMPOSITES

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Abstract. The properties of a composite material made out of stalk-based maize fiber and unsaturated polyester resin polymer as matrix are investigated in this paper by using the Extended Finite Element Method (XFEM) [1]. The systems of equations for both Euler-Bernoulli and the Shear models with damping are written as [2, 3]

$$\rho A\frac{\partial^2 v(x,t)}{\partial t^2}+EI\frac{\partial^4 v(x,t)}{\partial x^4}+\Upsilon=0\,. \tag{1}$$

$$\begin{cases}\rho A\dfrac{\partial^2 v(x,t)}{\partial t^2}-k'GA\left(\dfrac{\partial^2 v(x,t)}{\partial x^2}-\dfrac{\partial \alpha^*(x,t)}{\partial x}\right)+\Upsilon=0,\\ EI\dfrac{\partial^2 \alpha^*(x,t)}{\partial x^2}+k'GA\left(\dfrac{\partial v(x,t)}{\partial x}-\alpha^*(x,t)\right)=0,\end{cases} \tag{2}$$

where E is the longitudinal modulus of elasticity, I is the inertia moment of the section, $v(x,t)$ is the transversal displacement, ρ is the density of the beam, α^* the rotation angle of the cross-section due to the bending moment, and A is the cross sectional area. The term Υ represents the nonlocal damping force defined by

$$\Upsilon=\int_{-\infty}^{t}H(x)g_0\mu\delta(x-\xi)\exp(-\mu(t-\tau))\frac{\partial v(\xi,\tau)}{\partial t}\mathrm{d}\tau\mathrm{d}\xi\,, \tag{3}$$

with μ the relaxation constant and g_0 a constant. The method XFEM enables the accurate approximation of the solutions of (1)-(3) with different initial and boundary conditions, in the case of jumps, discontinuities or general high gradients across interfaces between the fiber and the matrix.
Mathematics Subject Classification (2010): 70Kxx, 74A45, 74A60
Key words: nonlocal theory, composites, cracks, fracture, XFEM.

1. Formulation of the problem

Let us consider a composite beam made out of unidirectional in-line maize fibers embedded into the unsaturated polyester resin polymer as matrix with methyl ethyl ketone peroxide as a catalyst and Cobalt Octoate as a promoter [4, 5]. Such composites are considered by researchers to be biocompatible and so, they are called *green composites*. These composites are lightweight having low density. For example, the maize stalk has the density [g/cm^3] 1.4; the flax and hemp have density [g/cm^3] 1.4; the jute 1.46, coconut 1.25 and cotton 1,51.
The friction coefficient μ depends on the material properties and it is determined experimentally. Thus, for polyester we have $\mu=0.9$-0.95, for cotton/polyester $\mu>1$, for

palm treated with oil/polyester $\mu = 0.45\text{-}0.57$, for palm untreated with oil/polyester $\mu = 0{,}2\text{-}0{,}65$ [6].
The maize fibers are non-porous with no gaps and at the interface with the matrix has a coefficient of friction from 0.51 to 0.67. The density of the resin is 1.12 g/ cm^3. The beam has the length L and a rectangle cross section $a \times b$. The volume fraction of the fibers is $V_f = 20 - 25\%$.
The system of equations (1)-(3) are obtained from the classical form of the Euler-Bernoulli and the Shear equations by including into the equations the nonlocal damping force Υ defined by (3) [7, 8]. This force allows to describing the damping in the behavior of the fiber-based composites. The general form of Υ is given by

$$\Upsilon = \int_{-\infty}^{t} C(x, \xi, t-\tau) v_{,t}(\xi, \tau) \mathrm{d}\tau \mathrm{d}\xi, \tag{4}$$

with $C(x, \xi, t-\tau)$ the kernel damping function. If the kernel function is assumed to be separable in space and time, we can write it in general form

$$C(x, \xi, t-\tau) = H(x) c(x-\xi) g(t-\tau). \tag{5}$$

The (5) represents the general expression of the nonlocal viscoelastic damping kernel. The function $H(x)$ denotes the presence of damping. We have $H(x) = H_0$ (constant) if x is within the fiber, and $H(x) = 0$ otherwise. The spatial function $c(x-\xi)$ is normalized to satisfy the condition

$$\int_{-\infty}^{\infty} c(x) \mathrm{d}x = 1, \tag{6}$$

and can be chosen as the Dirac delta function $\delta(x-\xi)$ to reflect the reacting character of the damping force

$$c(x-\xi) = \delta(x-\xi). \tag{7}$$

In the case of (7), we consider two cases:
(i) viscoelastic damping with the kernel depending on the past time histories

$$C(x, \xi, t-\tau) = H(x) \delta(x-\xi) g(t-\tau), \tag{8}$$

(ii) viscous damping with the force depending only on the instantaneous value of the velocity or strain rate

$$C(x, \xi, t-\tau) = H(x) \delta(x-\xi) \delta(t-\tau). \tag{9}$$

The model (9) represents the well-known viscous damping model. For $g(t-\tau)$ from (8), we consider

$$g(t-\tau) = g_0 \mu \exp(-\mu(t-\tau)), \tag{10}$$

with μ the relaxation constant and g_0 a constant.
The initial conditions attached to (1)-(3) are

$$v(x,0) = v_0(x), \quad \frac{\partial v}{\partial t}(x,t)\,|_{t=0} = v_1(x), \tag{11}$$

where $v_0(x)$ and $v_1(x)$ are the initial displacement and velocity. The boundary conditions are written for a simple supported beam

$$v(x,t)=0,\ \frac{\partial^2 v(x,t)}{\partial x^2}=0 \text{ for } x=0,\quad x=L, \tag{12}$$

2. The extended finite element method (XFEM)

A decade after its introduction (2002), the XFEM has now become the primary numerical approach for analysis of a wide range of discontinuity applications, including behavior and the failure of composites. The idea is to reproduce the singular/discontinuous nature of the field variable.

The level of accuracy with minimum additional degrees of freedom has transformed XFEM into the most efficient computational approach for solving various complex discontinuous problems [9]. A perfect bonding between the fibre and the matrix is necessary to ensure the composite action. Any debonding may affect the strength of the material. It is accepted that the composite material should be designed such that the fibre/matrix debonding never occurs before matrix cracking and delamination. The fibre breakage is the last mode of failure of a composite prior to its collapse. Once the fibres are broken, the strength of the material suddenly drops.

In applications, the fiber reinforced composite structures may be bars or plates, and the fibers are aligned unidirectionally or randomly. For this reason, analysis of these composites requires a calculation method suitable to develop solutions in which the singularities, dislocations, discontinuities in the matrix-fiber interface can be modeled. Such a method that model discontinuities in the continuum mechanics is XFEM developed by Thomas-Peter Fries de la RWTH Aachen University [10-13]. Application of 1D and 2D XFEM can be performed in Matlab routines and can be free downloaded

http://www.xfem.rwth-aachen.de/Seminar/Overview/SEM_Overview.php

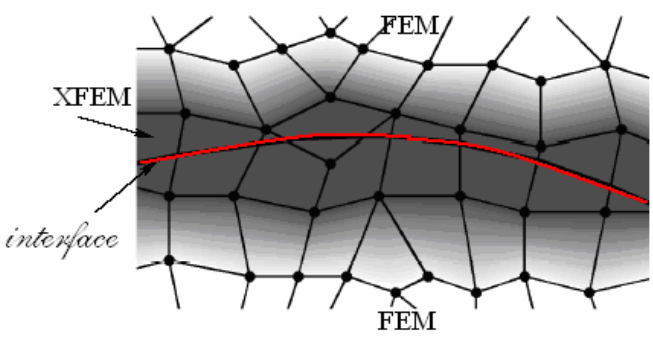

Fig. 1. Decomposition of a domain into FEM and XFEM elements.

To reduce the calculation volume, the new shape functions are used only for the elements in the vicinity of the interface between the fiber and matrix. Decomposition of a domain into FEM and XFEM elements is shown in Fig. 1, where the red line means the interface between the fiber and the matrix. The elements X and the nodes with additional degrees of freedom are shown in Fig. 2. For analysis we used elements XFEM - Q4 of size 0.05 mm. At the interface between the matrix and the fiber we use smaller elements.

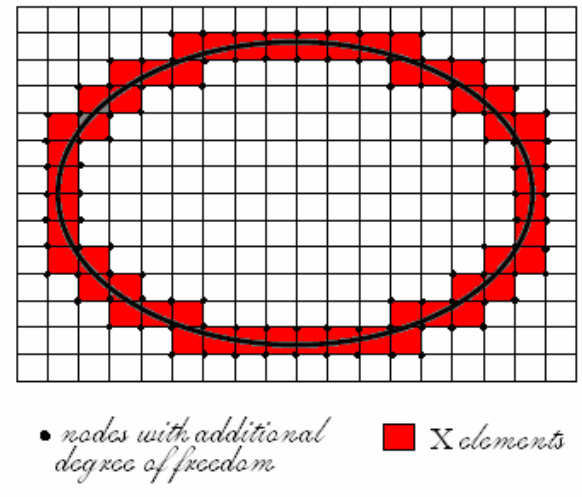

Fig. 2. X elements and nodes with additional DOF.

3. Failure of the beam

Consider now that the beam has an internal crack located at the interface between the matrix and the fiber, specifically in the plane of the cross section. Fig. 3 presents the cross section of the beam and the crack (green-matrix, yellow-fiber). The u_n is the normal displacement and u_t the tangential displacement at the crack tip. The crack is leading through its evolution and propagation to the separation of fibers in the matrix. Both models Euler-Bernoulli and Shear models are used.

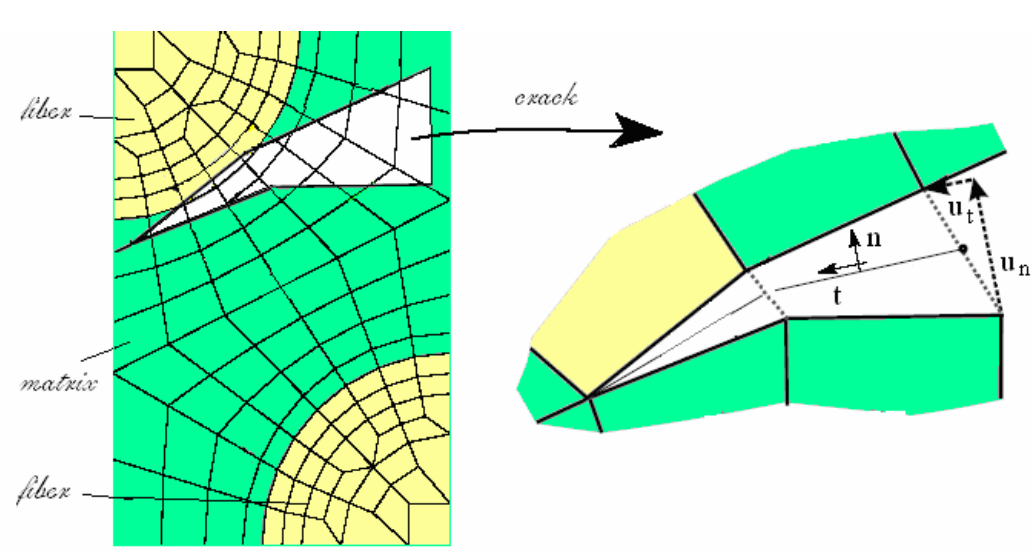

Fig. 3. Graphic illustration of the crack.

We apply the fracture criterion of Puck [14]. Puck assumed that the failure is caused by normal stress σ_n and shear stress τ_{1n} which acts in the plane of fracture. The separation of fibres into matrix can be described in terms of a deterioration parameter $\lambda \in [0,1]$ defined as [13]

$$\lambda = \sqrt{\left(\frac{u_n}{\delta_n^2}\right)^2 + \left(\frac{u_t}{\delta_t^2}\right)^2}\,, \tag{13}$$

where δ_n and δ_t are displacements that characterize the cohesion fiber-matrix. For $\lambda = 0$ the beam is not deteriorated, while for $\lambda = 1$ it appears the separation beyween the fiber and the matrix. The variation of the comprerssive strength with respect to λ is defined as (fig. 4)

$$\sigma(\lambda) = \begin{cases} \dfrac{\sigma_{\max}}{\lambda_1}\lambda, & 0 \le \lambda < \lambda_1 \\ \sigma_{max} & \lambda_1 \le \lambda < \lambda_2 \\ \dfrac{\sigma_{\max}}{1-\lambda_2}(1-\lambda), & \lambda_2 < \lambda \le 1 \end{cases} \tag{14}$$

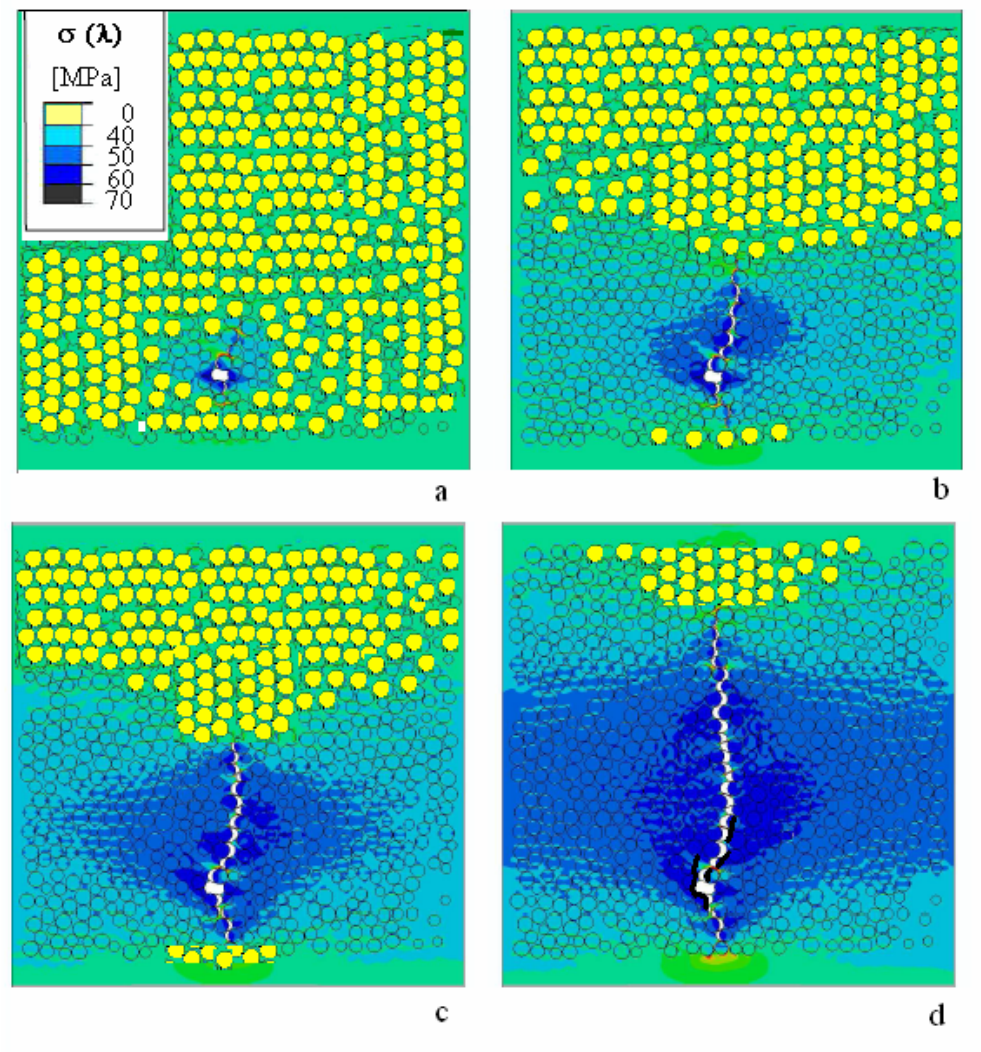

Fig. 4. The propagation of the crack; a) $\lambda = 0,3$; b) $\lambda = 0,4$; c) $\lambda = 0,5$; d) $\lambda = 0,6$.

The propagation of the crack in the cross sectional of the beam is displayed in în Fig. 4 for $0.4 \le \lambda < 0.6$ in the case of Euler-Bernoulli beam. The fibers are yelow, and the crack is white. The stress $\sigma(\lambda) = \sigma_{\max}$ has different values beginning with 40 MPa for $\lambda = 0.3$, (Fig.

4a); 50 MPa for $\lambda = 0.4$ (Fig. 4b) ; 60 MPa for $\lambda = 0.5$ (Fig. 4c) ands up to 70 MPa for $\lambda = 0.6$ (Fig. 4d).

Conclusion

The paper studies the propagation of a crack into a composite beam within XFEM framework. In this approach the crack is located at the interface between the fiber and the matrix. The damping is described by the nonlocal theory for both Euler-Bernoulli and Shear models. The difference in results consists in the compressive strength values for shear model are more closer to the experimental result than those given by the Euler Bernoulli model [15].

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PARTIAL DIFFERENTIAL EQUATIONS NUMERICALLY SOLVED USING TAYLOR EXPANSION

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Abstract: This paper presents a numerical method for solving linear PDE's. The method relies on the Taylor series expansion of analytical functions and transforms the PDE together with the initial / boundary conditions into a system of linear equations. Applications of the method are illustrated by three numerical examples.
Mathematics Subject Classification (2010): 41A10, 65N99
Key words: Taylor series, Initial - boundary-value problems, Solution of partial differential equations

1. Introduction

Partial differential equations are used to model physical phenomena described by functions which depend on two or more independent variables. The numerical method presented here consists in reducing the partial differential equation to a set of linear equations whose unknowns are the coefficients of the Taylor series expansion of the searched function. The method can be applied to obtain a polynomial approximation for the solution of initial - boundary-value problems when the solution is analytic.

2. Mathematical preliminaries

If $\Omega \subset \mathbb{R}^2$ is an open set which contains the origin $(0,0)$ and $u:\Omega \to \mathbb{R}$ is an analytic function on Ω, then it can be expanded in power series as follows:

$$u(x,y) = \sum_{i,j=0}^{\infty} u_{i,j} x^i y^j \,, \tag{1}$$

where $u_{i,j} = \dfrac{1}{i!\,j!}\dfrac{\partial^{i+j}u}{\partial x^i \partial y^j}(0,0)$.

The partial derivatives $\dfrac{\partial^{m+n}u}{\partial x^m \partial y^n} \overset{not}{=} u^{(m,n)}(x,y)$ can be also expanded in power series as:

$$u^{(m,n)}(x,y) = \sum_{i,j=0}^{\infty} u_{i,j}^{(m,n)} x^i y^j \,,\ \forall m,n \geq 0\,, \tag{2}$$

where $u^{(0,0)}(x,y) = u(x,y)$ and $u_{i,j}^{(0,0)} = u_{i,j}$, $\forall i,j \geq 0$.

Theorem 1
With the above notations we have the following recurrence relations:

$$u_{i,j}^{(m+1,n)} = (i+1)u_{i+1,j}^{(m,n)}, \tag{3}$$

$$u_{i,j}^{(m,n+1)} = (j+1)u_{i,j+1}^{(m,n)}, \tag{4}$$

for any $i, j, m, n \geq 0$.

Let us approximate $u(x,y)$ by the Taylor polynomial of degree N, $T_N(x,y)$. Consequently, we take $u_{i,j}=0$ for all i, j s.t. $i+j>N$, so we can write:

$$u(x,y) \approx \bar{u}(x,y) = T_N(x,y) = X^T A Y, \tag{5}$$

where $X=\left(1,x,x^2,\ldots,x^N\right)^T$, $Y=\left(1,y,y^2,\ldots,y^N\right)^T$ and $A=\left(u_{i,j}\right)_{i,j=\overline{0,N}}$. Note that A is a triangular matrix.

The partial derivatives of *u* can be approximated by

$$u^{(m,n)}(x,y) \approx \bar{u}^{(m,n)}(x,y) = X^T A^{(m,n)} Y, \tag{6}$$

where $A^{(m,n)}=\left(u_{i,j}^{(m,n)}\right)_{i,j=\overline{0,N}}$ and $A^{(0,0)}=A$.

By using the recurrence relations from Theorem 1, we obtain that:

$$A^{(m,n)} = C^m A (C^n)^T, \tag{7}$$

where *C* is the matrix below:

$$C=\begin{pmatrix} 0 & 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 2 & 0 & \ldots & 0 \\ 0 & 0 & 0 & 3 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & N \\ 0 & 0 & 0 & 0 & \ldots & 0 \end{pmatrix}, \tag{8}$$

For instance, let us write the matrices used in the approximation of the second order partial derivatives: $u^{(2,0)}(x,y) \approx \bar{u}^{(2,0)}(x,y) = X^T A^{(2,0)} Y$, and $u^{(0,2)}(x,y) \approx \bar{u}^{(0,2)}(x,y) = X^T A^{(0,2)} Y$:

$$A^{(2,0)} = C^2 A = \begin{pmatrix} 2u_{2,0} & 2u_{2,1} & 2u_{2,2} & \ldots & 2u_{2,N-2} & 0 & 0 \\ 6u_{3,0} & 6u_{3,1} & 6u_{3,2} & \ldots & 0 & 0 & 0 \\ 12u_{4,0} & 12u_{41} & 12u_{4,2} & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ N(N-1)u_{N,0} & 0 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 0 & 0 & \ldots & 0 & 0 & 0 \end{pmatrix}, \tag{9}$$

$$A^{(0,2)} = A(C^2)^T = \begin{pmatrix} 2u_{0,2} & 6u_{0,3} & 12u_{0,4} & \ldots & N(N-1)u_{0,N} & 0 & 0 \\ 2u_{1,2} & 6u_{1,3} & 12u_{1,4} & \ldots & 0 & 0 & 0 \\ 2u_{2,2} & 6u_{2,3} & 12u_{2,4} & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 2u_{N-2,2} & 0 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 0 & 0 & \ldots & 0 & 0 & 0 \end{pmatrix}, \tag{10}$$

Consider the following initial-value problem for linear partial differential equations with constant coefficients:

$$A\frac{\partial^2 u}{\partial x^2}+B\frac{\partial^2 u}{\partial x\partial y}+C\frac{\partial^2 u}{\partial y^2}+D\frac{\partial u}{\partial x}+E\frac{\partial u}{\partial y}+Fu=f(x,y), \tag{11}$$

$$u(x,0)=g(x), \tag{12}$$

$$\frac{\partial u}{\partial y}(x,0)=h(x), \tag{13}$$

Suppose that

$$f(x,y)=\sum_{i,j=0}^{\infty} f_{i,j}x^i y^j,\ g(x)=\sum_{i=0}^{\infty} g_i x^i,\ h(x)=\sum_{i=0}^{\infty} h_i x^i, \tag{14}$$

are analytic functions. Then, from Cauchy-Kovalevsky Theorem, it follows that the above initial-value problem has a unique local analytic solution [2].

Let us use the Taylor polynomials (around the origin) to approximate the unknown function $u(x,y)$ (see relation (5)):

$$u(x,y)\approx T_N(x,y)=\sum_{i=1}^{N}\sum_{j=1}^{N-i} u_{i,j}x^i y^j$$

and the known functions $f(x,y)$, $g(x)$, $h(x)$:

$$f(x,y)\approx\sum_{i=1}^{M}\sum_{j=1}^{M-i} f_{i,j}x^i y^j,\ g(x)\approx\sum_{i=0}^{N} g_i x^i,\ h(x)\approx\sum_{i=0}^{N} h_i x^i, \tag{15}$$

where $M\le N$ and $f_{i,j}=\frac{1}{i!\,j!}\cdot\frac{\partial^{i+j}f}{\partial x^i\partial y^j}(0,0)$, $g_i=\frac{g'(0)}{i!}$, $h_i=\frac{h'(0)}{i!}$.

By replacing in the partial differential equation (11) and in the initial conditions (12) and (13), we obtain a system of linear equations whose unknowns are the coefficients $u_{i,j}$.

3. Numerical examples

3.1. Example 1

Consider the following hyperbolic equation

$$\frac{\partial^2 u}{\partial y^2}-\frac{\partial^2 u}{\partial x^2}=2x-2,\quad (x,y)\in\mathbb{R}^2, \tag{16}$$

with the initial conditions:

$$u(x,0)=x^2, x\in\mathbb{R}, \tag{17}$$

$$\frac{\partial u}{\partial y}(x,0)=4x+1, x\in\mathbb{R}, \tag{18}$$

We search for $u(x,y)\approx T_3(x,y)=X^T AY$, where $X=\left(1,x,x^2,x^3\right)^T$, $Y=\left(1,y,y^2,y^3\right)^T$ and $A=(u_{i,j})_{i,j=\overline{0,3}}$ is the matrix of the coefficients ($u_{i,j}=0$ for $i+j>3$). Since $\frac{\partial^2 u}{\partial x^2}=X^T A^{(2,0)}Y$ and $\frac{\partial^2 u}{\partial y^2}=X^T A^{(0,2)}Y$, the equation (16) can be written:

$$X^T\left(A^{(0,2)}-A^{(2,0)}\right)Y=2x-2, \tag{19}$$

and we obtain the linear equations:

$$2u_{0,2}-2u_{2,0}=-2;\ 2u_{1,2}-6u_{3,0}=2;\ 6u_{0,3}-2u_{2,1}=0$$

Now, from the initial conditions (17), (18) we have:

$$u(x,0)=u_{0,0}+u_{1,0}x+u_{2,0}x^2+u_{3,0}x^3=x^2\Rightarrow u_{0,0}=0,u_{1,0}=0,u_{2,0}=1,u_{3,0}=0$$

$\frac{\partial u}{\partial y}(x,0) = u_{0,1} + u_{1,1}x + u_{2,1}x^2 = 4x+1 \Rightarrow u_{0,1} = 1, u_{1,1} = 4, u_{2,1} = 0$. We obtain that

$$u(x,y) = y + x^2 + 4xy + xy^2,$$

which is the exact solution of the problem.

3.2. Example 2

Consider the Laplace equation on the rectangle $[0,2]\times[0,1]$:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,\ (x,y) \in (0,2)\times(0,1), \tag{20}$$

with the boundary conditions:

$$u(x,0) = x, u(x,1) = x+1,\ x \in [0,2], \tag{21}$$

$$u(0,y) = y, u(2,y) = y+2,\ y \in [0,1], \tag{22}$$

We approximate the solution by the Taylor polynomial of the second degree:

$$u(x,y) \approx T_2(x,y) = u_{0,0} + u_{1,0}x + u_{0,1}y + u_{2,0}x^2 + u_{1,1}xy + u_{0,2}y^2.$$

From the PDE we find:

$$2u_{2,0} + 2u_{0,2} = 0$$

and from the boundary conditions we obtain

$$u_{0,0} = 0, u_{1,0} = 1, u_{0,1} = 1, u_{2,0} = 0, u_{1,1} = 0, u_{0,2} = 0,$$

so $u(x,y) = x + y$, which is the exact solution of the problem.

Note that the "classic" method of solving the above Dirichlet problem is to search for the function *u* as a sum of two functions, $u = v + w$, where *v* is the solution of the problem (23) and *w* is the solution of the problem (24):

(23)
$$\begin{cases} \Delta v = 0,\ (x,y) \in (0,2)\times(0,1) \\ v(x,0) = x,\ x \in [0,2] \\ v(x,1) = x+1,\ x \in [0,2] \\ v(0,y) = v(2,y) = 0,\ y \in [0,1] \end{cases}$$

(24)
$$\begin{cases} \Delta w = 0,\ (x,y) \in (0,2)\times(0,1) \\ w(0,y) = y,\ y \in [0,1] \\ w(2,y) = y+2,\ y \in [0,1] \\ w(x,0) = w(x,1) = 0,\ x \in [0,2] \end{cases}$$

Solving each problem by the Fourier method of variable separation we obtain that

$$v(x,y) = \sum_{n=1}^{\infty} \frac{\sin\frac{n\pi x}{2}}{n\pi \sinh\frac{n\pi}{2}} \left(4(-1)^{n-1} \sinh\frac{n\pi(1-y)}{2} + \left(6(-1)^{n-1} + 2\right)\sinh\frac{n\pi y}{2} \right), \tag{25}$$

$$w(x,y) = \sum_{n=1}^{\infty} \frac{\sin n\pi y}{n\pi \sinh 2n\pi} \left(2(-1)^{n-1} \sinh n\pi(2-x) + \left(6(-1)^{n-1} + 4\right)\sinh n\pi x\right), \tag{26}$$

so the searched function is the sum of (25) and (26): $u(x,y) = v(x,y) + w(x,y)$.

3.3. Example 3 [3]

Let us find a polynomial approximation for the solution of the following elliptic equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -2\sin x \sin y,\ (x,y) \in \left(0,\frac{\pi}{6}\right)\times\left(0,\frac{\pi}{6}\right), \tag{27}$$

with the boundary conditions:

$$u(x,0) = 0, \quad u\left(x,\frac{\pi}{6}\right) = \frac{e^{-x} + \sin x}{2}, \quad x \in \left[0,\frac{\pi}{6}\right], \tag{28}$$

$$u(0,y)=\sin y,\quad u\left(\frac{\pi}{6},y\right)=\left(e^{-\frac{\pi}{6}}+\frac{1}{2}\right)\sin y,\quad y\in\left[0,\frac{\pi}{6}\right], \tag{29}$$

We approximate the solution $u(x,y)$ by the corresponding Taylor polynomial of the 6th degree:

$$u(x,y)\approx\bar{u}(x,y)=T_6(x,y)=X^TAY, \tag{30}$$

and the function $f(x,y)=-2\sin x\sin y$ by the Taylor polynomial of the 4th degree:

$$f(x,y)\approx -2xy+\frac{1}{3}x^3y+\frac{1}{3}xy^3,$$

and from the PDE (27) we obtain:

$$X^T\left(A^{(2,0)}+A^{(0,2)}\right)Y=f(x,y)\approx 2xy+\frac{1}{3}x^3y+\frac{1}{3}xy^3, \text{ or:}$$

$$\begin{pmatrix} 2u_{2,0} & 2u_{2,1} & 2u_{2,2} & 2u_{2,3} & 2u_{2,4}\\ 6u_{3,0} & 6u_{3,1} & 6u_{3,2} & 6u_{3,3} & 0\\ 12u_{4,0} & 12u_{4,1} & 12u_{4,2} & 0 & 0\\ 20u_{5,0} & 20u_{5,1} & 0 & 0 & 0\\ 30u_{6,0} & 0 & 0 & 0 & 0 \end{pmatrix}+\begin{pmatrix} 2u_{0,2} & 6u_{0,3} & 12u_{0,4} & 20u_{0,5} & 30u_{0,6}\\ 2u_{1,2} & 6u_{1,3} & 12u_{1,4} & 20u_{1,5} & 0\\ 2u_{2,2} & 6u_{2,3} & 12u_{2,4} & 0 & 0\\ 2u_{3,2} & 6u_{3,3} & 0 & 0 & 0\\ 2u_{4,2} & 0 & 0 & 0 & 0 \end{pmatrix}=\begin{pmatrix} 0 & 0 & 0 & 0 & 0\\ 0 & -2 & 0 & 1/3 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 1/3 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

From the first boundary condition, $u(x,0)=0$, we obtain that

$$u_{i,0}=0,\quad i=\overline{0,6}.$$

In order to use the other boundary conditions we approximate the functions by the 6th degree Taylor polynomials:

$$u(0,y)=\sin y=y-\frac{1}{3!}y^3+\frac{1}{5!}y^5\Rightarrow u_{0,1}=1,u_{0,2}=0,u_{0,3}=-\frac{1}{6},u_{0,4}=0,u_{0,5}=\frac{1}{120},u_{0,6}=0.$$

$$u\left(x,\frac{\pi}{6}\right)=\frac{e^{-x}+\sin x}{2}\approx\frac{1}{2}+\frac{1}{4}x^2-\frac{1}{6}x^3+\frac{1}{48}x^4+\frac{1}{1440}x^6$$

$$u\left(\frac{\pi}{6},y\right)=\left(e^{-\frac{\pi}{6}}+\frac{1}{2}\right)\sin y\approx\left(e^{-\frac{\pi}{6}}+\frac{1}{2}\right)\left(y-\frac{1}{3!}y^3+\frac{1}{5!}y^5\right).$$

We obtain a system of linear equations whose unknowns are the coefficients $u_{i,j}$. So, we find the approximate solution:

$$\bar{u}(x,y)=y-0.000341xy+\frac{1}{2}x^2y-\frac{1}{6}y^3-0.333541x^3y+0.000207xy^3+$$
$$+\frac{1}{24}x^4y-\frac{1}{12}x^2y^3+\frac{1}{120}y^5+\frac{1}{18}x^3y^3.$$

The exact solution of the problem is:

$$u(x,y)=(e^{-x}+\sin x)\sin y$$

and one can see that most of the coefficients of $\bar{u}(x,y)$ correspond to the coefficients of the 6th degree Taylor polynomial of the exact solution $u(x,y)$:

$$T_6(x,y)=y+\frac{1}{2}x^2y-\frac{1}{6}y^3-\frac{1}{3}x^3y+\frac{1}{24}x^4y-\frac{1}{12}x^2y^3+\frac{1}{120}y^5+\frac{1}{18}x^3y^3.$$

The absolute error is:

$$\sup_{x,y\in\left[0,\frac{\pi}{6}\right]}\left|u(x,y)-\bar{u}(x,y)\right|\le 5\cdot 10^{-5}.$$

4. Conclusion

The Taylor series are used to solve initial – boundary-value problems for linear partial differential equations with two independent variables. The solution of partial differential equations is reduced to the problem of solving a system of linear equations. Numerical examples are given to show the validity of the method. In the first two examples, the exact (polynomial) solution was found, while in the last example, a polynomial approximation of the exact solution was created. The method can be extended to PDEs with three or more independent variables.

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APPLICATIONS OF SAWYER'S DUALITY PRINCIPLE

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Abstract: In this paper, using Sawyer's duality principle we study some properties of a class of linear operators defined on the cone of all sequences from ℓ^p with the property $|x_n| \cdot a_n \downarrow 0$, for $1 \leq p < \infty$ and $a = (a_n)_n$ a positive weight.
Mathematics Subject Classification (2010): 15A60, 47B35, 26D15.
Key words: Infinite matrices, Toeplitz matrices, Schur multipliers, discrete Sawyer duality principle.

1. Introduction

Sawyer's formula, also known as Sawyer's duality principle was first formulated by Sawyer in [14]. It is known Hölder's inequality and its converse in the form

$$\sup_{f \text{ nonnegative}} \frac{\int_0^\infty f(x)g(x)dx}{\left(\int_0^\infty f(x)^p v(x)dx\right)^{1/p}} = \left(\int_0^\infty g(x)^{p'} v(x)^{1-p'}\right)^{1/p'} \tag{1}$$

for all $v, g \geq 0$, where $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$. In [14], Sawyer obtained an analogue of (1) when the supremum is restricted to nonnegative and nonincreasing functions f. Then by duality he obtained a characterization of the inequality

$$\left(\int_0^\infty \left(\frac{1}{x}\int_0^\infty f(t)dt\right)^q w(x)dx\right)^{1/q} \leq C\left(\int_0^\infty f(x)^p v(x)dx\right)^{1/p}$$

for nonnegative and nonincreasing f on $(0,\infty)$, where v, w are nonnegative functions on $(0,\infty)$.

Sawyer's formula provides a simple and explicit equivalent norm for the dual space of classical Lorentz space $\Lambda^p(v)$, $1 < p < \infty$ which consists of those measurable functions on $\mathbb{R}^n$ such that $\left(\int_0^\infty f^*(x)^p v(x)dx\right)^{1/p} < \infty$, where $f^*(x) = \inf\{\lambda : |\{t \in \mathbb{R}^n : |f(t)| > \lambda\}| \leq x\}$ is the nonincreasing rearrangement of f on $(0,\infty)$ with respect to Lebesgue measure on $\mathbb{R}^n$. Another important applications of Sawyer's formula are some inclusions between the Lorentz spaces (see e.g. [8]). The discrete variant of this principle may also be useful for (discrete) inequalities.

In this paper we apply the duality formula in the study of a class of Banach spaces of infinite matrices. We introduce this space motivated by the previous papers [2], [3], [5], [10], [11], [12] and the book [13]. On the other hand, we are interested in this space for the potential applications.

In the paper [10], the authors introduced the space $B_w(\ell^2)$ consisting of infinite

matrices A such that $A(x) \in \ell^2$ for every $x = (x_n)_n \in \ell^2$ with $|x_n| \downarrow 0$. This space can be considered as a "weak" variant of the classical space $B(\ell^2)$. Of course the spaces $B_w(\ell^2)$ and $B(\ell^2)$ does not coincide in general. This fact can be remarked if we consider the "diagonal" matrices. One interesting observation is that the "diagonal" matrices from the space $B_w(\ell^p)$ can be characterized in nice terms of some sequence spaces (for details see [4] and [10]). In fact, this space was inspired by the study of the matricial analogue of Fejer's theory ([1], [3] and [2]).

Let $p \geq 1$ and $a = (a_n)_n$ be a positive weight. We define

$$B_w^a(\ell^p) = \{A \text{ infinite matrix}; Ax \in \ell^p \text{ for every } x = (x_n)_n \in \ell^p \text{ such that } |x_n| \cdot a_n \downarrow 0\}.$$

Then $B_w^a(\ell^p) = B_w(\ell^p)$, when $a_n = 1$ for every $n \geq 1$. For $p = 2$ this space was studied in [12]. For $a = (1,1,\ldots)$ the space of infinite matrices was studied in [10] and [11].

In the following we recall some basic definitions. We define the Schur product of two matrices as

$$A * B = \left(a_{ij} \cdot b_{ij}\right)_{i,j \geq 1},$$

where $A = \left(a_{ij}\right)_{i,j\geq 1}, B = \left(b_{ij}\right)_{i,j\geq 1}$.

We denote by

$$M\left(\ell^2\right) = \left\{M : M * A \in B\left(\ell^2\right) \text{for every } A \in B\left(\ell^2\right)\right\}$$

the space of Schur multipliers which is a Banach space with the norm

$$\left\|M\right\| = \sup_{\|A\|_{B(\ell^2)} \leq 1} \left\|M * A\right\|_{B(\ell^2)}.$$

More generally, if X and Y are two linear spaces of infinite matrices, the space of *Schur multipliers* from X to Y is the space

$$(X, Y) := \{M : M * A \in Y \text{ for every } A \in X\}.$$

If X and Y are Banach spaces, then we consider on the space (X, Y) the natural norm

$$\left\|M\right\|_{(X,Y)} := \sup_{\|A\|_X \leq 1} \left\|M * A\right\|_Y.$$

For an infinite matrix $A = (a_{ij})$ and an integer k we denote by $A_k = (a'_{ij})$, where

$$a'_{ij} = \begin{cases} a_{ij} & \text{if } j - i = k, \\ 0 & \text{otherwise.} \end{cases}$$

A_k is called Fourier coefficient of kth order associated to matrix A (see e.g. [3] and [2]).

The paper is organized as follows. In Section 2 we present a duality formula which is an important tool for our proofs. In Secton 3, the main result is a characterization of diagonal matrices from $B_w^a(\ell^p)$. We also state some remarks about Schur multipliers.

2. A duality formula

In this Section we present a duality formula due to E. Sawyer in [14]. Other proofs can be found in [6] and [15] (see also [9]). For the convenience of the reader we present here both, the result in the case of functions and of sequences (see [14] and [7]).

Let w be a weight function in $\mathbb{R}_+$ (nonnegative locally integrable function). For a given weight w we denote by

$$W(r)=\int_0^r w(t)dt<\infty,\quad 0\le r<\infty.$$

Theorem 2.1 *If* $1<p<\infty$ *we have that*

$$\sup_{f\downarrow}\frac{\|f\|_{L^1(w_1)}}{\|f\|_{L^p(w_0)}}\approx\left(\int_0^\infty\left(\frac{W_1(t)}{W_0(t)}\right)^{p'-1}w_1(t)dt\right)^{1/p'}$$

$$\approx\left(\int_0^\infty\left(\frac{W_1(t)}{W_0(t)}\right)^{p'}w_0(t)dt\right)^{1/p'}+\frac{W_1(\infty)}{W_0^{1/p}(\infty)},$$

where $W(\infty)=\int_0^\infty w(t)dt$.

Next we state the discrete result.

Theorem 2.2 *Let* $w=(w(n))_n$, $v=(v(n))_n$ *be two weights in* N^* *and let*

$$S=\sup_{f\,]0}\frac{\sum_{n=0}^\infty f(n)v(n)}{(\sum_{n=0}^\infty f(n)^p w(n))^{\frac{1}{p}}}.$$

(i) If $0<p\le 1$, *then*

$$S=\sup_{n\ge 0}\frac{V(n)}{W^{\frac{1}{p}}(n)},$$

with W *defined by* $W(n)=\sum_{k=0}^n w(k)$, $n=0,1,2,...$ *and* V *defined in the same way.*

(ii) *If* $1<p<\infty$, *then*

$$S\approx\left(\int_0^\infty\left(\frac{\widetilde{V}(t)}{\widetilde{W}(t)}\right)^{p'-1}\widetilde{v}(t)dt\right)^{\frac{1}{p'}}$$

$$\approx\left(\int_0^\infty\left(\frac{\widetilde{V}(t)}{\widetilde{W}(t)}\right)^{p'}\widetilde{w}(t)dt\right)^{\frac{1}{p'}}+\frac{\widetilde{V}(\infty)}{\widetilde{W}^{\frac{1}{p}}(\infty)},$$

where $\widetilde{v}$ *is a weight in* R_+ *defined by*

$$\widetilde{v}=\sum_{n=0}^\infty v(n)\chi_{[n,n+1)}$$

and $\widetilde{V}(t)=\int_0^t\widetilde{v}(s)ds$. *In the same way are defined* $\widetilde{w}$ *and* $\widetilde{W}$.

Moreover, the implicit constant in the symbol $\approx$ *depends only on* p .

2. An application

In the following we state and prove the main result of this paper.

Theorem 3.1 *Let* $a=(a_n)_n$ *be a strictly positive weight,* $1\le p<\infty$ *and let the matrix* $M=M_0$ *be a diagonal matrix defined by the sequence* $m=(m_n)_n$,

$$M = \begin{pmatrix} m_1 & 0 & 0 & \cdots \\ 0 & m_2 & 0 & \ddots \\ 0 & 0 & m_3 & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix}.$$

Then we have that

$$M \in B_w^a(\ell^p) \text{ if and only if } \sup_{n\geq 1} \frac{\sum_{k=1}^{n} |m_k|^p v_k^p}{\sum_{k=1}^{n} v_k^p} < \infty,$$

where $v_n = \frac{1}{a_n}$ *for* $n = 1,2,\ldots$. *Moreover the norm is*

$$\|M\|_{B_w^a(\ell^p)} = \sup_{n\geq 1} \left(\frac{\sum_{k=1}^{n} |m_k|^p v_k^p}{\sum_{k=1}^{n} v_k^p} \right)^{\frac{1}{p}}.$$

Proof. We evaluate the term $\sup_{|x_n|\cdot a_n\downarrow} \frac{\|Mx\|_{\ell^p}}{\|x\|_{\ell^p}}$. Then we have that

$$\sup_{|x_n|\cdot a_n\downarrow} \frac{\|Mx\|_{\ell^p}}{\|x\|_{\ell^p}} = \sup_{|x_n|\cdot a_n\downarrow} \frac{\left(\sum_{n=1}^{\infty} |m_n|^p |x_n|^p\right)^{\frac{1}{p}}}{\left(\sum_{n=1}^{\infty} |x_n|^p\right)^{\frac{1}{p}}}$$

$$= \sup_{|y_n|\downarrow} \frac{\left(\sum_{n=1}^{\infty} |m_n y_n v_n|^p\right)^{\frac{1}{p}}}{\left(\sum_{n=1}^{\infty} |y_n v_n|^p\right)^{\frac{1}{p}}}$$

where $y_n = \frac{x_n}{v_n} = x_n \cdot a_n$, for all $n \geq 1$.

We denote by

$$S := \sup_{|y_n|\downarrow} \frac{\left(\sum_{n=1}^{\infty} |m_n y_n v_n|^p\right)^{\frac{1}{p}}}{\left(\sum_{n=1}^{\infty} |y_n v_n|^p\right)^{\frac{1}{p}}}.$$

Applying Theorem 2.2 (i), with $f(n) = |y_n|^p$, $v(n) = v_n^p |m_n|^p$, $w(n) = v_n^p$ we obtain for every n,

$$V(n) = \sum_{k=1}^{n} v(k) = \sum_{k=1}^{n} v_k^p \mid m_k \mid^p$$

and

$$W(n) = \sum_{k=1}^{n} w(k) = \sum_{k=1}^{n} v_k^p .$$

It follows that

$$S = \sup_{n\geq 1} \left(\frac{\sum_{k=1}^{n} v_k^p \mid m_k \mid^p}{\sum_{k=1}^{n} v_k^p} \right)^{\frac{1}{p}}$$

and

$$\left\| M \right\|_{B_w^a(\ell^p)} = \sup_{n\geq 1} \left(\frac{\sum_{k=1}^{n} v_k^p \mid m_k \mid^p}{\sum_{k=1}^{n} v_k^p} \right)^{\frac{1}{p}} .$$

The proof is complete.

Remark 3.2 *In the case of* $B_w(\ell^2)$ *we know that this space coincides with* $B(\ell^2)$ *in the case of Toeplitz matrices. We conjecture that our space, introduced in this paper, coincides with* $B(\ell^p)$, $1 \leq p < \infty$ *at least for suitable choice of the weight in the case of Toeplitz matrices. Moreover, if we consider the the Schur multipliers we conjecture that*

$$M(\ell^p) = (B_w^a(\ell^p), B_w^a(\ell^p)).$$

This fact it is still not know in particular cases when $p = 2$ *and* $a_n = 1$, $n = 1,2,\ldots$.

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BOUNDEDNESS ON SOME WEIGHTED SPACES

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Abstract: In this paper we study the boundedness of infinite matrices on some weighted spaces.
Mathematics Subject Classification (2010): 47B35, 47B99
Key words: infinite matrices, Schur multipliers, Toeplitz matrices

1. Introduction

Let $w=(w_n)_n$ be a sequence of nonnegative numbers. We denote by $B_w(\ell^2(w),\ell^2)$ the space of all infinite matrices defined on all sequences from $\ell^2(w)$ with the property that $|x_n|\downarrow 0$ to ℓ^2. Let $A=\left(a_{ij}\right)_{i,j\geq 1}$ be an infinite matrix. We recall the definition of

$$B_w\left(\ell^2\right)=\left\{A\,\text{infinite matrix}: Ax\in\ell^2 \text{for every } x\in\ell^2 \text{with } \left|x_k\right|\downarrow 0\right\},$$

where

$$\ell^2=\left\{x=\left(x_k\right)_k:\sum_{k=0}^{\infty}\left|x_k\right|^2<\infty\right\}$$

is the classical Hilbert space of sequences and

$$\ell^2(w)=\left\{x=\left(x_k\right)_k:\sum_{k=0}^{\infty}\left|x_k\right|^2 w_n<\infty\right\}.$$

The above space of matrices has appeared in the study of the matriceal analogue of some well-known Banach spaces as $C(\mathsf{T})$, $M(\mathsf{T})$, $L^1(\mathsf{T})$ (see [3], [2], [4] and [7]).

A similarity between the functions defined on T and the infinite matrices was remarked for the first time in 1978 by Arazy [1] and exploited later on by Shields starting with a few constructs used in harmonic analysis together with their matriceal analogues [9]. In [2], [3], [4] (see also the book [8]) the Fejer's theory developed for Fourier series was extended in the framework of matrices.

In this analogy we identify a function f with the Toeplitz matrix $A=\left(a_{ij}\right)_{i,j\geq 1}$,

$$a_{ij}=a_{i-j} \text{ for all } i,j\in\mathsf{N}^*$$

where $(a_k)_{k\in\mathsf{Z}}$ is the sequence of Fourier coefficients of the function f.

The *Schur product* (sometimes called Hadamard product) of two infinite matrices is defined by

$$A*B=\left(a_{ij}\cdot b_{ij}\right)_{i,j\geq 1},$$

where $A=\left(a_{ij}\right)_{i,j\geq 1}, B=\left(b_{ij}\right)_{i,j\geq 1}$.

Following [5], we denote by

$$M\left(l^2\right)=\left\{M: M*A\in B\left(l^2\right)\text{for every } A\in B\left(l^2\right)\right\}$$

the *space of all Schur multipliers* equiped with the following norm

$$\|M\| = \sup_{\|A\|_{B(l^2)} \le 1} \|M * A\|_{B(l^2)}.$$

Let us recall the definition of "diagonal" matrices, $A_k = (a_{ij}')_{i,j\ge 1}$, $k \in Z$ to be the matrix with the elements

$$a_{ij}' = \begin{cases} a_{ij} & \text{if } j - i = k \\ 0 & \text{otherwise} \end{cases}.$$

A_k is called the *Fourier coefficient of* k *-order of the matrix* A (see e.g. [8]).

2. Two spaces of sequences

In this Section we recall the well known theorem of "factorization" due to Bennett [6]. First, we have to define two spaces of sequences.

Let $a = (a_1, a_2, \ldots)$ be a fixed sequence with nonnegative terms. We suppose that $a_1 > 0$. Then the partial sum

$$A_n = a_1 + \cdots + a_n$$

never vanish. For $p > 0$ we define

$$d(a, p) = \{x : \sum_{n=1}^{\infty} a_n \sup_{k\ge n} |x_k|^p < \infty\}$$

and

$$g(a, p) = \{x : \sum_{k=1}^{n} |x_k|^p = O(A_n)\}.$$

These spaces are BK-spaces if $p \ge 1$. In this case the norms are:

$$\|x\|_{d(a,p)} = \left(\sum_{n=1}^{\infty} a_n \sup_{k\ge n} |x_k|^p \right)^{1/p}$$

and

$$\|x\|_{g(a,p)} = \sup_n \left(\frac{1}{A_n} \sum_{k=1}^{n} |x_k|^p \right)^{1/p}.$$

We observe that $d(a, p)$ is a "small" space and $g(a, p)$ is a "large" space. When $A_n \to \infty$ then $g(a, p)$ contains unbounded sequences. Next theorem, due to Bennett ([6], p. 9, Th. 3.8) shows us that a perfect balance exists between the two spaces of sequences.

Theorem 1 *If* $0 < p \le \infty$, *then*

$$d(a, p) \cdot g(a, p) = \ell^p.$$

More precisely, given any sequence x, *we have*

$$\inf\{\|y\|_{d(a,p)} \|z\|_{g(a,p)} : y \cdot z = x\} = \|x\|_p.$$

3. Main results

The first result characterize the diagonal matrices from $B_w(\ell^2(w), \ell^2)$.

Theorem 2 *Let* A *be an infinite matrix of the form*

$$B = \begin{pmatrix} b_1 & 0 & 0 & \cdots \\ 0 & b_2 & 0 & \cdots \\ 0 & 0 & b_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Then $B \in B_w(\ell^2(w), \ell^2)$ *if and only if* $b = (b_n)_n \in g(w,2)$. *Moreover*

$$\|B\|_{B_w(\ell^2(w),\ell^2)} = \sup_n \left(\frac{1}{W_n} \sum_{k=1}^{n} |b_k|^2 \right)^{1/2}.$$

Proof. The proof of the theorem is based on the "factorization" theorem presented in the above Section. Let us take a sequence $x = (x_n)_n \in \ell^2(w)$ such that $|x_n| \downarrow 0$ and B, a matrix from $B_w(\ell^2(w), \ell^2)$. Using the "factorization" with suitable choice of weights we obtain the desired result. The proof is complete.

If we translate this sequence $b = (b_k)_k$ above or below of the main diagonal we obtain the following similar result.

Corollary 3 *a) Let* $k > 0$ *and* $B = B_k$ *given by* $b = (b_n)_n$. *Then* $B \in B_w(\ell^2(w), \ell^2)$ *if and only if*

$$\sup_n \left(\frac{1}{W_{n+k}} \sum_{i=1}^{n} |b_i|^2 \right)^{1/2} < \infty$$

b*) Let* $k < 0$ *and* $B = B_k$ *given by* $b = (b_n)_n$. *Then* $A \in B_w(\ell^2(w), \ell^2)$ *if and only if*

$$\sup_n \left(\frac{1}{W_n} \sum_{i=1}^{n} |b_i|^2 \right)^{1/2} < \infty$$

Remark 4 *Using the above theorem we can see that* $B(\ell^2) \subseteq B_w(\ell^2(w), \ell^2)$ *and the inclusion is proper when* $W_n = \sum_{k=1}^{n} w_k \to \infty$ *for* $n \to \infty$.

In this last case, $B_w(\ell^2(w), \ell^2)$ *is not closed under Schur multiplication. On the other hand, there are simple examples to show that* $B_w(\ell^2(w), \ell^2) \not\subset M(\ell^2)$ *and* $M(\ell^2) \not\subset B_w(\ell^2(w), \ell^2)$. *Let us take for instance a diagonal matrix from* $B_w(\ell^2(w), \ell^2)$ *such that the sequence on the main diagonal is unbounded. In this case, the matrix failed to be in* $M(\ell^2)$ *since it is well known that the space of Schur multipliers contains only bounded sequences on diagonals.*

On the other hand, if we take

$$M = \begin{pmatrix} 1 & 1 & 1 & \cdots \\ 1 & 1 & 1 & \cdots \\ 1 & 1 & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

is is clear that M *is a Schur multiplier. However, simple computations shows us that* M *fails to be in* $B_w(\ell^2(w), \ell^2)$.

The last theorem characterize diagonal matrices which are Schur multipliers from $B_w(\ell^2(w),\ell^2)$ into $B_w(\ell^2(w),\ell^2)$.

Theorem 5 *Let* $M=\begin{pmatrix} m_1 & 0 & 0 & \cdots \\ 0 & m_2 & 0 & \cdots \\ 0 & 0 & m_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$ *be a diagonal matrix. Then* M *is a Schur multiplier from* $B_w(\ell^2(w),\ell^2)$ *into* $B_w(\ell^2(w),\ell^2)$ *if and only if* $m=(m_k)_k\in\ell^\infty$.

Proof. Let M be a Schur multiplier as in the statement of the theorem. Then

$$M\in M(B_w(\ell^2(w),\ell^2),B_w(\ell^2(w),\ell^2))$$

if and only if

$$M*B\in B_w(\ell^2(w),\ell^2) \text{ for every } B\in B_w(\ell^2(w),\ell^2).$$

We denote with $b=(b_k)_k$ the sequence from the main diagonal of the matrix B and applying Theorem 2, it follows that $b\in g(w,2)$. Using the factorizations

$$\ell^\infty\cdot g(w,2)=g(w,2)$$

and

$$\ell^2=d(w,2)\cdot g(w,2),$$

proved in [6] we get that $(m_k)_k\in\ell^\infty$ and the proof is complete.

We finish the paper with a conjecture. It is known that $B(\ell^2)\subset M(B_w(\ell^2))$. We conjecture that the same is true for our more general space, namely

$B(\ell^2)\subset M(B_w(\ell^2(w),\ell^2),B_w(\ell^2(w),\ell^2))$.

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DYNAMIC WEB PAGES

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Abstract: There are many ways to create a dynamic content on Web. Two major approaches consists into PHP programming and Java script components. By using PHP programming tools, we may create HTML code based pages. Some dynamic effects may be created through CSS classes and Java script tools. We present some features useful in applications.
Mathematics Subject Classification (2010): 68M10, 68M11, 68M12
Key words: html, css, java script

1.Introduction

The aim of this paper is to create dynamic content as a drop-down menu using classes and *Java Script* functions. There is a rich online content dedicated to such matters but we think a very concise illustration may be useful for a web programmer.
Usually, a dropdown menu is created as a table whose lines are hidden. A *Java Script* function will activate menu lines and another function will hide the lines.

2.CSS classes

The css file contains definitions of classes *prezentareNormal* and *prezentareTrecere* that apply to *div* section.
The section *prezentareNormal* will initially be hidden:

```
div.prezentareNormal
{
display: none; /* the object will be hidden */
position: static;
}
```

One mouse event will activate the object:

```
div.prezentareTrecere
{
background-color: #CCFFFF;
display: inline;
position: absolute;
}
```

The file that contains the above classes is *clase.css*.

3.Java Script functions

The following function will apply to an event

```
function extinde(s)
{
  var d = s.getElementsByTagName("div").item(0);
  d.className = "prezentareTrecere";
}
```

which means that by moving the mouse into the element region, it will become visible.

```
function restrange(s)
{
 var d = s.getElementsByTagName("div").item(0);
 d.className = "prezentareNormal";
}
```

which means that by moving the mouse out of the element region, it will become hidden.
In applications, the *s* parameter will be the *this* pointer.
The file which contains the above functions is *functii.js*.

4.Web page

First, the *head* section will contain the references to *css* classes and *Java Script* functions:

```
<link rel="StyleSheet" href="clase.css">
<script language="javascript" src="functii.js"></script>
```

Next, we will define a table where the rows are initially hidden.

```
<body>
 <table width="30%" bgcolor="#FFFFFF">
```

The initial table will be 30% width of the screen. In fact, a single line of text will be displayed, containing the text “Despre noi”, the meaning being of a menu title.

```
<tr>
```

One row will be one object which contains a table on which apply two different rules depending on the mouse position.

```
<td onmouseover="extinde(this);" onmouseout="restrange(this);"
        style="background-color: #CCFFFF"  >
 <p align="center" >Despre noi</p>
 <div  class="prezentareNormal" style="width: 200px; height: 125px">
```

Now, the menu's items are rows in a table:

```
<table >
 <tr>
        <p align="center"><a href="item11.htm" >Vocatie</a>
 </tr>
 <tr>
        <p align="center"><a href="item12.htm" >Pilotaj</a>
 </tr>
 <tr>
        <p align="center"><a href="item13.htm" >Retea</a>
 </tr>
```

```
        </table>
       </div>
      </td>
     </tr>
    </table>
   </body>
```

The above description may be expanded easily to more menus. This may be done by adding a *td* tag into the main *tr* section. The new menu may have, of course, a different number of selections (items, lines).

```
   <body>
    <table width="30%" bgcolor="#FFFFFF">
     <tr>
      <td onmouseover="extinde(this);" onmouseout="restrange(this);"
                style="background-color: #CCFFFF"  >
       <p align="center" >Despre noi</p>
       <div  class="prezentareNormal" style="width: 200px; height: 125px">
        <table >
         <tr>
                <p align="center"><a href="item11.htm" >Vocatie</a>
         </tr>
         <tr>
                <p align="center"><a href="item12.htm" >Pilotaj</a>
         </tr>
         <tr>
                <p align="center"><a href="item13.htm" >Retea</a>
         </tr>
        </table>
       </div>
      </td>
```

Here it starts the new menu:

```
      <td onmouseover="extinde(this);" onmouseout="restrange(this);"
                style="background-color: #CCFFFF"  >
       <p align="center" >Activitati</p>
       <div  class="prezentareNormal" style="width: 200px; height: 125px">
        <table >
         <tr>
                <p align="center"><a href="item21.htm" >Publicatii</a>
         </tr>
         <tr>
                <p align="center"><a href="item22.htm" >Cursuri</a>
         </tr>
        </table>
       </div>
      </td>
     </tr>
    </table>
   </body>
```

5.Conclusions

Even there exists a huge content of web pages on the Internet, we belive that the very concise prezentation may be usefull. Thus, some standard *html* tags: *table*, *td*, *tr* may be combined very easy with some simple *css* classes and *Java Script* functions. Finally, still the basic pieces are well known, the result may be spectacular and usefull.

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A VARIATIONAL METHOD FOR THE $p(\bullet)$-LAPLACIAN

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Abstract: A direct variational method is used in proving the existence of a solution in $W_0^{1,p(\cdot)}(\Omega)$ to the operator equation $J_\varphi u = N_g u$, where J_φ is the duality mapping on $W_0^{1,p(\cdot)}(\Omega)$, corresponding to the gauge function φ, and N_g is the Nemytskij operator generated by a Carathéodory function g satisfying an appropriate growth condition ensuring that N_g may be viewed as acting from $W_0^{1,p(\cdot)}(\Omega)$ into its dual.

Mathematics Subject Classification (2010): 47J30, 35J60, 35B38
Key words: duality mapping; critical point; Sobolev space with variable exponent.

1. Introduction

In recent years there has been an extensive interest in the field of operator equations involving various forms of the $p(\cdot)$-Laplacian. It is well known that the $p(\cdot)$-Laplacian is the operator

$$-\Delta_{p(\cdot)} : W_0^{1,p(\cdot)}(\Omega) \to \left(W_0^{1,p(\cdot)}(\Omega)\right)^*,\ \Delta_{p(\cdot)} u := div\left(|\nabla u|^{p(\cdot)-2}\nabla u\right) \text{ for } u \in W_0^{1,p(\cdot)}(\Omega).$$

Many properties of the classical p-Laplacian may be retrieved, except that of being a duality mapping on $W_0^{1,p(\cdot)}(\Omega)$. So, in this paper, we will use a natural version of the $p(\cdot)$-Laplacian which is appropriate from the standpoint of duality mappings (see [5] or [8, Section 9.3]): considering φ a gauge function, then the $(\varphi, p(\cdot))$-Laplacian is the operator

$$-\Delta_{(\varphi,p(\cdot))} : W_0^{1,p(\cdot)}(\Omega) \to \left(W_0^{1,p(\cdot)}(\Omega)\right)^*,\ -\Delta_{(\varphi,p(\cdot))} u := J_\varphi u \text{ for } u \in W_0^{1,p(\cdot)}(\Omega),$$

where J_φ is the duality mapping on $W_0^{1,p(\cdot)}(\Omega)$, corresponding to the gauge function φ.

In particular, if $p(x)$ is constant and $\varphi(t) := t^{p-1}$, $t \geq 0$, then $\Delta_{(\varphi,p(\cdot))}$ coincides with Δ_p (see [9, Remark 4.1, p. 10]).

We mention that in [2], the existence of suitable solutions to equation $J_\varphi u = N_g u$ is proven by three different methods based on the reflexivity and the smoothness of the space, the Schauder fixed point theorem or the Leray-Schauder degree, respectively.

2. An abstract result

The main result of this paper is obtained by applying the following theorem.

Theorem 2.1. *Let* $\left(X,\|\cdot\|_X\right)$ *be a real reflexive and smooth Banach space, compactly embedded in the real Banach space* $\left(V,\|\cdot\|_V\right)$ *with the compact injection* $X \xrightarrow{i} V$. *Let* $H \in \mathsf{C}^1\left(X,\boldsymbol{R}\right)$ *be a functional given by*

$$H(u) := \Psi(u) - G(iu),\ u \in X, \tag{2.1}$$

where:

(i) $\Psi : X \to \boldsymbol{R}$ *satisfies:*

$(\mathrm{i}.1)$ *at any* $u \in X$,

$$\Psi(u) := \Phi\left(\|u\|_X\right), \tag{2.2}$$

with

$$\Phi(t) := \int_0^t \varphi(\tau)\mathrm{d}\tau \ \text{ for any } t \geq 0, \tag{2.3}$$

φ *being a gauge function which satisfies*

$$\varphi^* := \lim_{t\to\infty} \frac{t\varphi(t)}{\Phi(t)} < \infty. \tag{2.4}$$

(ii) $G : V \to \boldsymbol{R}$ *satisfies:*

$(\mathrm{ii}.1)$ $G \in \mathsf{C}^1\left(V,\boldsymbol{R}\right)$;

$(\mathrm{ii}.2)$ *there exist positive constants* c_1, c_4, c_5 *and* $q < \varphi^*$ *such that*

$$G(u) < c_1\|u\|_V^q + c_4\|u\|_V + c_5 \ \text{ for all } u \in V. \tag{2.5}$$

Then, the functional H *possesses a critical value.*

Before proving of Theorem 2.1 we list some of the results to be used.

A real Banach space X is said to be *smooth* if for any nonzero $x \in X$ there exists a unique support functional, i.e., there exists a unique $x^*(x) \in X^*$ having the metric properties $\langle x^*(x), x\rangle = \|x\|$ and $\|x^*(x)\| = 1$. According to Diestel [4, Theorem 1, p. 22], X is smooth if and only if its norm is Gâteaux differentiable.

A function $\varphi : \mathbf{R}_+ \to \mathbf{R}_+$ is said to be a *gauge function* if φ is continuous, strictly increasing, $\varphi(0) = 0$ and $\varphi(r) \to \infty$ as $r \to \infty$. Taking into account Asplund's result in [1], it follows that, on a smooth real Banach space X, the *duality mapping corresponding to a gauge function* φ is a single valued operator

$$J_\varphi : X \to X^*,\ J_\varphi u := \Psi'(u), \text{ where } \Psi(u) := \int_0^{\|u\|_X} \varphi(\tau)\mathrm{d}\tau.$$

The functional $H : X \to \boldsymbol{R}$ is said to be *weakly sequentially lower semicontinuous* if and only if for any $u \in X$, any sequence $(u_n)_n$ in X such that $u_n \xrightarrow{w} u$ as $n \to \infty$,

$$H(u) \leq \liminf_{n\to\infty} H(u_n).$$

The functional $H : X \to \boldsymbol{R}$ is said to be *coercive* if $\lim_{\|u\|_X\to\infty} H(u) = +\infty$.

The main tool used in proving Theorem 2.1 is the well known classical result.

Theorem 2.2. *Let* $\left(X,\|\cdot\|\right)$ *be a real reflexive Banach space and let* $H: X \to \boldsymbol{R}$ *be a coercive and (sequentially) weakly lower semicontinuous functional. Then* H *is bounded from below on* X *and attains its minimum in* X *.*

Proof of Theorem 2.1. We show that under the assumptions of Theorem 2.1, the functional H satisfies the hypotheses of Theorem 2.2. The function Φ given by (2.3) is continuous and convex, therefore the functional Ψ given by (2.2) is continuous and convex. Consequently the functional Ψ is sequentially weakly lower semicontinuous in X. The compact injection $X \xrightarrow{i} V$ implies that the functional H is sequentially weakly lower semicontinuous in X. We will show that the functional is H coercive. Let $\varepsilon > 0$ be such that

$$\varphi^* - \varepsilon > q. \tag{2.6}$$

Based on the definition of φ^*, there exists a $\delta > 0$ such that

$$\frac{\tau\varphi(\tau)}{\Phi(\tau)} > \varphi^* - \varepsilon \text{ for any } \tau \geq \delta$$

or

$$\frac{\varphi(\tau)}{\Phi(\tau)} > \frac{\varphi^* - \varepsilon}{\tau} \text{ for any } \tau \geq \delta.$$

Integrating this inequality on $[\delta, t]$, we obtain that for $t > \delta$ we have

$$\Phi(t) > c_2 t^{\varphi^* - \varepsilon}, \text{ with } c_2 := \Phi(\delta)/\delta^{\varphi^* - \varepsilon}. \tag{2.7}$$

On the other hand there exists a constant $c_3 > 0$ such that

$$\left\|i(u)\right\|_V \leq c_3 \|u\|_X \text{ for any } u \in X.$$

Therefore, from (2.5) and (2.7) we derive that

$$H(u) \geq c_2 \|u\|_X^{\varphi^* - \varepsilon} - c_1 c_3^q \|u\|_X^q - c_3 c_4 \|u\|_X - c_5 \quad \text{for any } u \in X.$$

Taking into account (2.6) it follows that the functional H is coercive. Consequently H is bounded from below on X and attains its minimum in a point $u \in X$. □

Remark. The functional H being C^1 on X, we conclude that the point u satisfies the equation $H'(u) = 0$. Taking into account the form (2.1) of the functional H, this equation can be rewritten as

$$J_\varphi u - G'(u) = 0.$$

3. Lebesgue and Sobolev spaces with variable exponent

This section gathers various needed definitions and basic properties related to Lebesgue and Sobolev spaces with variable exponents. For proofs and references, see e.g. [7].

Let Ω be a bounded and smooth domain in $\boldsymbol{R}^N$, $N \geq 2$ and let $p \in L^\infty(\Omega)$ be such that

$$1 \leq p^- := \operatorname*{ess\,inf}_{x \in \Omega} p(x) \leq p^+ := \operatorname*{ess\,sup}_{x \in \Omega} p(x).$$

The *Lebesgue space* $L^{p(\cdot)}(\Omega)$ *with variable exponent* $p(\cdot)$ is defined as

$$L^{p(\cdot)}(\Omega) := \left\{ u : \Omega \to \boldsymbol{R};\, u \text{ is d}x\text{-measurable and } \rho_{p(\cdot)}(u) := \int_\Omega |u(x)|^{p(x)} \,\mathrm{d}x < \infty \right\}$$

and it is endowed with the norm

$$\|u\|_{p(\cdot)} := \inf\left\{\lambda > 0 ; \rho_{p(\cdot)}\left(\lambda^{-1}u\right) \le 1\right\}.$$

The space $L^{p(\cdot)}(\Omega)$ is a separable Banach space. If $p^- > 1$, the space $L^{p(\cdot)}(\Omega)$ is uniformly convex, hence reflexive. For any $u \in L^{p(\cdot)}(\Omega)$ with $p \in L^{\infty}(\Omega)$ satisfying $p^- > 1$ and $v \in L^{p'(\cdot)}(\Omega)$ $\left(\frac{1}{p(x)} + \frac{1}{p'(x)} = 1\right)$, one has

$$\int_{\Omega} u(x)v(x)\,dx \le \left(\frac{1}{p^-} + \frac{1}{(p')^-}\right)\|u\|_{p(\cdot)}\|v\|_{p'(\cdot)}. \tag{3.1}$$

The following result will be useful ([5, Remark 1]):

Lemma 1. *Let* $p \in L^{\infty}(\Omega)$ *be such that* $p^- \ge 1$ *and* $u \in L^{p(\cdot)}(\Omega)$. *Then*

$$\rho_{p(\cdot)}(u) \le 1 + \|u\|_{p(\cdot)}^{p^+}. \tag{3.2}$$

Given a function $p \in L^{\infty}(\Omega)$ that satisfies $p^- \ge 1$, the *Sobolev space* $W^{1,p(\cdot)}(\Omega)$ *with variable exponent* $p(\cdot)$ is defined as

$$W^{1,p(\cdot)}(\Omega) := \left\{u \in L^{p(\cdot)}(\Omega); \partial_i u \in L^{p(\cdot)}(\Omega), 1 \le i \le N\right\},$$

where for each $1 \le i \le N$, ∂_i denote the distributional derivative operator with respect to the i-th variable. This space may be equivalently defined as

$$W^{1,p(\cdot)}(\Omega) := \left\{u \in L^{p(\cdot)}(\Omega); |\nabla u| \in L^{p(\cdot)}(\Omega)\right\},$$

where $\nabla u(x) := \left(\partial_1 u(x), \ldots, \partial_N u(x)\right)$. It is a Banach space with respect to the norm

$$\|u\| := \|u\|_{p(\cdot)} + \|\nabla u\|_{p(\cdot)}.$$

Let $p \in C(\bar{\Omega})$ be such that $p^- \ge 1$. Given any $x \in \bar{\Omega}$, let

$$p^*(x) := \frac{Np(x)}{N - p(x)} \text{ if } p(x) < N, \text{ and } p^*(x) := \infty \text{ if } p(x) \ge N, \tag{3.3}$$

and let there be given a function $q \in C(\bar{\Omega})$ that satisfies

$$1 \le q(x) < p^*(x) \text{ for each } x \in \bar{\Omega}. \tag{3.4}$$

Then the following compact injection holds:

$$W^{1,p(\cdot)}(\Omega) \subset L^{q(\cdot)}(\Omega), \tag{3.5}$$

so that, in particular,

$$W^{1,p(\cdot)}(\Omega) \subset L^{p(\cdot)}(\Omega).$$

We consider the Sobolev space $W_0^{1,p(\cdot)}(\Omega)$, the closure of $C_0^{\infty}(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$. This space can be equivalently renormed via the Poincaré inequality (see e.g. [7]) with the norm

$$\|u\|_{1,p(\cdot)} := \|\nabla u\|_{p(\cdot)}.$$

Moreover this norm is Fréchet differentiable on $W_0^{1,p(\cdot)}(\Omega)$ (see [3]).

In [5, Section 4] and [2], it is established that any duality mapping corresponding to the gauge function φ on $\left(W_0^{1,p(\cdot)}(\Omega),\|\cdot\|_{1,p(\cdot)}\right)$ has the form

$$\left\langle J_\varphi u,h\right\rangle=\frac{\varphi\left(\|u\|_{1,p(\cdot)}\right)\int_{\Omega\setminus\Omega_{0,u}}p(x)\dfrac{|\nabla u(x)|^{p(x)-2}\nabla u(x)\cdot\nabla h(x)}{\|u\|_{1,p(\cdot)}^{p(x)-1}}\mathrm{d}x}{\int_\Omega p(x)\dfrac{|\nabla u(x)|^{p(x)}}{\|u\|_{1,p(\cdot)}^{p(x)}}\mathrm{d}x},\quad h\in W_0^{1,p(\cdot)}(\Omega). \quad (3.6)$$

Here $\Omega_{0,u}:=\left\{x\in\Omega;|\nabla u(x)|=0\right\}$.

4. The main existence result

The main result is

Theorem 4.1. *Let Ω be a domain in $\boldsymbol{R}^N$ ($N\ge 2$), let $p\in\mathsf{C}(\bar\Omega)$ be a function such that $p^->1$, and let $\varphi:\boldsymbol{R}_+\to\boldsymbol{R}_+$ be a gauge function which satisfies* (2.4)*, where Φ is given by* (2.3)*. Also, let us consider a Carathéodory function $g:\Omega\times\boldsymbol{R}\to\boldsymbol{R}$ satisfying the following hypothesis:*

(H1) *there exists a function $q\in\mathsf{C}(\bar\Omega)$ that satisfies* (3.4) *such that*

$$|g(x,s)|\le C_1|s|^{q(x)/q'(x)}+a(x)\ \text{for almost all } x\in\Omega \text{ and all } s\in\boldsymbol{R}, \quad (4.1)$$

where $\dfrac{1}{q(x)}+\dfrac{1}{q'(x)}=1$, *$a$ is a bounded function,* $a(x)\ge 0$ *for almost all* $x\in\Omega$, *for some constant* $C_1>0$*;*

Suppose that the following condition holds:

(H2) $\varphi^*>q^+$.

Considering the Nemytskij operator generated by g, $N_g:L^{q(\cdot)}(\Omega)\to L^{q'(\cdot)}(\Omega)$,

$$(N_g u)(x):=g(x,u(x))\ \text{for almost all } x\in\Omega,$$

then, the equation

$$J_\varphi u=N_g u,$$

where J_φ is given by (3.6)*, has solutions in* $W_0^{1,p(\cdot)}(\Omega)$.

To prove this theorem, Theorem 2.1 will be applied to the functional

$$H:X:=W_0^{1,p(\cdot)}(\Omega)\to\boldsymbol{R},\ H(u):=\Phi\left(\|u\|_{1,p(\cdot)}\right)-\mathsf{G}(i(u)), \quad (4.2)$$

where Φ is given by (2.3) and $\mathsf{G}:V:=L^{q(\cdot)}(\Omega)\to\boldsymbol{R}$,

$$\mathsf{G}(u):=\int_\Omega G(x,u(x))\mathrm{d}x,\ G(x,s):=\int_0^s g(x,\tau)\mathrm{d}\tau.$$

The space $W_0^{1,p(\cdot)}(\Omega)$ is reflexive ([7]) and smooth ([6], [5]). Also, according to [9, Proposition 4.3], it follows that, under the hypotheses of Theorem 4.1, the functional H given by (4.2) is well-defined and C^1 on $W_0^{1,p(\cdot)}(\Omega)$, with

$$H'(u) = J_\varphi(u) - g(x,u).$$

By using (4.1) it follows that

$$|G(x,s)| \le \frac{C_1}{q^-}|s|^{q(x)} + a(x)|s|,$$

therefore

$$|\mathsf{G}(u)| \le \frac{C_1}{q^-}\int_\Omega |u(x)|^{q(x)}\,dx + \int_\Omega a(x)|u(x)|\,dx.$$

Taking into account Lemma 1 and by using inequality (3.1), we have

$$|\mathsf{G}(u)| \le \frac{C_1}{q^-}\|u\|_{q(\cdot)}^{q^+} + \left(\frac{1}{q^-} + \frac{1}{(q')^-}\right)\|a\|_{q'(\cdot)}\|u\|_{q(\cdot)} + \frac{C_1}{q^-} \text{ for any } u \in L^{q(\cdot)}(\Omega),$$

that is (2.5).

Applying Theorem 2.1, the conclusion of Theorem 4.1 follows.

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IGUAL PROJECT AS A SPRINGBOARD TO MODERN EDUCATION IN LATIN AMERICA

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Abstract: Seeking a solution to the frequent problem with the quality of education in most Latin American countries, universities from several European countries come together around the idea of creating an alternative education, that meet the needs and quality of education offered by private schools in Latin America. The idea behind the IGUAL project is to ensure accessible and high-grade in a low-cost education to students in disadvantaged countries in the region by attempting to follow the 7 principles of Finnish education experties. An analysis of the need for change in the education of these countries has been made. It includes the description of the IGUAL project as part of the solution in the field of information technologies.
Mathematics Subject Classification (2010): 97Q60
Key words: information technologies, computer science education, teaching methods, classroom techniques

1. Introduction

The low quality of primary and secondary education in most Latin America countries is a well-known problem. The private educational market has been steadily growing in those countries. These private schools, in general, offer a higher quality and personalized education for the students that can afford it. The main selling point of these institutions is access to better resources: better teachers, technologies, materials and pedagogical methods. (Xavier Ochoa, Cristian Cechinel, Camilo Jimenez, Carlos Arévalo, Erick Araya, Sandro da Silva Camargo,Cláudia Camerini Perez, Katherine Chiluiza, Luis Alvarez, Jorge Morales, 2011)
On the other hand, for years, In a number criteria Finland already has one of the best education systems. It ranks among the math, reading and science in the PISA prestigious ranking (PISA, 2012) of the Organization for Economic Cooperation and Development. Teachers from other countries come to share experiences and learn strategies of training of teachers from Finnish schools. Students in Finland show the highest level of knowledge in the world and despite the excellent results they spend the least time involved in learning. There is 7 principles of school education in Finland: equality, cost-free, individuality, practicality, confidence, voluntariness, independence. (Kireeva, 2013)

2. Finland's 7 principles of school education

2.1. Equality

Finnish educational system provides equality of schools, teachers, subjects, students, even parents. There is a parallel system of education "from kindergarten to university" with no difference between state and private schools. All have exactly the same equipment, options and proportionate financing.

Finns do not sort students into classes according to ability or aspirations preferences. They are trying to fully integrate into society those who need special treatment. The difference in level between weak and strong students is the smallest in the world.

2.2. Cost-free

Invited on December 10, 2010 to reveal the pedagogical recipe in a broadcast on the American television network PBS, prof. Passy Salberg responds that "education is free for everyone, from the preparatory classes to university." Based such conditions it is difficult to make comparisons with the Latin American model.

Besides the training itself, free are: lunch; excursions, museums and all extracurricular activities; transport, carrying the child, if the nearest school is located at a distance farther than 2 kilometers; textbooks, all stationery, calculators and even laptops and tablets. Any parent funding for any purpose is prohibited.

2.3. Individuality

For each child is determined an individual plan of training and development. Individualization concerns the content of used textbooks, exercises, amount of class- and homework and time spent on them, but also the learning material. Some receive comprehensive and detailed exposure of the material and of those who are asked for only superficial knowledge – a summary of the most important and basic teaching material. Assessment is based on personal level.

2.4. Practicality

In school is taught only what will need into the child's life.

2.5. Confidence

On one hand is the confidence in school workers and teachers: each teacher uses that method of training, which is considered as the most appropriate. On the other hand is confidence in children: it is considered that the student alone chooses what will be more useful.

2.6. Voluntariness

Educators are trying to attract the attention of students, but if the child is absent absolutely no interest or ability to training, it will be oriented to the future practical and useful profession. The one and the other choice in the country valued equally.

2.7. Independence

School must teach the child the most important thing - to be alone in the future successful life. Therefore, children learn to think for themselves and to acquire knowledge.

But there are other changes too, not least to the traditional format that sees rows of pupils sitting passively in front of their teacher, listening to lessons or waiting to be questioned. Instead there will be a more collaborative approach, with pupils working in smaller groups to solve problems while improving their communication skills (independent.co.uk, 2015).

3. Latin America

In the opposite, in Latin America countries there is a problem with equality in education. The private education is growing in those countries. There is a difference in quality and personalized education between private and state schools. This difference in education quality creates a problem once students from public schools reach university. This problem is aggravated by the fact that the great majority of public schooled students belong to low-income families. All the problems that arise from this social status in Latin America (need to work at an early age, economical difficulties, etc) also conspire to reduce the probabilities of success of these students. In this light students that could afford private primary and secondary education have much better opportunities to have access to high quality universities and to complete successfully their studies. (ALFA Program, 2015)

On the other hand, students that due to their socio-economical status only had access to public education have, statistically, a lower chance to enter universities and to obtain a professional

degree. This has a negative impact on the competitiveness of Latin American countries, as only the middle- and high-income segments are fully contributing to the pool of specialized workforce while the talent and potential is uniformly distributed among the whole population. While scholarships and subsidized or free higher education could help to overcome the economic problems of low-income students, the lack of an adequate primary and secondary education has not been directly addressed in the region. While improving public basic education is the ideal solution to the problem, changing current educational structures have proved to be a long-term and difficult project for any country.

4. Need for alternative learning. IGUAL project.

There is much evidence that adolescents and youth who are disconnected from mainstream institutions and opportunities are likely to suffer significant, often long-term, negative effects as they enter adulthood. Many of the youth may reconnect to education and/or identify ways to be productive and creative if given the opportunity to do so, through alternative education strategies and settings. Such schools and programs are intended to serve this population and there are a variety of program models operating around the Latin American countries.

Thus, disconnected youth are a primary target group for alternative education schools and programs. There is a general sense in the youth development community that there is a great need for alternative education for 16 to 24 year old vulnerable youth, and that currently much of the need is not being met.

The Innovation for Equality in Latin American University (IGUAL) Project is a joint initiative between Latin American and European Universities, including the Finnish University of TAMK, to facilitate, through the use of innovative learning solutions, the accessibility of Latin American Higher Education Institutions (HEIs) to students that have received insufficient education during their primary and secondary phase, based on the Seven Finnish principles of school education experience and implementing them in programming courses.

The application area is introductory computer programming, where the gap between public and private basic education is the widest due to the limited access to technological resources in public schools. All the project results are made technically compatible to be shared in LACLO and Codewitz networks.

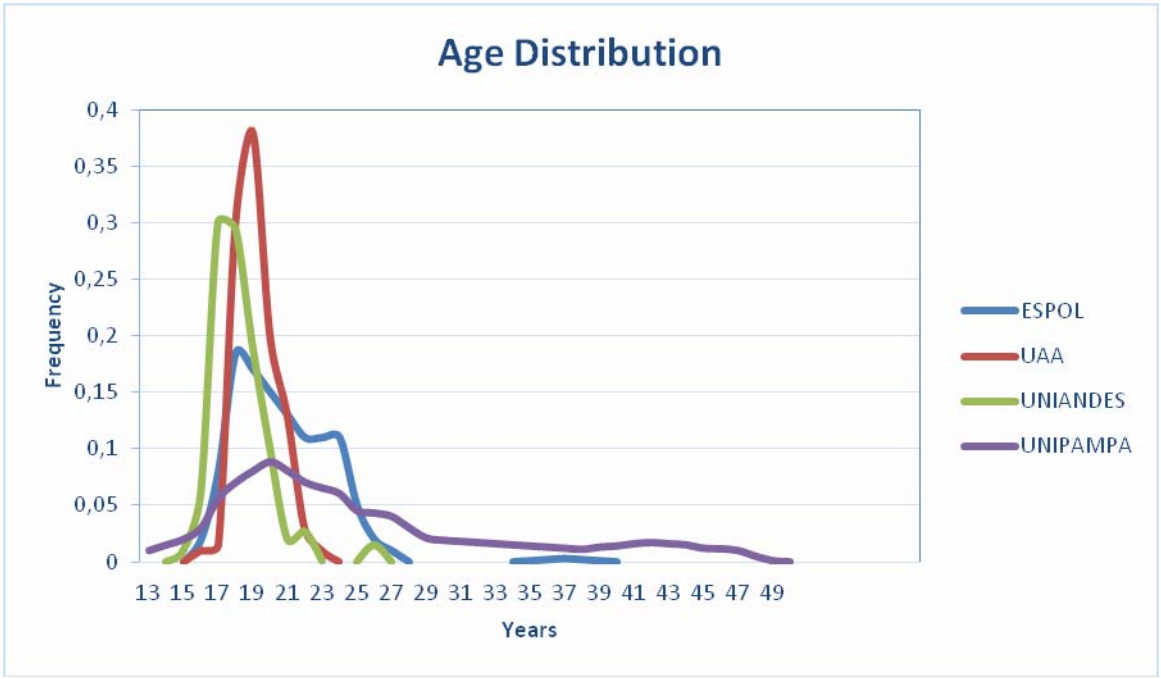

Figure 1 Age Distribution of Students

The first analysis (Xavier Ochoa, Cristian Cechinel, Camilo Jimenez, Carlos Arévalo, Erick Araya, Sandro da Silva Camargo,Cláudia Camerini Perez, Katherine Chiluiza, Luis Alvarez, Jorge Morales, 2011) performed on the project was to obtain the age distribution of students that that are taking or just passed the Introduction to Programming courses. This distribution for each participating University could be seen in Figure 1.

These distributions show that there are mainly two groups of students that will be affected by the project. One group is made out by students 18-21 years old. The second group have 22-30 years old students.

When surveyed about if they had access to a computer before entering the University, the students answered as shown in Figure 2. Most of them had regular access to a computer before entering the University.

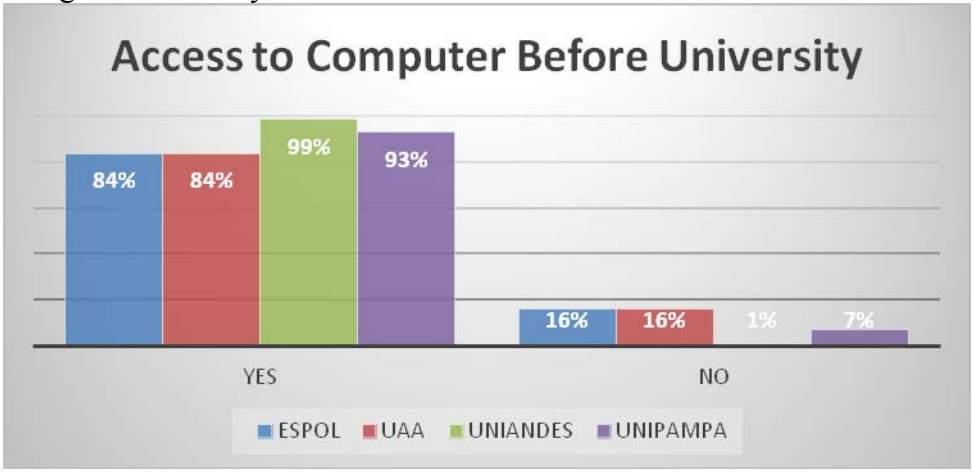

Figure 2 Access to Computer Before University

These findings prove that the access to technology problem is gradually disappearing in Latin America, but it is still present in a segment of the student population. This difference making the availability limited.

The students were surveyed about the type of High School (secondary education) that they attended. Their answer are summarized in Figure 3. In the case of ESPOL and UNIANDES, the majority of the students came from private schools, while in UAA and UNIPAMPA, the majority come from public schools. It is interesting to note that ESPOL is a public University, while UNIANDES is a private university.

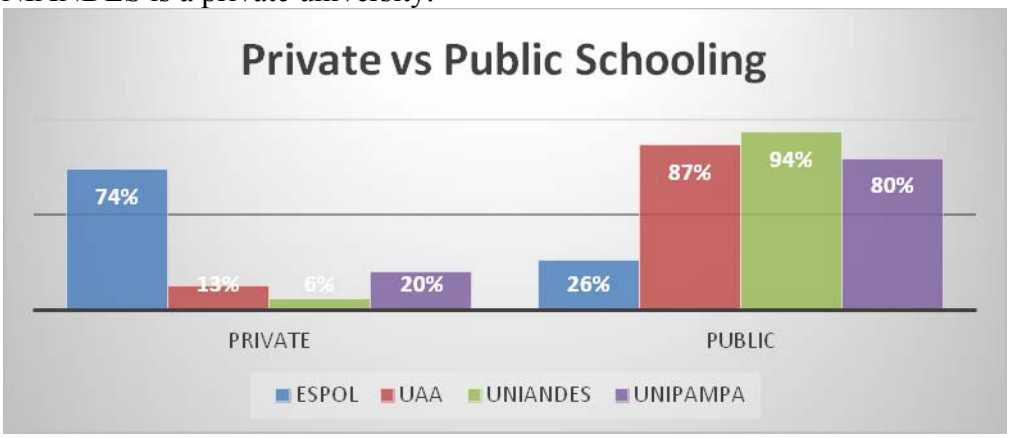

Figure 2 Private vs Public Schooling

In order to establish if the Programming Fundamentals course had a perceived impact in the level of knowledge that the student think that they posses, they were asked about that level before and after they have taken the course (in the case that they have already taken it). The results are presented in Figure 4.

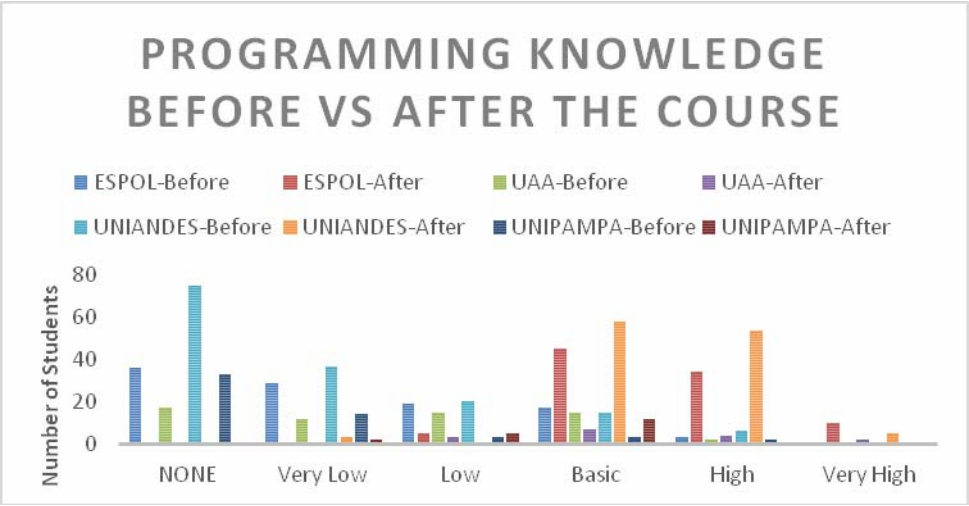

Figure 4 Programming Knowledge Before vs After the Course

The results suggest that most students think that the course has a positive impact in their knowledge level, but that impact is not as good as expected. Most of the students answer that they had a Basic level of programming after taking the course.

5. Conclusion

The main conclusions to be extracted from this Need Analysis are:

• There are two different groups taking the Programming Fundamentals course. One is composed by freshmen students that are in their first year at the University. The second group is made up by seniors that are about to graduate after 4 to 5 years of studies.

• The access problem seems to be disappearing, but there is still a minority of the students without access to computers before the University. The project should try to help those students.

• It is interesting to contrast the information about educational content and previous programming knowledge: the majority of surveyed students reported to have searched and found educational materials on line and that the quality of these was –to their perception– high, but most of them did not used the computer to learn to program. This can be an indicator that they did not found programming-related educational material that could have been useful for them.

• There is a perceive development in programming skills after taking the course, however, there still a lot of room for improvement.

• The technologies that students appreciate more for their studies are Email and Learning Management Systems. Although traditional, they are preferred to newer, more personal or more complex technologies. The project should try to make use of them.

• Most University students access the internet daily.

• Students are used to obtain learning materials online and use them for independent learning. This support the proposal of the project of using those materials to improve the learning process.

• To summarize the findings of the context survey, it can be concluded that the possible impact of the developed Learning Solutions of the IGUAL project can be high, because a large percentage of students already use computers at home, use the Internet, and they frequently search for and have used on-line learning materials in their learning environments.

• The Learning Styles results provide very clear guidelines as to which are the preferred ways of learning of the target population: they have a strong preference of visual learning materials and a moderate preference to Active, Sensing and Sequential ways of learning. So a premise in the context of the IGUAL project for designing Learning Solutions would be towards

software that let the students ‘do’ (active), ‘feel’ (sensing), and ‘see’ (visual) step by step (sequential) pedagogical examples.
• Results of the test show that students had trouble selecting a correct iteration structure and knowing how to create them (two questions of topic 6). They also had problems with questions related to creating functions, which is a way –in C and most structured programming languages – to divide a program into smaller modules. This is a basic skill that has to do with dealing with a complex problem and then breaking it into smaller, less complex parts.
Evaluating logical expressions is commonly a difficult subject and also a critical skill that is related to many programming topics (for example, making calculations and creating selection and iteration structures). Finally, at the most basic level, a surprising result is that students had also difficulties differentiating between a constant (a programming memory label whose value doesn’t change) and a variable (a memory label to store values that will be constantly changing), which intuitively seems to be a not too difficult subject. Results of the Content Survey clearly show what the most problematic topics are. This finding provides a useful design guideline to focus the development of Learning Solutions of the project.
The main problem to be addressed during this project is the increased level of difficulty that public schooled students in Latin America confront during their university studies, compared with their private schooled counterparts. This difficulty results in lower performance and a higher level of dropout (ALFA Programme, an initiative of EuropeAid, 2010). The solutions that this project is providing are (IGUAL project, 2010):
The application area is introductory computer programming, where the gap between public and private basic education is the widest due to the limited access to technological resources in public schools. There are also several activities such as a joint conference and virtual seminars during the project to create links between the members of LACLO and Codewitz. All the project results are made technically compatible to be shared in both networks.

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AFFINE SURFACES IN $\mathbf{R}^4$ AND $\mathbf{R}^5$. LAPLACE OPERATOR AND CORRESPONDING BELTRAMI FORMULAE

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Abstract: We consider affine surfaces in $\mathbf{R}^4$ and $\mathbf{R}^5$, under certain assumptions. The Wintgen inequalities are stated and Beltrami formulae are discussed for these surfaces; the results in codimensions 2 and 3 are compared.
Mathematics Subject Classification (2010): 53A15.
Key words: Affine surface; Wintgen inequality; Beltrami formula.

1. Introduction

In this short proceedings paper we recall only basic formulae for affine surfaces in $\mathbf{R}^4$ and $\mathbf{R}^5$. The References section contains necessary materials to understand the definitions of the notions and proofs of the used results.

The Gauss curvature G, the normal curvature $G^{\perp}$ and the squared mean curvature $\|H\|^2$ of any surface M^2 in the Euclidean space $\mathbf{E}^4$ always satisfy the inequality

$$G+\left|G^{\perp}\right|\leq\|H\|^2 \ .$$

This inequality was proved by P. Wintgen [12] and is known as the *Wintgen inequality*. The equality holds if and only if the ellipse of curvature of M^2 in $\mathbf{E}^4$ is a circle. This result was extended by Guadalupe and Rodriguez [3] to surfaces of higher codimension in Riemannian space forms.

In the submitted paper [9] the first author obtained the affine versions of the Wintgen inequalities for affine surfaces in $\mathbf{R}^4$ and $\mathbf{R}^5$, under certain assumptions. The used approach was given by Nomizu and Vrancken in [11], starting from the standard equiaffine structure in codimension *2* and from the metric defined by the authors of [1] for non-degenarate affine surfaces of codimension *3* with zero cubic form and Ricci tensor of rank *2*. The results are stated (without proof) in Section 4.

In Submanifold Theory of the classical Differential Geometry, the Beltrami formula is one of the most known and used. More precisely, $\Delta f=-nH$, where Δf is the immersion of the *n*-dimensional submanifold M^n into an arbitrary codimensional Euclidean space $\mathbf{E}^{\mathbf{m}}$, where H is the mean curvature vector field.

In Section 5 we compare the Beltrami formula in codimensions 2 and 3, obtained using the Levi-Civita connection and the connection of the induced affine metric (or the so-called the *relative metric*), whenever the affine fundamental form is non-degenerate.

2. Affine surfaces in $\mathbf{R}^4$

We recall the basic equations for a non-degenerate surface in $\mathbf{R}^4$.

Let M^2 be a surface immersed in $\mathbf{R}^4$, $f : M^2 \to \mathrm{R}^4$ and let σ be an arbitrary transversal plane bundle, i.e. $(f_*)(TM) \oplus \sigma = T\mathrm{R}^4$. For a given σ, the following two equations are fundamental

$$D_X Y = \nabla_X Y + h(X,Y) = \nabla_X Y + h^1(X,Y)\xi_1 + h^2(X,Y)\xi_2,$$
$$D_X \xi = -S_\xi X + \nabla^\perp_X \xi,$$

with

$$\nabla^\perp_X \xi_j = \tau^1_j(X)\xi_1 + \tau^2_j(X)\xi_2, \quad \forall j \in \{1,2\},$$

where h^1 and h^2 are symmetric bilinear forms and τ^j_i are local *1*-forms.

The *affine mean curvature* is defined by

$$H = \frac{1}{2}[(traceS_{\xi_1})\xi_1 + (traceS_{\xi_2})\xi_2].$$

Gauss equation has the expression

$$R(X,Y)Z = S_{h(Y,Z)}X + S_{h(X,Z)}Y .$$

From the previous two relations it follows that

$$h(X_1,X_1) = \xi_1, \;\; h(X_1,X_2) = \xi_2, \;\; h(X_2,X_2) = -\xi_1,$$

for $\{X_1, X_2\}$ a local frame field (orthonormal) and $\{\xi_1, \xi_2\}$ a basis of the normal bundle, where

$$h(X,Y) = h^1(X,Y)\xi_1 + h^2(X,Y)\xi_2.$$

Since the Gauss curvature is identically zero (see [11]), the shape operators at each point $p \in M^2$ are given by

$$S_{\xi_1}X_1 = \lambda_1 X_1 + \lambda_2 X_2,$$
$$S_{\xi_1}X_2 = -\lambda_2 X_1 + \lambda_1 X_2,$$
$$S_{\xi_2}X_1 = -\lambda_2 X_1 + \lambda_1 X_2 = S_{\xi_1}X_2,$$
$$S_{\xi_2}X_2 = -\lambda_1 X_1 - \lambda_2 X_2 = -S_{\xi_1}X_1,$$

where λ_1, λ_2 are real constants.

Then, the norm of the affine mean curvature of the affine surface *M* is

$$\| H \|^2 = \frac{1}{4}\{[g(S_{\xi_1}X_1, X_1) + g(S_{\xi_1}X_2, X_2)]^2 + [g(S_{\xi_2}X_1, X_1) + g(S_{\xi_2}X_2, X_2)]^2\} =$$
$$= \frac{1}{4}\{[g(\lambda_1 X_1 + \lambda_2 X_2, X_1) + g(-\lambda_2 X_1 + \lambda_1 X_2, X_2)]^2 +$$
$$+ [g(-\lambda_2 X_1 + \lambda_1 X_2, X_1) + g(-\lambda_1 X_1 - \lambda_2 X_2, X_2)]^2\} = \lambda_1^2 + \lambda_2^2.$$

3. Affine surfaces in $\mathbf{R}^5$

We consider affine surfaces in $\mathbf{R}^5$ with zero cubic form and the Ricci tensor of rank two. For details see [2].

The *cubic form* $C(X,Y,Z)$ is expressed by

$$C(X,Y,Z) = \nabla^\perp_X h(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z).$$

The condition of vanishing cubic form characterizes the complex curves and the product of two plane curves in $\mathbf{R}^4$. The meaning of the rank of the Ricci tensor is given in Theorem 4.3 from [8].

We recall the following expressions of the shape operators form, given in [8]

$$S_{\xi_1}X_1 = \alpha_1 X_1 + \alpha_2 X_2,$$

$$S_{\xi_1}X_2 = \alpha_3 X_1 + \alpha_4 X_2,$$
$$S_{\xi_2}X_1 = \beta_1 X_1 + \beta_2 X_2,$$
$$S_{\xi_2}X_2 = \beta_3 X_1 + \beta_4 X_2,$$
$$S_{\xi_3}X_1 = \gamma_1 X_1 + \gamma_2 X_2,$$
$$S_{\xi_3}X_2 = \gamma_3 X_1 + \gamma_4 X_2,$$
$$S_{\xi_1} = \begin{pmatrix} \alpha_1 & \alpha_3 \\ \alpha_2 & \alpha_4 \end{pmatrix} = \begin{pmatrix} 2\lambda & 0 \\ 0 & \lambda \end{pmatrix},$$
$$S_{\xi_2} = \begin{pmatrix} \beta_1 & \beta_3 \\ \beta_2 & \beta_4 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{2}\tau\lambda \\ \frac{1}{2}\lambda & 0 \end{pmatrix},$$
$$S_{\xi_1} = \begin{pmatrix} \gamma_1 & \gamma_3 \\ \gamma_2 & \gamma_4 \end{pmatrix} = \begin{pmatrix} \tau\lambda & 0 \\ 0 & 2\tau\lambda \end{pmatrix},$$

where λ is a real constant at each point $p \in M^2$.

It is shown in [8] that a metric g can be defined on $TM \oplus \sigma$ and can be extended to all of $\mathbf{R}^5$ for an affine surface M in $\mathbf{R}^5$ with zero cubic form and Ricci tensor of rank two such that

$$g(X_1, X_1) = \lambda,\ g(X_1, X_2) = 0,\quad g(X_2, X_2) = \tau\lambda,$$
$$g(X_i, \xi_j) = 0,\quad \forall i = 1,2,\ \forall j = 1,2,3,$$
$$g(\xi_1, \xi_1) = 2\lambda^2,\quad g(\xi_1, \xi_2) = 0,\quad g(\xi_1, \xi_3) = \tau\lambda^2,$$
$$g(\xi_2, \xi_2) = \frac{1}{2}\tau\lambda^2,\quad g(\xi_2, \xi_3) = 0,$$
$$g(\xi_3, \xi_3) = 2\lambda^2.$$

For $\tau = 1$ the metric is Riemannian and for $\tau = -1$ it is pseudo-Riemannian. The second fundamental form h is given by

$$h(X_1, X_1) = \xi_1,\ h(X_1, X_2) = \xi_2,\ h(X_2, X_2) = \xi_3.$$

4. Wintgen inequalities for affine surfaces in $\mathbf{R}^4$ and $\mathbf{R}^5$

In [9] the first author proved that in order to define the Gauss curvature and the normal curvature it is necessary to impose the condition that the metrics g and $g^\perp$ should be parallel with respect to the corresponding connections. The geometrical meaning of these conditions is given in [11].

The following results were obtained.

Proposition 4.1. [9] *An affine surface in* $\mathbf{R}^4$ *is flat and has constant affine mean curvature under the induced affine metric.*

Theorem 4.2. [9] *For an affine surface in* $\mathbf{R}^4$ *Wintgen inequality* $G + \left|G^\perp\right| \le \left\|H\right\|^2$ *is satisfied. Wintgen equality holds if and only if the shape operators vanish.*

For affine surfaces in $\mathbf{R}^5$ the *normalized normal scalar curvature* $\rho^\perp$ (the corresponding notion of normal Gauss curvature $G^\perp$ in higher codimension) of the affine normal bundle was defined by

$$\rho^{\perp} = \sqrt{(g(R^{\perp}(X_1, X_2)\xi_1, \xi_2)^2 + (g(R^{\perp}(X_1, X_2)\xi_2, \xi_3)^2 + (g(R^{\perp}(X_1, X_2)\xi_1, \xi_3)^2},$$

where $\{X_1, X_2\}$ is a local frame field (orthonormal) and $\{\xi_1, \xi_2, \xi_3\}$ a basis of the normal bundle.

The following result holds.

Theorem 4.3. i) *The Wintgen inequality* $G + \rho^{\perp} < \|H\|^2$ *for affine surfaces in* $\mathbf{R}^5$ *with zero cubic form and the Ricci tensor of rank two, endowed with the pseudo-Riemannian metric g it is always strictly satisfied.*

ii) *The Wintgen inequality* $G + \rho^{\perp} \leq \|H\|^2$ *for affine surfaces in* $\mathbf{R}^5$ *with zero cubic form and the Ricci tensor of rank two, endowed with the Riemannian metric g, it is satisfied if and only if the real constant* λ *satisfies* $|\lambda| \geq 1 + \sqrt{3}$.

5. Beltrami formulae for affine surfaces in $\mathbf{R}^4$ and $\mathbf{R}^5$

The classical formula for the Laplacian is:

$$\Delta f = -X_i X_i f + (\nabla_{X_i} X_i) f,$$

where $f : M^2 \to \boldsymbol{R^m}$, $m \in \{4,5\}$, is an immersion of a surface M^2.

On the other hand, the Gauss formula implies $\Delta f = -h(X_1, X_1) - h(X_2, X_2)$.

For $m = 4$:

$$\left.\begin{array}{l} h(X_1, X_1) = \xi_1,\ h(X_2, X_2) = -\xi_1 \Rightarrow \Delta f = 0, \\ S_{\xi_1} = \begin{pmatrix} \lambda_1 & \lambda_2 \\ -\lambda_2 & \lambda_1 \end{pmatrix} \Rightarrow traceS_{\xi_1} = 2\lambda_1 \\ S_{\xi_2} = \begin{pmatrix} -\lambda_2 & \lambda_1 \\ -\lambda_1 & -\lambda_2 \end{pmatrix} \Rightarrow traceS_{\xi_2} = -2\lambda_2 \end{array}\right] \Rightarrow H = \lambda_1\xi_1 - \lambda_2\xi_2.$$

It follows that, in this case, the classical Beltrami formula $\Delta f = -nH$ for $n = 2$, i.e. $\Delta f = -2H$, holds if and only if $\lambda_1 = \lambda_2 = 0 \Leftrightarrow S = 0 \Leftrightarrow M^2$ is a totally geodesic surface.

For $m = 5$:

$$\left.\begin{array}{l} h(X_1, X_1) = \xi_1,\ h(X_2, X_2) = \xi_3 \Rightarrow \Delta f = -\xi_1 - \xi_3, \\ S_{\xi_1} = \begin{pmatrix} 2\lambda & 0 \\ 0 & \lambda \end{pmatrix} \Rightarrow traceS_{\xi_1} = 3\lambda \\ S_{\xi_2} = \begin{pmatrix} 0 & \frac{1}{2}\lambda \\ \frac{1}{2}\lambda & 0 \end{pmatrix} \Rightarrow traceS_{\xi_2} = 0 \\ S_{\xi_3} = \begin{pmatrix} \lambda & 0 \\ 0 & 2\lambda \end{pmatrix} \Rightarrow traceS_{\xi_3} = 3\lambda \end{array}\right] \Rightarrow H = \frac{3\lambda}{2}(\xi_1 + \xi_3) \Rightarrow \Delta f = -\frac{2}{3\lambda}H.$$

Then the Beltrami formula $\Delta f = -2H$ holds if and only if $\lambda = \frac{1}{3}$.

An interesting point regarding the Beltrami formula is the comparison of our calculus with the following Remark 2 (page 89) from [2].

Remark 2 [2]. *Although there is defined an affine metric for hypersurfaces and for surfaces in* $\mathbf{R}^4$*, we can not use Beltrami's formula. Indeed, for hypersurfaces, the Laplacian of the affine metric, applied to the position vector, gives* $\Delta x = -n\xi$*, where* ξ *is the affine normal vector field. Hence* Δx *never vanishes. For surfaces in* $\mathbf{R}^4$*, the Laplacian of the affine metric, applied to the position vector, is always tangent to M and its vanishing does not characterize critical volume cf.* [4,5].

The expression of Δ_g, with g the affine metric, is given in [11], page 141. We could not find the full versions of [4] and [5] to completely understand the last affirmation from the previous remark. We don't know any similar statement(s) for affine surfaces in $\mathbf{R}^5$.

The difference from our result and that from the Remark 2 comes from the use of different connections.

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AN INTEGRAL FOR VECTOR FUNCTIONS WITH RESPECT TO VECTOR MEASURES

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Abstract: In this paper is given a generalization of the results obtained in [1] regarding an integral for vector functions with respect to vector measures. The Hilbert space used in [1] is replaced here by an arbitrarily Banach space, and the measures take values in the conjugate space of that Banach space. Some basic properties of this integral are given, and we also construct an example to illustrate the theory. Finally , using the Haar functions system as a Schauder base in $L^p([0,1])$, we prove a theorem that reduces the vector integral to scalar integrals in the case when the Banach space is $L^p([0,1])$, for $p \in (1,\infty)$.

Mathematics Subject Classification : 28B05, 28C20**,** 46E30.
Key words: vector integral, totally measurable function, vector measure, Haar functions.

1. Preliminary Facts

We recall here some definitions and results that we will use. Let X be a real Banach space and X' its conjugate. Let, also, (T,d) be a compact metric space. We denote by $\boldsymbol{B}$ the Borel subsets of T . For any σ-additive measure $\mu:\boldsymbol{B}\to X$, we define the variation of μ that is :
$|\mu|(A)=\sup\{\sum\|\mu(A_i)\|,(A_i)_i$ being a finite partition with Borel subsets of $A\}$, for any Borel subset A of T. If $|\mu|(T)<\infty$, we say that μ has bounded variation (see [2]). We will denote by: $cabv(X)=\{\mu:\boldsymbol{B}\to X,|\mu|(T)<\infty\}$. This set becomes a Banach space with the norm: $\|\mu\|=|\mu|(T)$ (it can be proved that this is really a norm). Now, we define some classes of functions on T and taking values in X :
$S(X)=\{f:T\to X, f$ simple function$\}$,
$TM(X)=\{f:T\to X,$ such that there exists a sequence $(f_n)_n$ of simple functions uniformly convergent to $f\}$ (the space of totally measurable functions),
$C(X)=\{f:T\to X, f$ is continuous$\}$. For any subset A of T , we denote by φ_A the characteristic function of A.

2. The definition and the properties of the integral

Let $f=\sum_{i=1}^{m}\varphi_{A_i}x_i$ be a simple function, where $x_i\in X$ and A_i are Borel subsets of T . Let, also, $\mu\in cabv(X')$. The integral of f with respect to μ is : $\int fd\mu=\sum_{i=1}^{m}\mu(A_i)(x_i)$. It is obvious that $\left|\int fd\mu\right|\le\|\mu\|\|f\|_\infty$, hence , the linear application $f\to\int fd\mu$ is continuous and has an

uniformly continuous extension to the closure of $S(X)$ with respect to $\|\ \|_\infty$, that is, the space $TM(X)$. Namely, if $(f_n)_n \subset S(X)$ is a sequence which converges uniformly to $f \in TM(X)$ then we define : $\int f d\mu = \lim_{n\to\infty} \int f_n d\mu$, and the limit doesn't depend on the sequence of simple function which is uniform convergent to f.

Proposition 1. *Let $f \in TM(\mathbf{R})$ and $\mu \in cabv(\mathbf{R}), x \in X, y \in X'$. Then :*
a) $fx \in TM(X)$; b) $\mu y \in cabv(X')$; c) $\int (fx) d(\mu y) = (\int f d\mu) y(x)$.
Proof. a) and *b)* are obvious;
c) Let $f \in S(\mathbf{R}), f = \sum_{i=1}^{m} \varphi_{A_i} a_i$, where $a_i \in \mathbf{R}$. We have :
$\int (fx) d(\mu y) = \sum_{i=1}^{m} \mu(A_i) y(a_i x) = (\sum_{i=1}^{m} \mu(A_i) a_i) y(x) = (\int f d\mu) y(x)$. Now, if $f \in TM(\mathbf{R})$, let $(f_n)_n \subset S(\mathbf{R})$, a sequence uniformly convergent to f. We can write :
$\int (fx) d(\mu y) = \lim_{n\to\infty} \int (f_n x) d(\mu y) = \lim_{n\to\infty} (\int f_n d\mu) y(x) = (\int f d\mu) y(x)$.

Corollary 2. *If $X = \mathbf{R}^n$ and $\{e_1, ..., e_n\}$ is its canonical basis, let $f \in TM(X)$, $\mu \in cabv(X')$, $f = \sum_{i=1}^{n} f_i e_i, \mu = \sum_{i=1}^{n} \mu_i e_i$. Then, $\int f d\mu = \sum_{i=1}^{n} \int f_i d\mu_i$.*

Example 3. Let $T = [0,1]$ with the metric $d(x,y) = |x - y|$, $X = L^{\frac{4}{3}}(T), X' = L^4(T)$,
$f : T \to X, f(t) = (\frac{t}{n})_{n\geq 1}, \mu : \boldsymbol{B} \to X', \mu(A) = (\frac{\lambda(A)}{n})_{n\geq 1}$ where A is a Borel subset of T and λ is the Lebesgue measure. We deduce : $\|f(t) - f(s)\| = |t - s| (\sum_{n=1}^{\infty} \frac{1}{n^{\frac{4}{3}}})^{\frac{3}{4}}$, for any $t, s \in T$, proving that f is continuous, hence, totally measurable. For any Borel subset A of T, we have : $\|\mu(A)\|^4 = \lambda(A)^4 \sum_{n=1}^{\infty} \frac{1}{n^4}$, that implies $\|\mu(A)\| = \lambda(A) \frac{\pi}{\sqrt[4]{90}}$. Using this equality, it is easy to see that $|\mu|(T) = \frac{\pi}{\sqrt[4]{90}}$, so $\mu \in cabv(X')$ and $\|\mu\| = \frac{\pi}{\sqrt[4]{90}}$. We will compute $\int f d\mu$. For doing this, let the sequence $(f_k) \subset S(X)$, $f_k = (\sum_{i=1}^{k} \varphi_{A_i} \frac{i}{k} \frac{1}{n})_{n\geq 1}$, where $A_i = [\frac{i-1}{k}, \frac{i}{k}]$. It is not difficult to see that this sequence converges uniformly to f. Hence, we have :
$\int f d\mu = \lim_{k\to\infty} \int f_k d\mu = \lim_{k\to\infty} \sum_{i=1}^{k} (\sum_{n=1}^{\infty} \frac{\lambda(A_i)}{n} \frac{i}{k} \frac{1}{n}) = \frac{\pi^2}{6} \lim_{k\to\infty} (\sum_{i=1}^{k} \lambda(A_i) \frac{i}{k}) = \frac{\pi^2}{6} \int_0^1 x dx = \frac{\pi^2}{12.}$.
For the following theorem , we need to recall some more basic facts. More precisely, we will construct a sequence of functions that represent a Schauder basis in $L^r([0,1])$ for any $r \in (1, \infty)$. Let $\chi_n^k : [0,1] \to \mathbf{R}$, $\chi_0^0 = 1$, $\chi_0^1(x) = \{1,$ for $x \in [0, \frac{1}{2}), 0,$ for $x = \frac{1}{2}, -1$ for

$x \in (\frac{1}{2},1]\},.....,\chi_n^k(x) = \{\sqrt{2^n}$, for $x \in [\frac{2k-2}{2^{n+1}},\frac{2k-1}{2^{n+1}}]$, $-\sqrt{2^n}$, for $x \in (\frac{2k-1}{2^{n+1}},\frac{2k}{2^{n+1}}]$, 0 for any other values of $x \in [0,1]\}$, $k \in \{1,...2^n\}$. We can write this functions system as a sequence $(g_j)_{j\geq 1}$, writing the functions increasingly with respect to k for any fixed n, then increasingly with respect to n. One can prove the following two results (see[3]) :

Lemma 4. $\int\limits_{[0,1]} g_i g_j d\lambda = \delta_{i,j}$, *for any* $i, j \geq 1$.

Lemma 5. *For any* $r \in (1,\infty)$, *the functions system* $(g_j)_{j\geq 1}$ *is a Schauder basis for the space* $L^r([0,1])$.

Definition 6. The functions system $(g_j)_{j\geq 1}$ is called Haar functions system.
Using this results, we will prove the following theorem :

Theorem 7. *Let* $p, q \in (1,\infty)$ *such that* $\frac{1}{p}+\frac{1}{q}=1$, $X = L^p([0,1])$, $X' = L^q([0,1])$, $f \in TM(X)$, $\mu \in cabv(X')$, $f = \sum_{n=1}^{\infty} f_n g_n$, $\mu = \sum_{m=1}^{\infty} \mu_m g_m$. *Then, we have:*

$$\int f d\mu = \sum_{n=1}^{\infty} \int f_n d\mu_n .$$

Proof. We can write :

$$\int f d\mu = \int (\sum_{n=1}^{\infty} f_n g_n) d(\sum_{m=1}^{\infty} \mu_m g_m) = (\sum_{n=1}^{\infty} (\sum_{m=1}^{\infty} (\int f_n d\mu_m) \int\limits_{[0,1]} g_n g_m d\lambda)) = \sum_{n=1}^{\infty} \int f_n d\mu_n$$,using lemma 5.

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A DENSITY THEOREM IN THE SET OF CONTINUOUS FUNCTIONS WITH VALUES IN THE UNIT INTERVAL

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Abstract: Here we characterize the uniform closure of a subset M of $C(X,[0,1])$, containing 0 and 1. This is a Bishop–type theorem.
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Key words: (VN)-property, antisymmetric sets, density theorem.

Introduction

In 1981, Bruno Brosowski and Frank Deutsch gave an elementary proof to Stone-Weierstrass' theorem. Using the same technique, T. Ransford found an elementary proof for Bishop's approximation theorem**.** In 1992, J. Prolla generalizes Von Neumann's variant of Stone-Weierstrass' theorem for continuous functions with range in [0,1], using Ransford's tools. In this paper we obtain a generalization of Prolla's result.

Let X be a compact Hausdorff space, and let $C(X,\mathbb{C})$ be the algebra of all continuous complex-valued functions on X, equipped with the uniform topology. For every closed subset F of X, and every $f \in C(X,\mathbb{C})$, we shall denote by:

$f|F$ -the restriction of f to F and by :

$\|f\|_F = \sup\{|f(x)|; x \in F\} = \|f|F\|$.

Also, for every subset M of $C(X,\mathbb{C})$, the distance of $f|F$ to $M|F$, is defined by:

$d_{f,M}(F) = \inf\{\|f-m\|_F; m \in M\}$.

Lemma 1. *The collection of all closed subsets of X with the property: $d_{f,M}(F) = d_{f,M}(X)$, has a minimal element.*

Proof. Let $\Im$ be the collection of all subsets of X, such that $d_{f,M}(F) = d_{f,M}(X)$, and let $\Im_0$ be a subset of $\Im$, totally ordered by inclusion. Obviously, $X \in \Im$, and therefore $\Im \neq \phi$.
For every $m \in M$, and every $F \in \Im$, we shall denote by:

$F^m = \{x \in F; |f(x) - m(x)| \geq d_{f,M}(X)\}$.

We observe that $F^m \neq \phi$, and $F_1^m \cap F_2^m = (F_1 \cap F_2)^m \neq \phi$, for every $F_i \in \Im_0$, $i = 1,2$.
By the compactness of X it follows that $\cap\{F^m; F \in \Im_0\} \neq \phi$ .On the other hand, if we write:

$E = \cap\{F; F \in \Im_0\}$, then $E^m = \cap\{F^m; F \in \Im_0\} \neq \phi$ and therefore $E \in \Im$.

By Zorn's lemma we infer that $\Im$ has minimal elements.

Definition 1. *We shall say that a subset* $M \subset C(X,[0,1])$ *has the property (VN) if* $\forall f,g,h \in M$ *involves* $fg+(1-f)g \in M$.

Remark 1. *If* M *has the property (VN), and* $0,1 \in M$ *, then* $\phi^n \in M$ *and* $(1-\phi^n)^m \in M, \forall \phi \in M$ *, and* $\forall n,m \in \mathbb{N}$.

Indeed, $\phi^2 = \phi.\phi+(1-\phi)0 \in M$ and $1-\phi = \phi.0+(1-\phi)1 \in M$. If we suppose that: $\phi^k \in M$, then $\phi^{k+1} = \phi.\phi^k+(1-\phi).0 \in M$, therefore, $\phi^n \in M, \forall n \in \mathbb{N}$. Also, if $1-\phi^k \in M$ then $1-\phi^{k+1} = \phi(1-\phi^k)+(1-\phi).1 \in M$, therefore $1-\phi^n \in M, \forall n \in \mathbb{N}$.

Definition 3 . *Let* M *be a subset of* $C(X,[0,1])$ *containing 0 and 1. We shall say that a subset* $S \subset X$ *is antisymmetric with respect to* M *, if every* $\phi \in M$ *, with the property:* $\phi f+(1-\phi)g \,|S \in M\,|S, \forall f,g \in M$, *is constant on* S .

Further, we shall denote by $\sum$ the collection of all subsets of X, antisymmetric with respect to M .

Remark 2. *The collection* $\sum$ *has the following properties:*

1) $\{x_0\} \in \sum$, $\forall x_0 \in X$.
2) *If* $S_i \in \sum$, *i=1,2, and* $S_1 \cap S_2 \neq \phi$ *, then* $S_1 \cup S_2 \in \sum$.
3) *If* $S \in \sum$ *, then the closure* $\overline{S}$ *of* S *belongs to* $\sum$.
4) *Every* $x \in X$ *belongs to a maximal element of* $\sum$*, denoted by* S_x .
5) $X = \cup\{S_x \,|x \in X\}$ *, and if* $x \neq y$ *, either* $S_x \cap S_y = \phi$ *, or* $S_x = S_y$.

The following theorem is the main result of this paper.

Theorem 1 . *Let* $M \subset C(X,[0,1])$*, with the property that* $0,1 \in M$ *, and let* $f \in C(X,[0,1])$. *Then there is a subset* $S \subset X$ *, antisymmetric with respect to* M *, such that:*

$$dist(f,M) = dist(f\,|S, M\,|S).$$

Proof. If we denote by $\Im$ the family of all closed subsets $F \subset X$, with the property: $d_{f,M}(F) = d_{f,M}(X) = d$, then, according to Lemma 1, $\Im$ has minimal elements . Let S be a minimal element of $\Im$. If we will show that $S \in \sum$, then the proof is finished.
If we suppose on the contrary, that $S \notin \sum$, then there is a $\phi \in M$ such that:

$$\phi f+(1-\phi)g \,|S \in M\,|S, \forall f,g \in M \text{ and } \phi \text{ is non constant on S.} \quad (1)$$

$$\text{Let } y,z \in S \text{ , and } a,b \in \mathbb{R} \text{ , such that: } 0 \le \phi(y) < a < b < \phi(z) \le 1 \quad (2)$$

Now , we can suppose that $2a < b$. Indeed, since $\frac{a}{b} < 1$, then there exists a $n \in \mathbb{N}^*$,

such that: $(\frac{a}{b})^n < \frac{1}{2}$ and therefore, $2a^n < b^n$. (3)

From (2) it follows : $0 \le \phi^n(y) < a^n < b^n < \phi^n(z) \le 1$. (4)

Further we shall denote : $\phi = \phi^n$, $a = a^n$ and $b = b^n$. We have $\phi \in M$, by Remark 1, and also $0 \le \phi(y) < a < b < \phi(z) \le 1$ and $2a < b$ (from (3) and (4)).

Now we observe that $\frac{1}{a} - \frac{1}{b} > \frac{1}{2a} > \frac{1}{b} > 1$. Therefore there is a $k > 1, k \in \mathbb{N}$, such that, $\frac{1}{a} > k > \frac{1}{b}$ and thus $a < \frac{1}{k} < b$.

If we denote by $p_n = (1-\phi^n)^{k^n}$, then, according to Remark 1, we have $p_n \,|\, S \in M \,|\, S$.

Let Y= $\{x \in S \,|\, 0 \le \phi(x) \le b\}$ and $Z = \{x \in S \,|\, a \le \phi(x) \le 1\}$.

Obviously, $Y \subset S$, and $Y \ne S$ .As $S \in \Im$ is minimal, it follows that $Y \notin \Im$, and therefore there is a function $m_Y \in M$, such that $\|f - m_Y\| < d$.

Similarly, there exists $m_Z \in M$, such that $\|f - m_Z\| < d$.

Now we observe that if $x \in Y \setminus Z$, then $0 \le \phi(x) \le a < \frac{1}{k}$.

According to Theorem 1 of [5] we infer that $p_n \to 1,$ uniformly on $Y \setminus Z$.

Similarly, if $x \in Z \setminus Y$, then $\frac{1}{k} < b \le \phi(x) \le 1,$ and $p_n \to 0$, uniformly on Z\Y.

Let $h_n = p_n m_Y + (1-p_n) m_Z$. Obviously $h_n \,|\, S \in M \,|\, S$, $h_n \to m_Y$ on $Y \setminus Z$, and $h_n \to m_Z$ on $Z \setminus Y$, respectively.

From this, it follows that $\|f - h_n\|_{Y \setminus Z} < d$ and $\|f - h_n\|_{Z \setminus Y} < d$ for n sufficient large.

Now we observe that, if $x \in Y \cap Z$, then:

$$|f(x) - h_n(x) = | p_n(x) f(x) + (1 - p_n(x) f(x) - h_n(x) \; | \le$$

$$\le p_n(x) \|f - m_Y\|_Y + (1 - p_n(x)) \|f - m_Z\|_Z < d .$$

As, $S = (Y \setminus Z) \cup (Y \cap Z) \cup (Z \setminus Y),$ it results that $\|f - h_n\|_S < d$, therefore $S \notin \Im$, which is contradictory.

Example 1. Let M be the set of continuous functions $g : [0,1] \to [0,1]$, such that g is affine on $J = [0, \frac{1}{2}]$, and $g(1) \in \mathbb{Z}$. We observe that:

$S_x = S_0 = J$, for every $x \in J$, and $S_x = \{x\}$ for every $x \in (\frac{1}{2}, 1]$.

Let $f : [0,1] \to [0,1]$ be a continuous function.

Obviously , for $x \in (\frac{1}{2}, 1]$, we have: $dist(f(x), M(x)) = 0$.

From Theorem 1 it results:

$dist(f, M) = \max\{dist(f \,|\, J, M \,|\, J) \;\; ; \; dist(f(1), \{0,1\})\}$.

Further, we shall denote by $[x] = \{y \in X \,|\, m(x) = m(y), \forall m \in M\}$, for every $x \in X$.

Theorem 2. *(Prolla 1992). Let* $M \subset C(X,[0,1])$ *be a subset , containing 0 and 1, with the property (VN), and let* $f \in C(X,[0,1])$. *Then there exists a* $x \in X$ *such that:*

$$dist(f;M) = dist\ (f|[x];M|[x]).$$

The proof follows from Theorem 1, since, in this case, the family $\sum$ of all subsets of X, antisymmetric with respect to M, coincides with the family $\{[x];x \in X\}$.

Theorem 3. *(Prolla 1992) Let* $M \subset C(X,[0,1])$ *be a subset with the property (VN), containing 0 and 1. Assume that* M *separates the points of* X *, and, for* $\forall x \in X, \exists \phi \in M$ *such that* $0 < \phi(x) < 1$. *Then* M *is dense in* $C(X,[01])$.

Proof. Since M separates the points of X, it follows that $[x] = \{x\}, \forall\, x \in X$. From Theorem 2 it results that there is a $x_0 \in X$ such that:

$dist\ (f;M) = dist(f(x_0);M(x_0)) = inf\{|f(x_0) - m(x_0)|; m \in M\}$.

If $f(x_0) = 0$, then $d(f(x_0);M(x_0)) = 0$, since $0 \in M$. It follows that $d(f;M) = 0$, therefore $f \in M$. The same situation occurs if $f(x_o) = 1$.

We assume now that $0 < f(x_o) < 1$. By hypothesis, there is a $\phi \in M$ such that $o < \phi(x_0) < 1$.

Let $\varepsilon > 0$. Obviously, there exists a $m \in \mathbb{N}$, such that $\phi^m(x_o) < \varepsilon$. Let $d = 1 - \phi^m(x_0) > 1 - \varepsilon$, and let $k \in \mathbb{N}$, with the property: $d^{k+1} < f(x_0) \le d^k$.

We observe that:

$$d^k - d^{k+1} = d^k(1-d) = d^k\phi^m(x_0) < \phi^m(x_0) < \varepsilon\ .$$

It results that $0 < f(x_0) \le d^{k+1} + \varepsilon$ and, so,

$$0 < f(x_0) - d^{k+1} < \varepsilon\ .$$

Since $\psi = (1-\phi^m)^k \in M$ and $\psi(x_0) = d^{k+1}$, it follows that :

$dist(f(x_0;M(x_o)) < \varepsilon$.

As $\varepsilon > 0$ is arbitrary, we conclude that $d(f;M) = 0$, therefore, $f \in \bar{M}$.

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SOME CONSERVATIVE NUMERICAL APPROXIMATIONS OF DYNAMICAL SYSTEMS

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Abstract: We consider some geometric invariant numerical schemes of higher order.
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1. Geometric background

Many differential equations arise from variational principles, that is to maximize or minimize some kind of integral $\int_a^b L\left(x, y^1, y^2, ..y^N, \frac{dy^1}{dx}, \frac{dy^2}{dx}, ..\frac{dy^N}{dx}, ...\frac{d^k y^N}{dx^k}\right)dx$. The unknown function $x \to \left(y^1(x), y^2(x), ..y^N(x)\right)$ enters in the variational problem with its derivatives up to order k and may suppose that the values

$$\left(x, y^1(x), y^2(x), ..y^N(x), \frac{dy^1(x)}{dx}, \frac{dy^2(x)}{dx}, ..\frac{dy^N(x)}{dx}, ...\frac{d^k y^N(x)}{dx^k}\right) \qquad (1)$$

are the coordinates of a point in some differential manifold of dimension $1+N(k+1)$. From geometric point of view we consider a fibration $\pi_{XY} : Y \to X$ where X is the real line and the fiber is diffeomorphic to some finite dimensional manifold Q, $\dim(Q) = N$. By $J^k Y$ we understand the equivalence classes of sections $s : X \to Y$ where two sections are equivalent iff the Taylor expansions coincide up to order k. $J^k Y$ has a natural differential structure and assuming x is the local coordinate in X and $\left(y^1, y^2, ..y^N\right)$ is a system of local coordinates in the fiber Q then $\left(x, y^A, y^A_i\right)_{A=1..N,\ i=1..k}$ is a system of local coordinates in $J^k Y$. $J^k Y$ is a fiber bundle over X with projection $\left(x, y^A, y^A_i\right)_{A=1..N,\ i=1..k} \to x$. For a section $s : X \to Y$ we define the k jet of s by $j^k s : X \to J^k Y$, $j^k s(x) = \left(x, \frac{ds^A(x)}{dx}, \frac{d^2 s^A(x)}{dx^2}, ...\frac{d^k s^A(x)}{dx^k}\right)$, that is $y^A_i\left(j^k s\right) = \frac{d^i s^A(x)}{dx^i}$. Various jet spaces are connected by $\pi_{J^l Y, J^k Y} : J^k Y \to J^l Y$ for k>l which consists in the truncation of the Taylor expansion from order k to order l. Using local coordinates we define the total derivative $D_x = \frac{\partial}{\partial x} + y^A_1 \frac{\partial}{\partial y^A} + y^A_2 \frac{\partial}{\partial y^A_1} + ... + y^A_k \frac{\partial}{\partial y^A_{k-1}} + y^A_{k+1} \frac{\partial}{\partial y^A_k}$ as a function from $J^{k+1} Y$ to the tangent space $TJ^k Y$ and we define the differential forms $\theta^A_0 = \theta^A = dy^A - y^A_1 dx$, $\theta^A_j = dy^A_j - y^A_{j+1} dx$, for $j \neq 1$. For any section s we have $j^k s^*\left(\theta^A_j\right) = 0$, and $\theta^A_j\left(D_x\right) = 0$. Using this formalism, the lagrangian is a differential form on

J^kY which in canonical local coordinates is expressed as

$$\mathrm{Lag} = L\left(x, y^A, y_1^A, ..y_k^A\right)dx \tag{2}$$

One proves that the following form is well defined on $J^{2k-1}Y$, independent of the local coordinates

$$\begin{aligned}\theta_{\mathrm{Lag}} &= Ldx + \sum_{A=1}^{N}\sum_{j=1}^{k}\sum_{m=0}^{j-1}(-1)^m D_{x,m}\frac{\partial L}{\partial y_j^A}\cdot\theta_{j-m-1}^A \\ &= Ldx + \sum_{A=1}^{N}\sum_{b=0}^{k-1}\sum_{m=0}^{k-b-1}(-1)^m D_{x,m}\frac{\partial L}{\partial y_{m+b+1}^A}\cdot\theta_b^A\end{aligned} \tag{3}$$

and is called the Poincaré-Cartan form of the lagrangian. The Euler-Lagrange equations give a flow $F_\tau : J^{2k-1}Y \to J^{2k-1}Y$ for $\tau \in R$ enough small in the following way: the initial data $\left(x, y^1(x), y^2(x), ..y^N(x), \frac{dy^1(x)}{dx}, \frac{dy^2(x)}{dx}, ..\frac{dy^N(x)}{dx}, ...\frac{d^{2k-x}y^N(x)}{dx^k}\right)$ are transformed by F_τ to $\left(x+\tau, y^1(x+\tau), y^2(x+\tau), ..y^N(x+\tau), \frac{dy^1(x+\tau)}{dx}, \frac{dy^2(x+\tau)}{dx}, ..\frac{dy^N(x+\tau)}{dx}, ...\frac{d^{2k-x}y^N(x+\tau)}{dx^k}\right)$ and the following conservation holds $F_\tau^*\left(d\theta_{\mathrm{Lag}}\right) = d\theta_{\mathrm{Lag}}$. This invariant form with respect to F_τ is called the symplectic form associated to the lagrangian. One finds this sort of theories in [1]-[4].

2. Conservative numerical schemes

A numerical scheme for a dynamical system is called conservative if something is preserved from a time step to the next time step. We shall explain according to [1] and [4] how this works for the Euler-Lagrange equations. Let $y = \left(y^1, y^2, ...y^N\right)$, $y' = \left(y^{1\prime}, y^{2\prime}, ...y^{N\prime}\right)$, and so on. The Euler-Lagrange equations are of order 2k so the solution is uniquely determined by $\left(x, y, y', y'', ...y^{(2k-1)}\right) \in J^{2k-1}Y$, the Cauchy data at x. But the solution is uniquely determined by to points $q_0 = \left(x_0, y_0, y_0', y_0'', ...y_0^{(k-1)}\right) \in J^{k-1}Y$ and $q_1 = \left(x_1, y_1, y_1', y_1'', ...y_1^{(k-1)}\right)$ if they are not too far each other [4]. Let the trajectory passing by q_0 and q_1 with $x_1 - x_0 = h$ and let q_2 on the trajectory with $x_2 - x_1 = h$. Let $S_e(q_0, q_1, h) = \int_{x_0}^{x_1} L(x, y(x), y'(x), ...)dx$ be the action along the trajectory connecting q_0, q_1. Analogously one defines the action with respect to any two points in $J^{k-1}Y$. If we take the trajectory from q_0 to q_2 then the action is

$$S_e(q_0, q_2, 2h) = \int_{x_0}^{x_2} L(x, y(x), y'(x), ...)dx = S_e(q_0, q_1, h) + S_e(q_1, q_2, h) \tag{4}$$

and for fixed endpoints q_0, q_2 the minimum of S_e implies

$$\frac{\partial S_e(q_0, q_1, h)}{\partial q_1} + \frac{\partial S_e(q_1, q_2, h)}{\partial q_1} = 0 \tag{5}$$

The equation (4) allows to pass from (q_0, q_1) to (q_1, q_2) that is from a pair of neighbors on the trajectory to the next pair of neighbors. The only problem is the fact that we do not know the solution and consequently we do not know the exact action $S_e(q_0, q_1, h)$. Here begins numerical approximations. From q_0 and q_1 we interpolate the trajectory and find $y_{q_0,q_1}(x)$. For example we can use the Hermite interpolation with data to $x = x_0$ and $x = x_1$. Thereafter

we approximate the integral

$$\int_{x_0}^{x_1} L\left(x, y_{q_0,q_1}(x), y'_{q_0,q_1}(x),...\right)dx \approx I_a\left(L\left(x, y_{q_0,q_1}(x), y'_{q_0,q_1}(x),...\right)\right) = S_d(q_0,q_1,h) \quad (6)$$

This approximate integral is the discrete action $S_d(q_0,q_1,h)$. Minimizing the discrete action

$$S_d(q_0,q_2,2h) = S_d(q_0,q_1,h) + S_d(q_1,q_2,h) \quad (7)$$

as in (4) with respect to q_1 for fixed q_0, q_2 we get the discrete counterpart of (5)

$$\frac{\partial S_d(q_0,q_1,h)}{\partial q_1} + \frac{\partial S_d(q_1,q_2,h)}{\partial q_1} = 0 \quad (8)$$

The discrete dynamics is the function $(q_0,q_1) \to (q_1,q_2)$ and we write $(q_1,q_2) = F_h(q_0,q_1)$, F_h being defined in an open set of $J^{k-1}Y \times J^{k-1}Y$ of pairs of points which define uniquely the real trajectory. We define

$$\Theta_d^+(q_0,q_1,h) = \frac{\partial}{\partial q_1} S_d(q_0,q_1,h)dq_1,\ \Theta_d^-(q_0,q_1,h) = -\frac{\partial}{\partial q_0} S_d(q_0,q_1,h)dq_0 \quad (9)$$

It follows

$$dS_d = \Theta_d^+ - \Theta_d^- \quad (10)$$

and the form

$$\Omega_d = d\Theta_d^+ = d\Theta_d^- \quad (11)$$

Is well defined by Θ_d^+ or by Θ_d^+. We have from (8)

$$\begin{aligned} &d\left(S_d(q_0,q_1,h) + F_h^* S_d(q_0,q_1,h)\right) = dS_d(q_0,q_1,h) + dS_d(q_1,q_2,h) \\ &= \frac{\partial}{\partial q_0} S_d(q_0,q_1,h)dq_0 + \frac{\partial}{\partial q_2} S_d(q_1,q_2,h)dq_2 \\ &= \left(F_h^* \Theta_d^+\right)(q_0,q_1,h) - \Theta_d^-(q_0,q_1,h) \end{aligned} \quad (12)$$

Taking exterior derivative we get

$$F_h^* \Omega_d = \Omega_d \quad (13)$$

We have the following result

Proposition 1. *Let a generic approximation of a function* $y:[x_0,x_1]$, $x_2 - x_1 = h$, *by* $y_{q_0,q_1}(x)$ *knowing* $q_0 = \left(x_0, y(x_0), y'(x_0),..y^{(k-1)}(x_0)\right)$ *and* $q_1 = \left(x_1, y(x_1), y'(x_1),..y^{(k-1)}(x_1)\right)$ *and let* $S_d(q_0,q_1,h) = I_a\left(L\left(x, y_{q_0,q_1}(x), y'_{q_0,q_1}(x),...\right)\right) \approx \int_{x_0}^{x_1} L\left(x, y_{q_0,q_1}(x), y'_{q_0,q_1}(x),...\right)dx$, *where* I_a *is a formula for approximate integral (trapezoidal rule for example). Then starting from two points* q_0, q_1 *there it exists a unique solution* $y(x)$ *of the Euler-Lagrange equations such that the k jet of* $y(x)$ *passes trough* q_0, q_1. *Let* $x_0, x_1, x_2,...$ *a sequence of real points such that* $x_i - x_{i-1} = h$, *constant. Then starting from* (q_0,q_1) *the next pair of the points on the k jet of the solution can be approximated by* (q_1,q_2) *where* q_2 *is the solution of the nonlinear equation (8). This correspondence* $(q_0,q_1) \overset{F_h}{\to} (q_1,q_2)$ *preserves the differential form*

$$\Omega_d = \frac{\partial^2 S_d(q_0,q_1,h)}{\partial q_0 \partial q_1} dq_0 \wedge dq_1 \quad (14)$$

QED.

The benefit of such approach is double: if the approximate action S_d is enough close to the exact action S_e then the equation (8) is enough close to (5) and there is a possibility to determine the order of approximation of the numerical solution and on the other side the

conservation law (14) prevents the divergence of the numerical solution for a large number of steps of computation. In [4] one finds some examples.

3. Some polynomials useful for interpolation

To approximate by a polynomial a function $f:[a,b]\to R$ if we know $f(a),f'(a),f''(a)$, $f(b),f'(b),f''(b)$ we can use the Hermite interpolation: $P(x)=f(a)P_{a0}(x)+f'(a)P_{a1}(x)+f''(a)P_{a2}(x)+f(b)P_{b0}(x)+f'(b)P_{b1}(x)+f''(b)P_{b2}(x)$ where for a=0, b=1 we have

$$P_{00}(x)=1-x^3+3x^3(x-1)-6x^3(x-1)^2$$

$$P_{01}(x)=x-x^3+2x^3(x-1)-3x^3(x-1)^2$$

$$P_{02}(x)=\frac{1}{2}x^2-\frac{1}{2}x^3+\frac{1}{2}x^3(x-1)-\frac{1}{2}x^3(x-1)^2$$

$$P_{10}(x)=1+(x-1)^3+3(x-1)^3x+6(x-1)^3x^2$$

$$P_{11}(x)=(x-1)-(x-1)^3-2(x-1)^3x-3(x-1)^3x^2$$

$$P_{12}(x)=\frac{1}{2}(x-1)^2+\frac{1}{2}(x-1)^3+\frac{1}{2}(x-1)^3x+\frac{1}{2}(x-1)^3x^2$$

For interval the [0,h] the above polynomials are easy to translate. This kind of interpolation is useful for lagrangians $L(x,y,y',y'',y''')$.

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SYMMETRIES IN BUCKINGHAM-TYPE PROBLEMS

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Abstract: We study the symmetries that characterize the two-body problem associated to the potential proposed by R. Buckingham. We show that the corresponding vector fields in different coordinates present symmetries that form isomorphic Abelian groups endowed with an idempotent structure.
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Keywords: two-body problem, Buckingham potential, collision, escape, symmetries.

1. Basic equations of motion

The Buckingham potential is a function proposed by Richard Buckingham in a theoretical study of the equation of state for gaseous helium, neon and argon ([1]), which describes the Pauli repulsion energy and van der Waals energy for the interaction of two atoms that are not directly bonded, as a function of the interatomic distance r,

$$U(r)=A\exp(-Br)-(M/(r^6)),$$

where A, B and M are constants. Because the Buckingham potential is central, the associated two-body problem can be reduced to a central-force problem. The motion is confined to a plane, where we fix one particle as centre at the origin of this plane $\mathbb{R}^2$ and study the relative motion of the other particle.

We denote the position (or configuration) vector of this particle by $\mathbf{q}=(q_1,q_2)$ and the momentum vector by $\mathbf{p}=\dot{\mathbf{q}}$, $\mathbf{p}=(p_1,p_2)$. Then, the Buckingham potential is

$$U(\mathbf{q})=A\exp(-B|\mathbf{q}|)-(M/(|\mathbf{q}|^6)), \tag{1}$$

where A, B and M are constants and the kinetic energy of the unit-mass particle is

$$T(\mathbf{p})=((|\mathbf{p}|^2)/2).$$

The motion is described by the equations

$$\dot{\mathbf{q}}=((\partial H)/(\partial \mathbf{p})),\quad \dot{\mathbf{p}}=-((\partial H)/(\partial \mathbf{q})) \tag{2}$$

for the Hamiltonian

$$H(\mathbf{q},\mathbf{p}):=T(\mathbf{p})-U(\mathbf{q})=((|\mathbf{p}|^2)/2)-A\exp(-B|\mathbf{q}|)+(M/(|\mathbf{q}|^6)).$$

The phase space is $Q\times P$, where $Q=\mathbb{R}^2-\{(0,0)\}$ is the configuration space and $P=\mathbb{R}^2$ is the momentum space. For given initial conditions $(q_1,q_2,p_1,p_2)\in Q\times P$, the existence and uniqueness of a real analytic solution (q_1,q_2,p_1,p_2) of the system are ensured by standard results of the theory of differential equations. The equations of motion have an isolated singularity at the origin, which corresponds to a collision.

The Hamiltonian function is a first integral of the system of equations, the integral of energy:

$$H(\mathbf{q},\mathbf{p})=h/2=\text{const.}, \tag{3}$$

where h is the energy constant.

The field $U(\mathbf{q})$ being central, the angular momentum is conserved, hence we obtain another first integral

$$L(\mathbf{q},\mathbf{p})=q_1p_2-q_2p_1=C=\text{const.}, \tag{4}$$

where C stands for the constant of angular momentum.

We observe that the potential $U(\mathbf{q})$, the motion equations and the energy integral have an isolated singularity for $t=t^*$, at the origin $\mathbf{q}=(0,0)$. This singularity corresponds to a collision particle-centre (see [9]).

We shall apply a sequence of McGehee-type transformations of the second kind ([2], [3]) to remove and to regularize the above differential equations

$$r=|q|$$

$$\theta=arctg\left(\frac{q_2}{q_1}\right)$$

$$\xi=\dot{r}=\frac{q_1p_1+q_2p_2}{|q|}$$

$$\eta=r\dot{\theta}=\frac{q_1p_2-q_2p_1}{|q|}$$

We obtain (see [10]):

$$\begin{aligned}\dot{r}&=\xi\\ \dot{\theta}&=\eta/r\\ \dot{\xi}&=((\eta^2)/r)-ABexp(-Br)+6M(1/(r^7)\\ \dot{\eta}&=-((\xi\eta)/r),\end{aligned} \tag{5}$$

We denote

$$x=r^3\xi, \qquad y=r^3\eta.$$

Rescaling the time through

$$ds=r^{-4}dt$$

and writing $'=d/ds$, the equations of motion become:

$$\begin{aligned}r'&=rx\\ \theta'&=y\\ x'&=3x^2+y^2-ABr^7\,exp(-Br)+6M\\ y'&=2xy.\end{aligned} \tag{6}$$

The first integrals now read respectively

$$x^2+y^2=hr^6+2Ar^6\,exp(-Br)-2M \tag{7}$$

$$y=Cr^2. \tag{8}$$

2. Symmetries

Examining the model based on Buckingham potential, we observe that the vector field characterized by the equations of motion presents symmetries that form isomorphic Abelian groups endowed with an idempotent structure. This approach has done for various fields in [8], [5], [6], [4] and it is also valid for Buckingham potential. For Cartesian coordinates and standard polar coordinates we use [10].

The vector field corresponding to Cartesian coordinates given by (2) exhibits eight symmetries $S_i=S_i(q_1,q_2,p_1,p_2,t)$, $i=0,1,...,7$, as follows:

$$S_0=(q_1,q_2,p_1,p_2,t)=I$$
$$S_1=(q_1,q_2,-p_1,-p_2,-t)$$
$$S_2=(q_1,-q_2,p_1,-p_2,t)$$
$$S_3=(-q_1,q_2,-p_1,p_2,t)$$
$$S_4=(q_1,-q_2,-p_1,p_2,-t)$$
$$S_5=(-q_1,q_2,p_1,-p_2,-t)$$
$$S_6=(-q_1,-q_2,-p_1,-p_2,t)$$
$$S_7=(-q_1,-q_2,p_1,p_2,-t).$$

These symmetries map solution onto solution.

Theorem 2.1. The set $G=\{S_i|i=0,7\}$ (where $S_0=I$) is the identity), endowed with the composition law "$\circ$", forms a symmetric Abelian group with an idempotent structure.

Proof. To prove the affirmation of the theorem, it is easy to construct the composition table, observing that every element of G is its own inverse.

$(G,\circ)$ is an Abelian group of order eight with three generators of order two, S_1, S_2, S_3:

$$S_4=S_1\circ S_2$$
$$S_5=S_1\circ S_3$$
$$S_6=S_2\circ S_3$$
$$S_7=S_1\circ S_2\circ S_3.$$

Thus, G is isomorphic to $Z_2\times Z_2\times Z_2$.

Theorem 2.2. The group contains seven proper subgroups of order four, isomorphic to Klein's group.

Proof. Consider S_k, S_i, S_j $(\neq I)$ G such that

$$S_k=S_i\circ S_j,\quad i\neq j\neq k.$$

Composing both members with S_i or S_j, we get $S_i\circ S_k=S_j$ or $S_j\circ S_k=S_i$. Thus the set

$$(H)_{ijk}=\{(I, S_i, S_j, S_k)|S_k=S_i\circ S_j,\quad i\neq j\neq k \text{ holds}\},\tag{9}$$

endowed with the same composition law forms a proper subgroup of G, isomorphic to Klein's group. It is easy to check that the only combinations (i, j, k) that fulfil the condition (9) are

$$(i, j, k)\in\{(1,2,4), (1,3,5), (1,6,7), (2,3,6), (2,5,7), (3,4,7), (4,5,6)\},$$

which finishes the proof.

The vector field (5), which is in polar coordinates, has eight symmetries $S_i^{pol}=S_i^{pol}(r,\theta,\xi,\eta,t)$, $i=0,1,...,7$, as follows:

$$S_0^{pol}=(r,\theta,\xi,\eta,t)=I^{pol}$$
$$S_1^{pol}=(r,\theta,-\xi,-\eta,-t)$$
$$S_2^{pol}=(r,-\theta,\xi,-\eta,t)$$
$$S_3^{pol}=(r,\pi-\theta,\xi,-\eta,t)$$
$$S_4^{pol}=(r,-\theta,-\xi,\eta,-t)$$
$$S_5^{pol}=(r,\pi-\theta,-\xi,\eta,-t)$$
$$S_6^{pol}=(r,\pi+\theta,\xi,\eta,t)$$
$$S_7^{pol}=(r,\pi+\theta,-\xi,-\eta,-t).$$

Theorem 2.3. The set $G^{pol}=\{S_i^{pol}\,|i=0,7\}$ (where I^{pol} is the identity), endowed with the composition law "$\circ$", forms a symmetric Abelian group with an idempotent structure. G^{pol} is isomorphic to $Z_2\times Z_2\times Z_2$ and contains seven proper subgroups of order four, isomorphic to Klein's group.

Proof. To prove the first affirmation, a composition table, wholly similar to that used in Theorem 2.1, can be constructed. The proof of the last affirmation is similar to the proof of the Theorem 2.2.

The vector field (6), which is in collision-blow-up McGehee-type coordinates, presents seven symmetries, $\overline{S}_i=\overline{S}_i(r,\theta,x,y,s)$, $i=1,\ldots,7$.

Theorem 2.4. The set $\overline{G}=\{I\}\bigcup\{\overline{S}_i\,|i=\overline{1,7}\}$, ended with the same composition law "$\circ$", forms a symmetric Abelian group with an idempotent structure, isomorphic to $Z_2\times Z_2\times Z_2$ and contains seven proper subgroups of order four, isomorphic to Klein's group.

Proof. The statement results making the change of notation

$$\overline{S}_i(r,\theta,x,y,s)=S_i^{pol}(r,\theta,\xi,\eta,t),\ i=0,1,...,7$$

and imitating the proofs of the above theorems.

These symmetries form isomorphic and diffeomorphic 8-elements Abelian groups with idempotent structure.

Theorem 2.5. The groups G, G^{pol}, and $\overline{G}$ are diffeomorphic each other.

Proof. Considering the real analytic diffeomorphisms

$$\mathrm{R}^2-\{(0,0)\}\times\mathrm{R}^3\to(0,\infty)\times S^1\times\mathrm{R}^3\to(0,\infty)\times S^1\times\mathrm{R}^3,$$
$$(q_1,q_2,p_1,p_2,t)\to(r,\theta,\xi,\eta,t)\to(r,\theta,x,y,s).$$

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BIFURCATION ANALYSIS OF NATURAL CONVECTION FLOWS - MATHEMATICAL MODELLING

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Abstract: This paper presents the mathematical modeling used to investigate by numerical simulation the stability of steady natural convective flows in cavities having different but uniform temperatures on the sidewalls and adiabatic walls for the top and bottom boundaries. The fluid under current investigation was considered incompressible and Newtonian along with the Boussinesq approximation. Boundary flow velocity is assumed to be of the no-slip type for the entire domain. For the analyzed configuration, the convective flow generated by vertical density gradients gives rise to unexpectedly variety of flow patterns that can exist simultaneously for specific or critical Grashof and Rayleigh numbers.
Mathematics Subject Classification (2010): 35Q79, 65N30, 65N08.
Key words: partial differential equations, heat transfer, numerical analysis, finite element method.

1. Introduction

The study of natural convection flows in enclosures is motivated by many heat transfer engineering applications such as solar energy collectors, passive energy storage, double glass façade or double pane windows as part of the general objective for energy efficient design of buildings. When a fluid is heated from below, the flow exhibits a very strong and complex nonlinear behavior. Initially, the flow is and remains symmetrical, with only one symmetric steady-state solution but, as a controlling parameter (Grashof or Rayleigh number) is increased beyond a critical value, it could become asymmetrical and the system bifurcates from one fixed-point solution to the two stationary solutions. This phenomenon of breaking flow and thermal symmetry, named pitchfork bifurcation, is influenced by the cavity geometry, thermo-physical properties of the fluid and thermal boundary conditions.

The Navier-Stokes equations are a set of coupled differential equations and could, in theory, be solved for a given flow problem by using methods from calculus. However, in practice, these equations are too difficult to be analytically solved. In the past, engineers made further approximations and simplifications to the equation set until they had a group of equations that could be solved. Recently, high speed computers have been used to solve approximations to the equations using a variety of techniques like finite difference, finite

volume, finite element, and spectral methods. This area of study is called Computational Fluid Dynamics [1].

2. Numerical methods for solving partial differential equations

Finite difference method is quite simple, easy to program and does not consume large computing resources. Initially, the method was used on orthogonal grid discretized areas and was difficult for treating curved areas. The development of the method for non-orthogonal network has complicated the situation, virtually negating initial advantages.

Finite element method [2] has applications in various fields of engineering (and not only) where physical phenomena are described by partial differential equations. Among the main areas using this method could be mentioned the structural mechanics, fluid flows, heat transfer, magnetic and electrical fields. There are three ways of formulating the finite element method: direct, variational and residual formulation.

Direct formulation is based on the matrix calculation of structures with displacement method.

Variational formulation is based on minimization of potential energy of the deformable solid, under a stationary potential energy criterion. Variational methods used in deformable solid mechanics use the principle of virtual work or energy theorems such as the minimum potential energy theorem (formulation trips), minimum complementary energy formulation (formulation voltages), Hellinger-Reissner theorem (mixed formulation in stress and strain) and Hamilton's theorem for dynamical problems. In the case of minimum potential energy theorem, the deformable solid body is meshed into finite elements and hypothetical field of trips inside each element is modeled using polynomial interpolation. By minimizing the potential energy of the deformable solid under a stationary principle, we obtain equilibrium elastic nodal equations system. Solving this equations system, displacements, strains and tensions of the deformable solids are obtained.

Residual formulation may be used if a functional form does not exists. It is a more general formulation than the variational formulation. To formulate residual finite element method least squares, Galerkin method or co-location method can be used.

Finite volume method uses small cells where the equations describing transport phenomena are solved for each control volume and the conservation of properties is required when passing through the area of the control volume. The amount of property coming out from a control volume falls (is transported) to the adjacent control volume. As a result, there is no generation or arbitrary disposal of properties. The method is resource consuming because the transported quantities through a joint surface should be equal in both adjacent volumes but having the opposite sign.

Finite volume method, as well as finite difference method and finite element method, is based on a numerical procedure which represents and evaluates the partial differential equations in the form of algebraic equations.

The computational domain V_h is divided into non-overlapping cells or finite volumes [3]

$$V_r, r = 1,..., N, \quad V_h = \bigcup_r V_r .$$

Usually these cells are polygons (triangles, quadrilaterals) in 2-D and polyhedra (tetrahedron, hexahedron, prism) in 3-D. The cell average value, which is the basic unknown quantity in the finite volume method, is introduced over the cell volume, V_r

$$u_r = \frac{1}{|V_r|}\int_{V_r} u(x)dx,$$

Let us consider

$$N(r) = \{Set\ of\ cells\ which\ share\ a\ common\ face\ with\ V_r\}.$$

The integral conservation law for cell V_r is

$$|V_r|\frac{du_r}{dt} + \sum_{s\in N(r)} \int_{V_r \cap V_s} f_i n_i ds = 0.$$

It remains to approximate the flux integral. This can be achieved using Gaussian quadrature. Taking p Gaussian points, we get

$$\int_{V_r \cap V_s} f_i n_i ds = \Delta s_{rs} \sum_{m=1}^{p} \omega_m F_{rs}^m ,$$

where ω_m are the Gaussian weights and F is an approximation to $f_i n_i$.

In all three above methods, the unknown values are ultimately calculated in the geometric mesh network nodes with the scope of interest.

3. Mathematical model

Consider the convective flow of a Boussinesq fluid in a two-dimensional rectangular cavity of width, δ and height, h. All the boundaries are rigid (no-slip velocity conditions imposed). The two vertical boundaries of the cavity have constant temperatures T_{hot} and T_{cold}, while the horizontal ones are adiabatic walls (perfectly thermally insulated). The geometry is fully symmetric and the flow is symmetrical with respect to the centre of the domain ($x=\delta/2$, $y=h/2$)

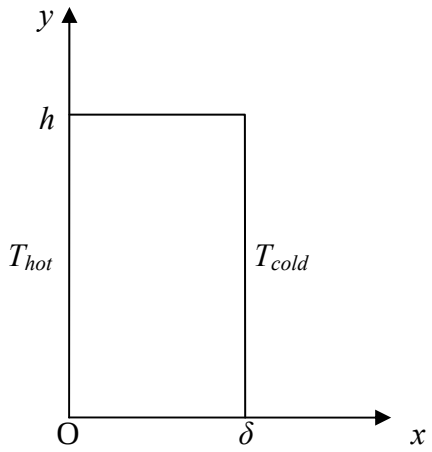

Fig.1 Geometry of the problem

Fundamental laws can be used to derive governing differential equations of fluid flow and heat transfer: conservation of mass, conservation of linear momentum (Newton's second law) and conservation of energy (first law of thermodynamics). Then the set of equations for velocity $\vec{w} = \{w_x, w_y\}$ [m/s], temperature T [K] and pressure p [N/m^2] is described by continuity, the momentum and energy equations. These equations are used for the analysis of natural convection behavior [4], [5], [6].

The continuity equation

$$\nabla \cdot \vec{w} = 0 . \qquad (1)$$

where ∇ is the nabla or Hamiltonian operator defined in cartesian coordinates as:

$$\nabla \equiv \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}$$

The momentum equation

$$\frac{\partial \vec{w}}{\partial t} + (\vec{w} \cdot \nabla)\vec{w} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \vec{w} + \vec{f}\,. \tag{2}$$

where t is the time [s], ρ is the density [kg/m^3], ν is the constant kinematic viscosity [m^2/s] and $\vec{f}$ is the body force per unit mass. If the only body force is the weight of the fluid, $\vec{f}$ can be witten as:

$$\vec{f} = \vec{g}\beta(T - T_{cold})\,,$$

where $\vec{g}$ is the acceleration due to gravity [m/s^2] and β is the thermal expansion coefficient [K^{-1}].

$\frac{\partial}{\partial t}$ is called the local derivative, which is physically the time rate of change at a fixed point, $\vec{w} \cdot \nabla$ is called the convective derivative, which is physically the time rate of change due to the movement of the fluid element from one location to another in the flow field where the flow properties are spatially different, $\nabla^2 \vec{w}$ is called the diffusion term, where ∇^2 is the Laplace operator defined in cartesian coordinates as:

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\,,$$

The energy equation

$$\rho c_p \left[\frac{\partial T}{\partial t} + (\vec{w} \cdot \nabla)T \right] = k \nabla^2 T\,. \tag{3}$$

where c_p is the specific heat at constant pressure [J/kgK] and k is the thermal conductivity of the fluid [W/mK].

No-slip boundary conditions for velocity are imposed at the contact between the fluid and the walls as follows:

$$\begin{aligned} \vec{w}(x = 0, 0 \le y \le h) &= \vec{0} \\ \vec{w}(x = \delta, 0 \le y \le h) &= \vec{0} \\ \vec{w}(y = 0, 0 \le x \le \delta) &= \vec{0} \\ \vec{w}(y = h, 0 \le x \le \delta) &= \vec{0} \end{aligned}\,. \tag{4}$$

Horizontal boundaries are thermally insulated:

$$\begin{aligned} \frac{\partial T}{\partial y}(y = 0, 0 \le x \le \delta) &= 0\,, \\ \frac{\partial T}{\partial y}(y = h, 0 \le x \le \delta) &= 0\,. \end{aligned} \tag{5}$$

Vertical boundaries are isothermal:

$$\begin{aligned} T(x = 0, 0 \le y \le h) &= T_{hot} \\ T(x = \delta, 0 \le y \le h) &= T_{cold}\,, \end{aligned} \tag{6}$$

The Navier-Stokes equations (1) – (3) describe the motion of fluid substances and establish that changes in momentum infinitesimal volumes of fluid are simply the sum of dissipative viscous forces, changes in pressure, gravity and other forces acting on the fluid.

It is possible to write governing equations in non-dimensional form. To do this, we need to select characteristic quantities that describe the flow problem (such as a characteristic length, δ and characteristic temperature, T_{cold}). Note that these non-dimensionalisations are

not unique. Using these characteristic quantities, the following non-dimensional parameters can be defined as:

$$\begin{gathered} x^* = \frac{x}{\delta},\ y^* = \frac{y}{\delta},\ t^* = \frac{t}{\delta^2/\nu}, \\ \vec{w}^* = \frac{\vec{w}}{\nu/\delta},\ p^* = \frac{p}{\rho(\nu/\delta)^2}, \\ T^* = \frac{T - T_{cold}}{T_{hot} - T_{cold}}, \\ \nabla^* = \delta \cdot \nabla, \\ \nabla^{*2} = \delta^2 \cdot \nabla^2. \end{gathered} \tag{7}$$

Using the above substitutions, the equations (1) – (3) can be written as follows.
Equation $\nabla \cdot \vec{w} = 0$ becomes

$$\frac{1}{\delta}\nabla^* \cdot \left(\frac{\nu}{\delta}\vec{w}^*\right) = 0.$$

Therefore,

$$\nabla^* \cdot \vec{w}^* = 0. \tag{8}$$

The momentum equation, writes as

$$\frac{\nu^2}{\delta^3}\frac{\partial \vec{w}^*}{\partial t^*} + \frac{\nu^2}{\delta^3}(\vec{w}^* \cdot \nabla^*)\vec{w}^* = -\frac{1}{\rho}\rho\frac{\nu^2}{\delta^3}\nabla^* p^* + \nu\frac{\nu}{\delta^3}\nabla^{*2}\vec{w}^* + \vec{g}\,\beta\,\Delta T\,T^*. \tag{9}$$

Simplifying this equation and using the Grashof number

$$\mathrm{Gr} = \frac{g\beta(T_{hot} - T_{cold})\delta^3}{\nu^2},$$

the nondimensional equation of momentum becomes:

$$\frac{\partial \vec{w}^*}{\partial t^*} + (\vec{w}^* \cdot \nabla^*)\vec{w}^* = -\nabla^* p^* + \nabla^{*2}\vec{w}^* + \mathrm{Gr}T^*\vec{e}_y, \tag{10}$$

where $\vec{e}_y$ the unit vector in the vertical direction.

The Grashof number is a dimensionless number in fluid dynamics and heat transfer which approximates the ratio of the buoyancy to viscous force acting on a fluid. It frequently arises in the study of situations involving natural convection.

Since $\frac{\partial T}{\partial t} + (\vec{w} \cdot \nabla)T = a\nabla^2 T$, we obtain

$$\frac{\partial T^*}{\partial t^*} + (\vec{w}^* \cdot \nabla^*)T^* = \frac{a}{\nu}\nabla^{*2}T^*. \tag{11}$$

Using the Prandtl number, $\mathrm{Pr} = \frac{\nu}{a}$, where $a = \frac{k}{\rho c_p}$ is the thermal diffusivity [m^2/s], the nondimensional equation of energy becomes:

$$\frac{\partial T^*}{\partial t^*} + (\vec{w}^* \cdot \nabla^*)T^* = \frac{1}{\mathrm{Pr}}\nabla^{*2}T^*. \tag{12}$$

The nondimensional boundaries conditions are

$$\begin{aligned}
&\vec{w}^*(x^* = 0,\ 0 \le y^* \le A) = \vec{0} \\
&\vec{w}^*(x^* = 1,\ 0 \le y^* \le A) = \vec{0} \\
&\vec{w}^*(y^* = 0,\ 0 \le x^* \le 1) = \vec{0} \\
&\vec{w}^*(y^* = A,\ 0 \le x^* \le 1) = \vec{0}
\end{aligned} \quad . \tag{13}$$

Horizontal boundaries are thermally insulated

$$\begin{aligned}
&\frac{\partial T^*}{\partial y^*}(y^* = 0, 0 \le x^* \le 1) = 0\,, \\
&\frac{\partial T^*}{\partial y^*}(y^* = A, 0 \le x^* \le 1) = 0\,.
\end{aligned} \tag{14}$$

Vertical boundaries are isothermal

$$\begin{aligned}
&T^*(x^* = 0, 0 \le y^* \le A) = 1 \\
&T^*(x^* = 1, 0 \le y^* \le A) = 0\,,
\end{aligned} \tag{15}$$

where A is the ratio h/δ.

4. Conclusions

Mathematical modeling is the first step for numerically solving the unsteady two-dimensional Navier-Stokes equations. Using finite element method, an analysis of the flow field and temperature distribution will be made for different geometries differentially heated over the vertical walls, concerning the stability of the steady laminar natural convection for several fluids. For each studied case, the critical Grashof number for which the flow looses stability and the temperature distribution becomes asymmetrically will be determined. Special attention will be given to the grid points and grid refinement to check whether the different flow behavior is physical or just of numerical nature.

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A COLLECTIVELY COINCIDENCE RESULT IN ALGEBRAIC TOPOLOGY AND ITS APPLICATION

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Abstract. In this paper we will give a *collectively coincidence result*, which has as the starting point, the Begle's Fixed-point Theorem for multimaps. Hence we will work in the *absence of convexity* assumptions. We will also give an *application* in the theory of the *general equilibrium*.
Mathematics Subject Classification: 54H25, 47H10.
Key words: multimap, coincidence theorem, acyclicity, ANR.

1. Preliminaries

Coincidence theory is, in most settings, a generalization of fixed-point theory. Coincidence points, like fixed-points, are today studied using many tools from mathematical analysis and topology, and have many applications in the game theory, optimization theory and economics.

Firstly, we recall some definitions and notations, reminding few of the mathematical concepts that we need. For a nonempty set X, 2^X denotes the class of all subsets of X.

Definition 1. If X and Y are two nonempty sets and $T:Y\to 2^X$ and $S:X\to 2^Y$ are nonempty-valued multimaps, an element $(x,y)\in X\times Y$ is called a *coincidence point* for T and S, if $x\in T(y)$ and $y\in S(x)$.

Definition 2. Let I be an index set and let $(X_i)_{i\in I}$, $(Y_i)_{i\in I}$ be two families of nonempty sets. Denote $X=\prod_{i\in I}X_i$, $Y=\prod_{i\in I}Y_i$. Let also $(T_i)_{i\in I}$ and $(S_i)_{i\in I}$ be two families of nonempty-valued multimaps with $T_i:Y\to 2^{X_i}$ and $S_i:X\to 2^{Y_i}$. An element $(x,y)\in X\times Y$, with $x=(x_i)_{i\in I}$ and $y=(y_i)_{i\in I}$ is called a *collectively coincidence point* for the families $(T_i)_{i\in I}$ and $(S_i)_{i\in I}$, if $y_i\in S_i(x)$ and $x_i\in T_i(y)$, for all $i\in I$.

A brief history of the theory of coincidence points can be found, for example, in [4].

The following theorem appeared in [2], as Theorem 3.1, p. 591.

Theorem 1. *Let I be an arbitrary index set, and for each $i \in I$ let $X_i \subseteq E_i$ and $Y_i \subseteq F_i$ be nonempty convex sets in some Fréchet spaces. Let also $X = \prod_{i\in I} X_i$, $Y = \prod_{i\in I} Y_i$ and $S_i : X \to 2^{Y_i}$, $T_i : Y \to 2^{X_i}$ ($i \in I$) two families of nonempty-valued and convex-valued multimaps. Suppose that:*

1) *for each $i \in I$, X can be covered with the interiors of all fibers of S_i, i.e.*

$$X = \bigcup_{y_i \in Y_i} \operatorname{int} S_i^{-1}(y_i)$$

and Y can be covered with the interiors of all fibers of T_i, i.e.

$$Y = \bigcup_{x_i \in X_i} \operatorname{int} T_i^{-1}(x_i);$$

2) *for each $i \in I$, S_i and T_i are compact, i.e. there exist two nonempty compact subsets $K_i \subseteq X_i$ and $L_i \subseteq Y_i$ such that $S_i(x) \subseteq L_i$ and $T_i(y) \subseteq K_i$, for all $x \in X$ and $y \in Y$.*

Then there exist $\tilde{x} = (\tilde{x}_i)_{i\in I} \in X$ and $\tilde{y} = (\tilde{y}_i)_{i\in I} \in Y$ such that $\tilde{x}_i \in T_i(\tilde{y})$ and $\tilde{y}_i \in S_i(\tilde{x})$, for each $i \in I$, that is, $(\tilde{x}, \tilde{y})$ is a collectively coincidence point for the families $(S_i)_{i\in I}$ and $(T_i)_{i\in I}$.

Proof. Let us define, for each $i \in I$, the multimap $U_i : X \times Y \to 2^{Y_i \times X_i}$ by the following formula $U_i(x, y) = S_i(x) \times T_i(y)$, $\forall x \in X$, $y \in Y$.

From the hypothesis "1)" it follows that, for each $i \in I$, we have:

$$X \times Y = \left(\bigcup_{y_i \in Y_i} \operatorname{int} S_i^{-1}(y_i) \right) \times \left(\bigcup_{x_i \in X_i} \operatorname{int} T_i^{-1}(x_i) \right) \subseteq \bigcup_{(y_i, x_i) \in Y_i \times X_i} \left(\operatorname{int} S_i^{-1}(y_i) \times \operatorname{int} T_i^{-1}(x_i) \right) \subseteq$$

$$\subseteq \bigcup_{(y_i, x_i) \in Y_i \times X_i} \operatorname{int} \left(S_i^{-1}(y_i) \times T_i^{-1}(x_i) \right) \subseteq X \times Y.$$

Hence

$$X \times Y = \bigcup_{(y_i, x_i) \in Y_i \times X_i} \operatorname{int} U_i^{-1}(y_i, x_i). \tag{1}$$

Now, by using the hypothesis "2)" we obtain that the operator

$$U_i : X \times Y \to 2^{Y_i \times X_i} \text{ is compact.} \tag{2}$$

By applying Lemma 1 below, we obtain the existence of a point $(\tilde{x}, \tilde{y}) \in X \times Y$, such that $\tilde{x} = (\tilde{x}_i)_{i\in I}$, $\tilde{y} = (\tilde{y}_i)_{i\in I}$ and $(\tilde{y}_i, \tilde{x}_i) \in U_i(\tilde{x}, \tilde{y}) = S_i(\tilde{x}) \times T_i(\tilde{y}) \Rightarrow \tilde{y}_i \in S_i(\tilde{x})$ and $\tilde{x}_i \in T_i(\tilde{y})$. □

Lemma 1. (see [2, Theorem 2.1]) *Let I be an arbitrary index set, and for each $i \in I$ let $X_i \subseteq E_i$ be a nonempty convex set in a Fréchet space. Let also $X = \prod_{i\in I} X_i$, and $T_i : X \to 2^{X_i}$ ($i \in I$) a nonempty-valued and convex-valued multimap. Suppose that:*

1) *for each $i \in I$, X can be covered with the interiors of all fibers of T_i, i.e.*

$$X = \bigcup_{y_i \in X_i} \operatorname{int} T_i^{-1}(y_i);$$

2) *for each $i \in I$, T_i is compact, that is, there exists $K_i \subseteq X_i$ a nonempty compact subset such that $T_i(x) \subseteq K_i$, for each $x \in X$.*

Then there exists $\tilde{x} = (\tilde{x}_i)_{i \in I} \in X$ *, with* $\tilde{x}_i \in T_i(\tilde{x})$*, for each* $i \in I$ (*hence* $\tilde{x}$ *is a collectively fixed-point for the family* $(T_i)_{i \in I}$).

In the previous results we can see that the *convexity* of some sets is essential. But when these sets fail to be convex, some problems arise. In this case we can use *other convexity type concepts.* In this paper we will use the *acyclicity* concept from the Algebraic Topology and we also use the *ANR concept for a class* $\mathcal{C}$ of topological spaces (usually, $\mathcal{C}$ is the class of metric spaces, when we speak of *ANR's* and do not specify $\mathcal{C}$).

Speaking informally (for details, see [3]) we can say that:
1) a *topological space* is said to be *acyclic* if its homology groups in all dimensions are the same as those of a point, for any homology theory (for us, for the Cěch homology); a subset of a topological space is *acyclic* if is an acyclic topological space with the induced topology.
2) an *ANR* (*'Absolute Neighborhood Retract'*) *for the class* $\mathcal{C}$ is a space Y from $\mathcal{C}$, such that, whenever Y is a closed subset of a space X from $\mathcal{C}$, then Y is a *'Neighborhood Retract'* of X, that is, there exists an extension of the inclusion $i: Y \to X$, to a neighborhood U of Y in X (i.e. U contains an open set in X that itself contains Y).

Concerning the notion of *acyclicity* (which is a concept weaker than the *convexity*) we notice more precisely the following definition: let X be a topological space and denote by $\check{H}^P(X, \mathbb{Z})$ its reduced Cěch homology group of X, in dimension p, with coefficients in $\mathbb{Z}$.

Then X is *acyclic* if $\check{H}^P(X, \mathbb{Z}) = 0$, for every $p \geq 0$. As examples of acyclic spaces we cite $\mathbf{R}^n$ and the R_δ-sets (see [3]).

Concerning the notion of *ANR spaces* we notice that the problem of extending a continuous map $f: Y \to Z$ (Z a topological space), from a *closed* subset Y of a topological space X to *all* of X, or *at least* to some *neighborhood* U of Y in X is very encountered in Topology. Karol Borsuk realized that the particular case when $Z = X$ and f is the inclusion $i: Y \to X$ deserves special attention. In his PhD thesis ("On retractions and related sets"), defended in 1930 at the University of Warsaw, Borsuk introduced and studied these basic notions as well as the *topologically invariant* notion of *'Absolute Retract'*. He thus laid the foundations of the theory of retracts. The very suggestive term *'Retract'* was proposed by Stefan Mazurkiewicz (1888-1945), who was Borsuk's PhD supervisor. The term *'Absolute Retract'* was suggested by Borsuk's colleague Nachman Aronszajn, also a student of Mazurkiewicz. It appears that the original Borsuk's thesis has been lost in the Second World War. *'Absolute Neighborhood Retracts'* (abbreviated as *ANR*) were introduced by Borsuk, in 1932.

The following result is a version of Lemma 1, a collectively fixed-point result, in Algebraic Topology and extends the Begle's Fixed-point Theorem to a finite family of multimaps (for the statement of the *Begle's Fixed-point Theorem*, see below, in the proof of Lemma 2).

Lemma 2. *Let* $I = \{1, ..., n\}$ *and* $(X_i)_{i \in I}$ *be a (finite) family of topological spaces. Assume that for all* $i \in I$, X_i *is a compact acyclic and finite dimensional ANR. Let* $X = \prod_{i=1}^{n} X_i$ *and for all*

$i \in I$, let $R_i : X \to 2^{X_i}$ be a nonempty-valued and acyclic-valued multimap with closed graph, i.e. the set $graph\, R_i = \{(x, x_i) \in X \times X_i \,|\, x_i \in R_i(x)\}$ is closed in the product topology of $X \times X_i$. Then there exists $\tilde{x} = (\tilde{x}_i)_{i \in I} \in X$ such that for all $i \in I$, $\tilde{x}_i \in R_i(\tilde{x})$, that is, $\tilde{x}$ is a collectively fixed-point for the family $(R_i)_{i=\overline{1,n}}$.

Proof. Firstly we remark that X is a compact, acyclic and finite dimensional ANR.
Consider $T : X \to 2^X$, $x = (x_i)_{i \in I} \mapsto T(x) = \underbrace{R_1(x)}_{\subseteq X_1} \times ... \times \underbrace{R_n(x)}_{\subseteq X_n} \subseteq X$. Because R_i is nonempty-valued and acyclic-valued for all $i \in I$, it follows that T has the same properties. Moreover, the graph of T is closed ($graph T = \{(x, y) \in X \times X \mid y \in T(x)\}$).
Now we will use the Begle's Fixed-point Theorem for multimaps: "*Let X be a compact acyclic finite-dimensional ANR. We suppose that $T : X \to 2^X$ is a multimap with closed graph (will we say, equivalently, that T is closed), that is, the set $graph T = \{(x, y) \in X \times X \,|\, y \in T(x)\}$ is closed in the product space $X \times X$. Moreover, we assume that T is nonempty-valued and acyclic-valued (or, equivalently, T has nonempty and acyclic values), i.e. for all $x \in X$, the set $T(x) \subseteq X$ is nonempty and acyclic. Then T has a fixed-point, that is, there exists $\tilde{x} \in X$ with $\tilde{x} \in T(\tilde{x})$.*" (for the proof of this theorem see, for example, [1]). It follows that there exists $\tilde{x} \in X$ such that $\tilde{x} \in T(\tilde{x})$. Denoting $\tilde{x} = (\tilde{x}_i)_{i \in I}$ we obtain that $\tilde{x}_i \in R_i(\tilde{x})$ for all $i \in I$.□

2. New result

Now we will give a new collectively coincidence result for two families of multimaps, without convexity assumptions.

Theorem 2. *Let $I = \{1, ..., n\}$ be an index set and $(X_i)_{i \in I}$, $(Y_i)_{i \in I}$ two families of compact acyclic finite dimensional ANRs. Denote $X = \prod_{i \in I} X_i$ and $Y = \prod_{i \in I} Y_i$. For each $i \in I$, let $S_i : X \to 2^{Y_i}$ and $T_i : Y \to 2^{X_i}$ be nonempty-valued and acyclic-valued multimaps having closed graphs (that is, the sets $graph\, S_i = \{(x, y_i) \in X \times Y_i \mid y_i \in S_i(x)\}$ and $graph\, T_i = \{(y, x_i) \in Y \times X_i \mid x_i \in T_i(y)\}$ are closed in the product topological spaces $X \times Y_i$ and $Y \times X_i$, respectively).*
Then there exist $(\tilde{x}, \tilde{y}) \in X \times Y$, $\tilde{x} = (\tilde{x}_i)_{i \in I}$ and $\tilde{y} = (\tilde{y}_i)_{i \in I}$ such that for all $i \in I$,

$$\tilde{x}_i \in T_i(\tilde{y}) \text{ and } \tilde{y}_i \in S_i(\tilde{x}),$$

that is, $(\tilde{x}, \tilde{y})$ is a collectively coincidence point for the families $(S_i)_{i \in I}$ and $(T_i)_{i \in I}$.

Proof. For all $i \in I$, define $U_i : X \times Y \to 2^{Y_i \times X_i}$ by the following formula

$$U_i(x, y) = S_i(x) \times T_i(y), \ \forall x \in X, \ y \in Y .$$

We will use Lemma 2 for U_i instead of R_i (and $X \times Y$ instead of X). For this we will go through the following steps:

1) Because X_i, Y_i are compact, then X and Y are compact, too. Consequently, $X \times Y$ is compact.
2) Also X, Y are acyclic spaces and therefore $X \times Y$ is acyclic, because the acyclicity is product-closed. Indeed in [8, see the proof of Theorem 1] and [9, before Lemma 3.1] was stated that, *a product of two acyclic sets is an acyclic set by the Künneth formula.* To recall this formula, notice that we can compute the singular homology of a product space $X \times Y$, with coefficients in a field F, by using the singular homology of X and Y, respectively:

$$H_n(X \times Y; F) \cong \underset{i+j=n}{\oplus} H_i(X; F) \otimes H_j(Y; F), \tag{3}$$

for all $n \in N^*$, though in general there are *torsion terms* that appear. More precisely, Künneth formula says that there is a cross product operation by which an i-cycle on X and a j-cycle on Y can be combined to create an $(i+j)$-cycle on $X \times Y$. This means that we have *chain maps* $F : C_*(X \times Y) \to C_*(X) \otimes C_*(Y)$ and $G : C_*(X) \otimes C_*(Y) \to C_*(X \times Y)$ such that $F \circ G$ is the identity and $G \circ F$ is *chain-homotopic* to the identity (see [6] for details).
3) The spaces X, Y and $X \times Y$ are finite dimensional ANRs because the product of a finite family of ANRs is an ANR according, for example, with the proof of Theorem 3.2.1 in [10].
4) Also the multimaps $(U_i)_{i=\overline{1,n}}$ are obvious nonempty-valued and acyclic-valued (because the acyclicity is product closed).
5) Also, U_i is closed for all $i \in I$, that is, its graph is a closed set, where

$$graph\ U_i = \left\{\left((x,v),(u,y)\right) \in (X \times Y) \times (X \times Y) \mid u_i \in S_i(v), y_i \in T_i(x)\right\}.$$

6) Now from Lemma 2 (the extension of the Begle's Fixed-point Theorem to a finite family of multimaps), there exists $(\tilde{x}, \tilde{y}) \in X \times Y$, $\tilde{x} = (\tilde{x}_i)_{i \in I}$ and $\tilde{y} = (\tilde{y}_i)_{i \in I}$ such that $(\tilde{y}_i, \tilde{x}_i) \in U_i(\tilde{x}, \tilde{y})$, that is, $\tilde{y}_i \in S_i(\tilde{x})$ and $\tilde{x}_i \in T_i(\tilde{y})$ for all $i \in I$.
So, $(\tilde{x}, \tilde{y})$ is a *collectively coincidence point* for the families $(S_i)_{i \in I}$ and $(T_i)_{i \in I}$. □

3. An application of our new coincidence result in the theory of general equilibrium

For the beginning we give some basic definitions.

Definition 3. A *generalized abstract economy* with two companies is a family $\Gamma = (X_i, S_i, Y_i, T_i)_{i \in I}$ such that for each $i \in I$, X_i and Y_i are two topological spaces, $X = \prod_{i \in I} X_i$, $Y = \prod_{i \in I} Y_i$, and for all $i \in I$, $S_i : X \to 2^{Y_i}$ and $T_i : Y \to 2^{X_i}$ are nonempty-valued multimaps (they will be the *constraints* of the economy Γ).

Definition 4. An *equilibrium point* for Γ is a point $(\tilde{x}, \tilde{y}) \in X \times Y$, $\tilde{x} = (\tilde{x}_i)_{i \in I}$, $\tilde{y} = (\tilde{y}_i)_{i \in I}$ such that for all $i \in I$,

$$\tilde{y}_i \in S_i(\tilde{x}) \text{ and } \tilde{x}_i \in T_i(\tilde{y})$$

(that is, $(\tilde{x}, \tilde{y})$ is a *collectively coincidence point* for the two families of constraints $(S_i)_{i \in I}$ and $(T_i)_{i \in I}$).

Now, based on our result from the previous section we will give a *new general equilibrium theorem* for a generalized abstract economy with two companies.

We will suppose that the index set I is finite, namely $I=\{1,...,n\}$, that is, each company in Γ has a finite number of firms.

Theorem 3. *Let* $I=\{1,...,n\}$ *and let* $\Gamma=(X_i, S_i, Y_i, T_i)_{i\in I}$ *be a generalized abstract economy. We consider* $X=\prod_{i\in I} X_i$ *and* $Y=\prod_{i\in I} Y_i$. *Let us suppose that, for each* $i\in I$ *:*

1) X_i *and* Y_i *are compact acyclic finite dimensional ANR;*

2) *The constraints* $S_i: X\to 2^{Y_i}$ *and* $T_i: Y\to 2^{X_i}$ *are nonempty-valued, acyclic-valued and closed.*

Then there exists $(\tilde{x},\tilde{y})\in X\times Y$ *an equilibrium point for* Γ.

Proof. The proof is immediate, because we can apply Theorem 2 finding a collectively coincidence point $(\tilde{x},\tilde{y})$ for the families $(S_i)_{i\in I}$ and $(T_i)_{i\in I}$.

Hence, if $\tilde{x}=(\tilde{x}_i)_{i\in I}\in X$ and $\tilde{y}=(\tilde{y}_i)_{i\in I}\in Y$, then $\tilde{y}_i\in S_i(\tilde{x})$ and $\tilde{x}_i\in T_i(\tilde{y})$ for all $i\in I$.

Therefore $(\tilde{x},\tilde{y})$ is an *equilibrium point* for Γ. □

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SHIFT OPERATORS ON SEQUENCES

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Abstract. Let K be a commutative field and let $S(A)$ be the K-vector space of all sequences $f:\mathbf{N}\to A$ with entries in a fixed K-algebra A (commutative or not). We shall always assume that $K\subseteq A$. Let $T:S(A)\to S(A), T(f)(n)=f(n+1)$ be the usual right shift operator and let $L=T^k+a_1T^{k-1}+...+a_kI,$ be a monic polynomial operator in $K[T],$ where I is the identity operator on $S(A)$. In this note we study the structure of the kernel of L in language of the multiplicative properties of L in $K[T],$ i.e. the structure of all recurrent sequences f with $L(f)=0$ in language of the polynomial L as an element in $K[T]$. Some interesting applications are also given.
Mathematics Subject Classification (2000) 11B37, 11B99, 47B37, 47B99.
Key words recurrent sequence, shift operator, algebraic equation, Hadamard product.

Let K be a commutative field and let $A\supseteq K$ be a K-algebra (commutative or not). Let $\overline{K}$ be a fixed algebraic closure of K (see [2] for the algebraic notions) and let $\overline{A}=A\otimes_K\overline{K}$ the least $\overline{K}$-algebra which contains A. A *sequence with values in A* is simply a function $f:N\to A$, where N is the additive monoid of natural numbers. Sometimes we write f as $\{x_n\}_n=\{x_0=f(0),...,x_n=f(n),...\}$. We denote by $S(A)$, the left A-module of all sequences with values in A. Recall that $(\alpha f+\beta g)(n)=\alpha f(n)+\beta g(n), n=0,1,...,\alpha,\beta\in A, f,g\in S(A).$ $S(A)$ is also a ring with the componentwise multiplication (Hadamard multiplication): $(fg)(n)=f(n)g(n), n=0,1,...$. By $\alpha\to f_\alpha$, $f_\alpha(n)=\alpha, n=0,1,...$ we can view A as a subring of $S(A)$. The A-linear operator $T:S(A)\to S(A), T(f)(n)=f(n+1), n=0,1,...$ is called the *simple shift operator of S(A)*. By $T^k:S(A)\to S(A)$ we mean the A-linear operator $T\circ T\circ...\circ T$, k-times, i.e. $T^k(f)(n)=f(n+k), n=0,1,...,k=0,1,...$. It is clear that $T^0=I$, the identity operator of $S(A)$. A K-linear combination of $I,T,T^2,...T^k$, $L=T^k+a_1T^{k-1}+...+a_{k-1}T+a_kI$, where k is a natural number greater than *1* and $a_1,a_2,...,a_k\in K$, is called a *K-(linear) shift operator on S(A)*. The natural number k is called the *order of L*. If $a_k\neq 0$ then we say that L is a *proper shift operator*.

A sequence $f\in S(A)$ is said to be *algebraic over K* if there exists a K-linear shift operator L such that $f\in\ker L,$ i.e. $L(f)=0.$ If L is a proper shift operator and $f\in\ker L,$ then f is called a *recurrent (recurring) sequence over K with values in A*. We denote by $S_K(A)$ the subset of all algebraic sequences over K and by $R_K(A)$ the subset of all recurrent sequences in $S_K(A)$ with values in A. Let $S_0(A)$ be the subset of elements f in $S_K(A)$ such that there exists a natural number $k\geq 1$ with $T^k(f)=0=(0,0,...,0,...)$. It is easy to see that $S_0(A)$ is an A-subalgebra of $S(A)$. Moreover, $f\in S_0(A)$ if and only if $f(n)=0, n\geq n_0$. Such a sequence

is said to *have a finite support* or that it is a *trivial sequence*. The least possible n_0 with $f(n)=0, n\geq n_0$ is called the *order of the trivial sequence f.*

From now on we consider a *K*-linear shift operator *L* as an element of the polynomial ring $K[T]$. If $L=T^k+a_1T^{k-1}+...+a_{k-1}T+a_kI$, the polynomial

(1) $$P_L(X)=X^k+a_1X^{k-1}+...+a_{k-1}X+a_k\in K[X],$$

is called the *(associated) characteristic polynomial of L*. If *f* is an algebraic element over *K*, i.e. if $f\in S_K(A)$, a *K*-linear shift operator L_f with $L_f(f)=0$ and with $\deg P_{L_f}(X)$ the least possible, is called the *minimal shift operator of f* and $k=\deg P_{L_f}(X)$ is called the *period of f*. In general, the minimal shift operator L_f of f is not irreducible in $K[T]$. For instance, $f(n)=2^n+3^n, n=0,1,...$ ($K=A=Q$, the rational number field) has as its minimal shift operator $L_f=T^2-5T+6I=(T-2I)(T-3I)$.

Proposition 1 A *K*-linear minimal shift operator L_f of an algebraic (over *K*) sequence $f\in S(A)$ is uniquely defined. Moreover, if *M* is another *K*-linear shift operator with $M(f)=0$, then *M* is a multiple of L_f in $K[T]$.

Proof Use the Euclidean division algorithm in $K[T]$ or in $K[X]$, etc.

Theorem 2 A sequence $f\in S(A)$ is algebraic over *K* if and only if the *K*-vector subspace $V=Sp\{f,T(f),T^2(f),...\}$ of *S(A)* has a finite dimension over *K*, i.e. if and only if there exists a natural number $k\geq 1$ such that $T^k(f)\in Sp\{f,T(f),...,T^{k-1}(f)\}$. The least such *k* is exactly the dimension of *V* and the period of *f*.

Proof We see that $T^k(f)=b_1T^{k-1}(f)+b_2T^{k-2}(f)+...+b_kf$ implies
$T^{k+1}(f)=b_1T^k(f)+b_2T^{k-1}(f)+...+b_kT(f)=b_1\left[b_1T^{k-1}(f)+b_2T^{k-2}(f)+...+b_kf\right]+...+b_kT(f)$
which is again a linear combination of $f,T(f),...,T^{k-1}(f)$ with coefficients in *K*. By a mathematical induction procedure we can also prove that

(2) $$T^{k+n}(f)\in Sp\{f,T(f),...,T^{k-1}(f)\}, n\geq 0.$$

Using this last result, we can easily prove a basic result for algebraic sequences (see also [1] for an analogous result relative to recurrent sequences).

Theorem 3 The subset $S_K(A)$ of all algebraic sequences over *K* is a *K*-subalgebra of *S(A).*

Proof Let us use formula (2) to see that if $f,g\in S_K(A)$, with $L_f=T^k+a_1T^{k-1}+...+a_{k-1}T+a_kI$ and $L_g=T^l+b_1T^{l-1}+...+b_{l-1}T+b_lI$, then $T^{k+l+n}(fg)\in Sp\{fg,T(f)T(g),...,T^i(f)T^j(g),...,T^{k-1}(f)T^{l-1}(g)\}, n\geq 0$. Thus the subspace $V=Sp\{fg,T(fg),T^2(fg),...\}$ is finitely generated, i.e. *fg* is algebraic over *K*. To practically find a basis in this last subspace is still an open problem. Since $L_f(f)=L_g(g)=0$, we see that $(L_f\circ L_g)(\alpha f+\beta g)=0$ for any $\alpha,\beta\in K$, so $\alpha f+\beta g\in S_K(A)$.

Proposition 4 Each algebraic sequence (over *K*) $f=\{x_n\}_n \in S_K(A)$ is of the form: $f=\{x_0, x_1, ..., x_l, y_0, y_1, ..., y_n, ...\}$, where $\{y_n\}_n$ is a recurrent sequence over *K* (this is why we also call an algebraic sequence *f*, an almost everywhere (a.e.) recurrent sequence). Moreover, $\dim_K \ker L = k$, where *L* is a *K*-linear shift operator of order *k*.

Proof Let *L* be a *K*-linear shift operator, such that $L(f)=0,$ or

(3) $$L(f)=T^k(f)+a_1T^{k-1}(f)+...+a_k f=0.$$

If $a_k \neq 0,$ then we write $l=-\infty$ and so, $f=\{y_0, y_1, ..., y_n, ...\}$ is a recurrent sequence over *K*. Let now *l* be the greatest number such that $a_k=a_{k-1}=...=a_{k-l+1}=0, a_{k-l}\neq 0$ in (3). Let us denote by $g=T^l(f)$ and by $L_1=T^{k-l}+a_1T^{k-l-1}+...+a_{k-l}I$. Since $L=L_1\circ T^l$, we find that $L_1(g)=0$, so $g=\{y_0, y_1, ...\}$ is a recurrent sequence over *K*. Since $g=T^l(f)$, we see that $f=\{x_0, x_1, ..., x_l, y_0, y_1, ..., y_n, ...\}$, where $x_0, x_1, ..., x_l$ are arbitrary elements in *A*. Moreover, even $y_0, y_1, ..., y_{k-l-1},$ are arbitrary elements in *A* and they do not depend on the choice of $x_0, x_1, ..., x_l$. Now, let *L* be as in (3). We see that $y_{k-l}, y_{k-l+1}, ..., y_n,$ are linear combinations of $y_0, y_1, ..., y_{k-l-1}$ over *K*, which are completely free. Thus, taking account that the first *l+1* elements $x_0, x_1, ..., x_l$ are free, we see that $\dim_K \ker L = k$.

Definition 1 Let *L* be a *K*-linear shift operator on $S(A)$ of order *k*, and let $g\in S(A)$ be a fixed sequence in $S(A)$. Let $f=\{x_n\}_n \in S(A)$ be un unknown sequence in $S(A)$ and let $y_0, y_1, ..., y_{k-1}$ be *k* fixed elements in *A*. A Cauchy problem for the equation $L(f)=g$ with the initial data $f(0)=y_0, f(1)=y_1, ..., f(k-1)=y_{k-1}$ means to find a solution $f=\{x_n\}_n \in S(A)$ of this last equation such that the first *k* terms of the sequence *f* are $y_0, y_1, ..., y_{k-1}$.

Proposition 5 Any Cauchy problem has a unique solution in $S(A)$. If $g\in S_K(A)$, then this unique solution is also in $S_K(A)$.

Theorem 6 Let *L* be a proper *K*-shift linear operator as in (7) (see Remark 1 bellow) and let $e_i, i=0,1,...,k-1$ be the unique solution of the Cauchy problem:

(4) $$L(f)=0,\ f(0)=0,...,\ f(i-1)=0,\ f(i)=1,\ f(i+1)=0,\ f(i+2)=0,...,\ f(k-1)=0.$$

Then $\{e_0, e_1, ..., e_{k-1}\}$ is a basis in the *K*-vector space $\ker L$, and any $h\in \ker L$ can be written:

(5) $$h=h(0)e_0+h(1)e_1+...+h(k-1)e_{k-1}\ .$$

Moreover, if h_P is a *particular solution* of the equation $L(f)=g$, i.e. $L(h_P)=g$, then the *general solution* of it is of the form:

(6) $$h=h_P+c_0e_0+...+c_{k-1}e_{k-1},$$

where $c_0, c_1, ..., c_{k-1}$ are arbitrary elements in *K*.

Here (5) is the *general solution of* $L(f)=0$ and $\{e_0, e_1, ..., e_{k-1}\}$ is called the *fundamental (canonical) system of solutions* of the same equation $L(f)=0$.

Remark 1 Proposition 4 says that we can assume that *L* is a *proper K-shift linear operator* i.e.

(7) $$L = T^k + a_1 T^{k-1} + ... + a_{k-1} T + a_k I ,$$

where $a_k \neq 0$. Moreover, since $S(A) \subseteq S(\overline{A})$ and $S_K(A) \subseteq S_{\overline{K}}(\overline{A})$, we can assume in the following that *K* is algebraically closed, i.e. that $K = \overline{K}$, $A = \overline{A}$.

It is easy to see that if $r_1, r_2, ..., r_t \in K$ are the distinct roots of the characteristic polynomial $P_L(X)$ with their corresponding algebraic multiplicities $k_1, k_2, ..., k_t$ ($\sum_{i=1}^{t} k_i = k$), one can write (in $K[T]$):

(8) $$L = \prod_{i=1}^{t} (T - r_i I)^{k_i} ,$$

this product being commutative in $K[T]$.

Example 1 For any $\alpha \in K, \alpha \neq 0$, the *infinite geometrical progression* $f_\alpha(n) = \alpha^n, n = 0, 1, ...$ is a recurrent sequence with the minimal operator $L_f = T - \alpha I$, i.e. its period is equal to *1*.

Example 2 For any $t \in \mathbf{N}$, the natural number set, we consider the *t power sequence*: $f_{[t]}(n) = n^t, n = 1, 2, ...$. If the characteristic of *K* is equal to zero, this sequence is a recurrent sequence of period equal to $t+1$. If the characteristic of *K* is a prime number *p*, then the period is less or equal to $t+1$. For instance, if $p = 2$, $f_{[2]}(n) = n^2$ satisfies the equation: $(n+2)^2 + n^2 = 0$; here the period is equal to $2 < t + 1 = 3$. The sequences of the form: $h_{\alpha,t}(n) = n^t \alpha^n$ are also recurrent sequences if $t \in \mathbf{N}$, and $\alpha \in K, \alpha \neq 0$. Indeed, it is easy to see that $L_{h_{\alpha,t}} = (T - \alpha I)^{t+1}$ is the minimal operator of $h_{\alpha,t}$. Moreover, since $\alpha^{t+1} \neq 0$, $L_{h_{\alpha,t}}$ is a proper operator; this means that $h_{\alpha,t}$ is also a recurrent sequence of period $t+1$.

Proposition 7 Let $r_1, r_2, ..., r_s$ be *s* distinct nonzero elements in *K* (this time an arbitrary field), let $P_1(X), ..., P_s(X)$ be *s* polynomials in $K[X]$ and let $f_i(n) = P_i(n) r_i^n, i = 1, ..., s$. Then these recurrent sequences (see Example 2) are linear independent in $R_K(A)$ over *K*.

Proof We use mathematical induction on $k = \sum_{i=1}^{s} \deg P_i(X)$. For $k = 0$, let $f_i(n) = \lambda_i r_i^n, \lambda_i \in K^*, i = 1, ..., s$ and let $\sum_{i=1}^{s} C_i \lambda_i r_i^n = 0, n = 0, 1, ...$, be a null linear combination over *K* of these recurrent sequences. It is sufficient to consider the first *s* equalities and to see that the Vandermonde determinant of this system in the unknowns $C_i \lambda_i, i = 1, 2, ..., s$ is not equal to zero. So, all the scalars C_i are equal to zero.
We assume now that we have proved the statement for all $h = 0, 1, ..., k-1$ and try to prove it for $h = k$. Let

(9) $$\sum_{i=1}^{s} C_i P_i(n) r_i^{\,n} = 0,\ n = 0, 1, ...$$

be a null linear combination of the recurrent sequences $f_i(n) = P_i(n) r_i^{\,n}$, $i = 1, ..., s$. We assume that $k = \sum_{i=1}^{s} \deg P_i(X)$. We replace n by $n+1$ in (9) and, from the resulting equality we subtract the equality (9) multiplied by r_1. We finally obtain a null linear combination of the following type:

(10) $C_1\left[r_1 P_1(n+1) - r_1 P_1(n)\right] r_1^{\,n} + C_2\left[r_2 P_2(n+1) - r_1 P_2(n)\right] r_2^{\,n} + ... + C_s\left[r_s P_s(n+1) - r_1 P_s(n)\right] r_s^{\,n} = 0$

Since the sum of the degrees of all the polynomials in the brackets is strictly less than *k*, we can apply the mathematical induction hypothesis and find that all C_i are zero.

Theorem 8 Let $L = \prod_{i=1}^{t} (T - r_i I)^{k_i}$ be a proper shift operator as in formula (8), where $r_1, r_2, ..., r_s$ are the distinct roots of the characteristic polynomial $P_L(X)$ associated to $L = T^k + a_1 T^{k-1} + ... + a_{k-1} T + a_k I$. Then the sequences $\left\{r_i^{\,n}\right\}_n, \left\{n r_i^{\,n}\right\}_n, ..., \left\{n^{k_i - 1} r_i^{\,n}\right\}_n$ make up a basis in the vector space $\ker(T - r_i I)^{k_i}$. Moreover, $\left\{r_i^{\,n}\right\}_n, \left\{n r_i^{\,n}\right\}_n, ..., \left\{n^{k_i - 1} r_i^{\,n}\right\}_n$, $i = 1, 2, ..., s$ is a basis in the vector space $\ker L$, i.e. $\ker L = \bigoplus_{i=1}^{t} \ker(T - r_i I)^{k_i}$.

Proof The previous Proposition 7 says that the set of sequences $\left\{r_i^{\,n}\right\}_n, \left\{n r_i^{\,n}\right\}_n, ..., \left\{n^{k_i - 1} r_i^{\,n}\right\}_n$, $i = 1, 2, ..., s$ are linear independent over *K*. It remains to prove that $\left\{r_i^{\,n}\right\}_n, \left\{n r_i^{\,n}\right\}_n, ..., \left\{n^{k_i - 1} r_i^{\,n}\right\}_n$ are in $\ker(T - r_i I)^{k_i}$. It is sufficient to remark that $(T - rI)\left(Q(n) r^n\right) = Q_1(n) r^{n+1}$, where $Q(X), Q_1(X) \in K[X]$ and $\deg Q(X) > \deg Q_1(X)$. Thus, if $Q(X)$ is a polynomial of degree less than k_i, then $Q(n) r_i^{\,n} \in \ker(T - r_i I)^{k_i}$.

Remark 2 This last theorem says that any recurrent sequence *f* such that $L(f) = 0$, where $L \in K[T]$, *K* being an arbitrary field, is of the form:

(11) $$f(n) = \sum_{i=1}^{s} P_i(n) r_i^{\,n},\ n = 0, 1, ...,$$

Where $P_i(X)$ is an arbitrary polynomial of degree less than k_i, the algebraic multiplicity of the root (of the characteristic polynomial $P_L(X), L(f) = 0$), $r_i \in \overline{K}$, an algebraic closure of *K*, $i = 1, ..., n$. We see from here that even if at the beginning the recurrent sequence *f* (over *K*) was considered with values in an arbitrary *K*-algebra *A*, we see that in fact its values are in $A \cap \overline{K}$, the intersection being realized in $\overline{A} = A \otimes_K \overline{K}$.

Corollary 9 (Hadamard) Let *f*, *g*, be two recurrent sequences over *K*, represented as in formula (11): $f(n) = \sum_{i=1}^{s} P_i(n) r_i^{\,n}, g(n) = \sum_{i=1}^{s} Q_i(n) u_i^{\,n}, n = 0, 1, ...$. Then the Hadamard product sequence *fg* is a recurrent sequence (over *K*) and it can be written:

(12) $$(fg)(n) = \sum_{i=1}^{s} \sum_{j=1}^{p} P_i(n) Q_j(n) r_i^{\,n} u_j^{\,n},\ n = 0, 1, ...$$

Proof By a simple multiplication of $f(n)$ and $g(n)$ in $\overline{K}$ we get (12). Let k_i be the algebraic multiplicity of r_i in the characteristic polynomial of L_f, the minimal shift operator of *f*, and let l_j be the algebraic multiplicity of u_j in the characteristic polynomial of L_g. The elements $\{r_i u_j\}, i = 1,..., s,\ j = 1,..., p$ are in $\overline{K}$, so they are algebraic over *K*. Let $P(X) \in K[X]$ be a minimal degree monic polynomial which has as roots all these elements $\{r_i u_j\}, i = 1,..., s,\ j = 1,..., p$. Let now $N = \max\{(k_i + l_j), i = 1,..., s,\ j = 1,..., p\}$ and let $R(X) = P^N(X) \in K[X]$ and

(13) $$R(X) = X^m + c_1 X^{m-1} + ... + c_m \in K[X]$$

and let $L = T^m + c_1 T^{m-1} + ... + c_m I \in K[T]$ be the linear shift operator which has as a characteristic polynomial the polynomial $R(X)$. Now, it is easy to see that $fg \in \ker L$ (look carefully at Theorem 8).

Remark 3 The linear shift operator *L* constructed in the proof of Corollary 9 is not in general the minimal shift operator L_{fg} of the Hadamard product *fg*. In order to obtain this last operator, we need to keep in formula (12) only the nonzero terms and to construct the minimal degree monic polynomial for $r_i u_j$, etc. Here is an explanation of the fact that it is extremely difficult to find an algorithm for an exact description of the shift operator L_{fg} (see also [1]). It is also difficult to find an exact description of the minimal degree polynomial with coefficients in *K* for a set of elements in $\overline{K}$ (see [2] for algebraic background).

Remark 4 Let *J* be an open subset in **R**, the real number field, let *K* = **R** and $A = C^\infty(J)$ be the *K*-algebra of all functions defined on *J* such that they have differentials of any order on *J*. Let $S_\infty(A) = \{Y(x) = (y(x), y'(x), ..., y^{(n)}(x), ...)\} \subset S(A)$ and *L* be the above proper shift operator restricted to $S_\infty(A)$. It is clear that $Y(x) \in \ker L$ if and only if *y(x)* is a solution of an *ODE-k.* So, the set of all solutions of a homogenous linear differential equation of order *k* with constant coefficients (in **R**) is an **R**-vector space of dimension *k*. Moreover, by using only the above technique for recursive sequences, one can prove the theorem of existence and uniqueness for a Cauchy problem for such an equation, without making use of Mathematical Analysis.

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AN APPLICATION IN SPSS FOR CALCULATE THE PEARSON CORRELATION COEFFICIENT

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Abstract: In any study which uses the statistical analysis there is a mathematical model. This model is based on specific mathematical instruments: correlation and regression. In regression analysis, one of the most used coefficients is the Pearson correlation coefficient. The SPSS program calculates the Pearson correlation coefficient and generates classification tables.
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1. Introduction

In any study which uses the statistical analysis there is a mathematical model, which is based on specific instruments: correlation and regression. The data is analysed and it is taken in consideration only the variables that have a significant correlation between them. Only the data related with these variables it will be used in the statistical analysis. For this, it is calculated the correlation coefficients for the independent variables, which were taken into account in the study. One of the most used coefficients is the Pearson correlation coefficient. The SPSS program calculates the Pearson correlation coefficient and generates classification tables. One cell of the table contains the value of correlation coefficient, the critical probability of the signification test and the number of values which were withheld for the calculus.

2. Pearson correlation coefficient

2.1. The correlation

Many times, we are interested in the way in which the variability of a data set it is reflected in the variability of another data set (if the data are correlated). For determinate this influence, we can perform a X^2 test for one sample. This completes a contingency table containing the data observed and the expecting data. The test result will indicate the types of existent correlations between variables. Then we will determine the correlation coefficients between the two variables.
The correlation coefficients represent the association grade between two variables. These coefficients indicate the way how the value of one variable it is modified by the value of the other variable (if these two variables are associated). The correlation coefficients shows if there is a correlation between these two variables and how tide it is the relation between them. Between two variables there are many correlation types:

- perfect positive;
- strong positive;
- weak positive;
- perfect negative;

- strong negative;
- weak negative;
- no correlation.

There are many correlation coefficients, and the most used are:

- *Pearson correlation coefficient* (detonated with r_{xy} or r), which is a parametric coefficient.
- *Spearman coefficient* (detonated with r_s), which is a non-parametric coefficient.

2.2. Pearson correlation coefficient

The correlation coefficient theoretically is denoted by ρ(X,Y) and it is define by relation:

$$\rho(X,Y) = \frac{cov(X,Y)}{\sigma_X \cdot \sigma_Y} = \frac{\sum_i (x_i - \mu_x) \cdot (y_i - \mu_y)}{N \cdot \sigma_X \cdot \sigma_Y} \quad , \; i = 1,\ldots,N \tag{1}$$

where:

- $cov(X,Y)$:covariance, defined by relation: $cov(X,Y) = \frac{\sum_i (x_i - \mu_x) \cdot (y_i - \mu_y)}{N}$;
- x_i , y_i : values of the correlate variables;
- μ_x , μ_y : the medium level of the values of the correlate variables;
- N : the number of the pairs of the values;
- σ_X , σ_Y : standard quadratic deviation for X, respective Y.

The correlation coefficient it is obtained by standardization of the covariance. The value of the correlation coefficient is value between -1 and +1:

$-1 \leq \rho \leq +1$.

If ρ takes the value zero, then between variables there is no correlation.

The sign of the ρ value gives the way of the relationship between the variables. The plus sign shows that there is a direct relationship (if the values of the variable X are increasing, also the values of the variable Y are increasing). The minus sign shows an inverse relationship (if the values of the variable X are increasing, then the values of the variable Y are decreasing).

The absolute value of ρ indicates the intensity of the relationship: the closer is getting by 1, the stronger is the relationship, respectively the closer is getting by 0, and the weaker is the relationship. A correlation coefficient with a value of 1 indicates a direct perfect relationship.

A correlation coefficient with a value of -1 indicates an inverse perfect relationship.

An estimator ρ* for the ρ correlation coefficient has as possible values empirics correlation coefficients (r_{xy}), which are determinate at the level of the sample that can be extracted by a survey method.

At a level of a n volume sample, the empiric Pearson correlation coefficient is:

$$r_{xy} = \frac{cov(X,Y)}{s_X \cdot s_Y} = \frac{\sum_i (x_i - x^*) \cdot (y_i - y^*)}{n \cdot \sigma_X \cdot \sigma_Y}, \tag{2}$$

which represent an estimation for parameter ρ.

After developing the relation (2), is obtained a simplified relation for calculate the empiric Pearson correlation coefficient:

$$r_{xy} = \frac{n\sum x_i y_i - \sum x_i \sum y_i}{\sqrt{\left[n\sum x_i^2 - \left(\sum x_i\right)^2\right]\left[n\sum y_i^2 - \left(\sum y_i\right)^2\right]}} \quad , \; i = 1, \ldots, n \tag{3}$$

Applications of correlation coefficients

The calculus of correlation coefficients is useful for understanding the links between variations of the different parameters which characterises a given population. From the many applications of correlation coefficients we present just a few examples: the relationship between bio productivity and the factors links to soil properties (humidity, concentration of different nutrients, temperature, granulocyte, pH); the relationship between the sliding speed of a glacier and the rate of rocks erosion); the relationship between the rate of soil erosion and the wind speed or the hardness of soil rocks.

The determination coefficient

The theory of the correlation coefficient "product-moment" leeway from the linear regression made with the least squares method. We must remember the fact that the value of correlation coefficient r_{xy} may be deduce from the quantity ware with a variation of a variable is "statistically explained" by the variation of each other variable. This quantity is named determination coefficient and is equal with the square of correlation coefficient, being denoted by r^2.
So the determination coefficient is $r^2 = r_{xy}^2$.

Example: if the Pearson correlation coefficient is having the value r_{xy} = 0,65, then the determination coefficient will have the value r^2 = 0,42. This number is representing the proportion of the variation of one variable explained by the other, or in other words, a statistical variable explains 42% from the variation of the other variable.

3. The Calculus of the correlation coefficient using SPSS

The **SPSS** program (**Statistical Package for the Social Sciences**) it is one of the most used programs in statistical analysis. First of his version was developed in 1965, at the Stanford University of California and evolved until 20th version. The application area extended from the first version, along with the operation modules and the offered facility. The program it is used today in marketing, experimental research, education, health. Beside possibility of statistical analysis, the program has powerful component for data management (selection, reconfiguration, creation of new data) and for data documentation (there is a meta data portfolio, which book data characteristics). It must be mentioned the flexibility of different data types accepted and also the modulus for making the rapports.
The SPSS application is permanently actualized both in terms of the range of the statistical techniques and also the detailing results.
In order to achieve statistical analysis, in this study, it was utilized the program IBM- SPSS Trial version 19.0.0.
For calculate the correlation coefficients between the interval or ordinal variables it will be choose the command **Analyse > Correlate > Bivariate**.
It will be displayed the dialogue **Bivariate Correlations** in which it will be fixed the variables to be analysed and the correlation coefficient to calculate: Pearson, Spearman and the significance test.
Thru **Options** dialogue it can be selected some elementary statistics wanted in results and the mode for deal the missing observations.
As a result it can get, in principal, the correlation matrix between selected variables for the analysis. One cell of the table contains the value of correlation coefficient, the critical probability of the signification test and the number of values which were withheld for the calculus (after dealing with missing cases), as in table 2.

According to the decision procedure in a statistic test it will be rejected the hypotheses of a null correlation if the critical probability it is equal or smaller than the chosen significant limit. The significant correlations can be automatically marked with * (α=0,05) or ** (α=0,01) by selecting from principal menu the option **Flag significant correlations**.

4. A study case

This study is about elaborate of a mathematical model that can predict the water quality in network distribution systems, and confirming the required national and international standards.

This prediction can be made by using statistical analysis of water quality, with specific instruments: correlation and logistic regression, these can be utilized in elaborate mathematical models.

For an existing network, if there are known data about the network structure, the materials of the equipment, the age of the network, it can be make a prediction about water quality in this network. In this kind of analysis can be use another types of data referring to network structure. The more data it is used, the more viable it is the study.

The developed mathematical model it was implemented in a study case that uses real data from ARA (Romanian Water Association, National Report 2012), about the Romanian networks, using SPSS program (Statistical Package for the Social Sciences), trail version.

It will be presented below the calculation of Pearson correlation coefficient for the specific data of this study. The study has as a base a real data referring to poor structure and the age of the networks from water distribution networks of Romania, and also the water quality, precisely the nonconforming sample of water quality, because from the many data referring to the actually state of a network, this in particular has the biggest impact to water quality.

In the first step of the algorithm it is analyse all the data and it is selected only the data that will be used in the following statistical analysis.

Once the data that will be used are established, the second step of the algorithm will determinate the correlations between them. In our case, the poor structure, the age of the network and the nonconforming sample of water quality have the most significant correlation between them.

In the next table, table 1, it is presented the real data from ARA National Report[4]. These data were used in this study case. These are data from 13 districts of Romania, exactly 23 towns, summing over 6.400 kilometres of water distribution networks

Table1. A synthesized data from water distribution networks of Romania [4]

No.	District	Town	Km of network	Poor structure [%]	Age of network > 30 years [%]	Nonconforming sample of water quality [%]
1.	Timiș	Deta	22,6	45,6	46	15
		Jimbolia	67,8	67,6	63,3	14,3
		Timișoara	617	27,7	63,4	2
2.	Bihor	Oradea	608	28,5	27,5	0
		Salonta	57,31	45,5	47,33	
3.	Mureș	Tg. Mureș	298,23	74,22	20	4,59
		Sighișoara	87	65	34	11
4.	Hunedoara	Hunedoara	187	95,3	17,5	0
		Hațeg	27,62	14	2	0
5.	Caraș Severin	Caransebeș	390	65	53	6
6.	Bistrița Năsăud	Bistrița	554,3	46	5	3,04
7.	Brăila	Brăila	400	44,4	50	16
8.	Buzău	Buzău	178,05	79,57	13,96	2
9.	Constanța	Constanța	1351,9	70,43	71,4	2,2
10.	Dolj	Craiova	427	68	73	7,9
11.	Cluj	Cluj Napoca	626	30	2	5
		Dej	100,7	36,43	13,9	8
12.	Sălaj	Zalău	102,6	1,4	0	0,19
		Jibou	29,91	24,9	0	5,2
		Simleul Silvanei	37,85	41,6	0	3,3
		Cehu Silvanei	21,052	18,3	0	2,9
13.	Vaslui	Bârlad	220,9	62,2	15	0,57
		Negrești	18	100	40	0

From the data mentioned above we will use the following parameters:

- poor structure (PS [%]),
- age of network (AN [%]);
- nonconforming sample of water quality (NS [%]).

With SPSS program we can generate classification table, and these tables will indicate predicting belonging to a certain category, based on predictor variable. In these tables it is precisely indicated the number of correct classifications, so they are a good indicator of the quality of prediction.

The next step of the algorithm is to calculate the correlation coefficient. With the help of logistic regression analysis it can be indicated an improved correspondence between predicted category appurtenance and the real appurtenance in a category.

In this study, were considered the independent variables PS (poor structure) and AN (age of network) and the dependent variable NS (nonconforming sample of water quality).

As a result it will be obtain the correlation matrix between selected variables for the analysis. One cell of the table contains the value of correlation coefficient, the critical probability of the signification test and the number of values which were withheld for the calculus (after treated the missing samples).
In the next table, table 2, it is presented the result from the SPSS program, with the obtained correlations between the variables:

Table 2. Correlations

Correlations		PS	AN	NS
PS	Pearson Correlation	1	0,416*	**0,064**
	Sig. (2-tailed)		0,048	0,773
	N	23	23	23
AN	Pearson Correlation	0,416*	1	**0,385**
	Sig. (2-tailed)	0,048		0,070
	N	23	23	23
NS	Pearson Correlation	0,064	0,385	1
	Sig. (2-tailed)	0,773	0,070	
	N	23	23	23
*. The correlations are significant from 0.05 (2-tailed).				

From the calculus presented in the table 2, it appears that the correlation (0,064) between PS (poor structure) and NS (nonconforming sample of water quality) it is lower than the correlation (0,385) between AN (age of network) and NS (nonconforming sample of water quality), which means that the age of network influence more the quality of water than poor structure of the distribution system.
Using mathematical models it can be accurately describe the correlation between different characteristics of water distribution network, such as poor structure or age equipment, and the quality of water. Statistical analysis, using logistic regression and correlations, is an excellent instrument for interpretation of experimental data and for prediction in water quality for distribution networks.

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A NORMAL SCORE PARAMETERIZATION FOR CHANNELIZED RESERVOIRS ESTIMATION USING THE ITERATIVE ADAPTIVE GAUSSIAN MIXTURE FILTER

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Abstract. In this paper we present an extension of a parametrization of the channelized reservoirs with two facies types, which is coupled with the iterative adaptive Gaussian mixture filter (IAGM) for assisted history matching (AHM) of production data. The main objectives of any AHM procedure are to match the past data within the model and measurement uncertainties and to preserve the geological realism in order to predict the future behavior of the reservoir. For generating an ensemble of channelized reservoirs, a multi-point geostatistical tool (SNESIM) has been used in combination with a training image. If in a previous study, the parametrization was defined by drawing from a conditional Gaussian distribution using a conditional mean; here we propose an unconditional sampling from the same conditional Gaussian distribution. In addition, we present a comparison of the two parameterizations. As AHM method, we are using the iterative version of AGM in combination with the SNESIM in the resampling step, in order to handle with the nonlinearities and preserve the channelized structure of the ensemble members. The results presented show that the both parameterizations in combination with the IAGM procedure were able to reduce the uncertainty in the updated ensemble providing a realistic geological structure.
Mathematics Subject Classification: 60G60, 86-08, 68U2
Keywords: history matching; facies estimation; facies estimation; the iterative adaptive Gaussian mixture filter

1. Introduction

The estimation of the spatial distribution of the channelized reservoirs within an assisted history matching (AHM) methodology using as engine the production data, has been the subject of various studies, but none solved this problem yet. If the generation of plausible channelized reservoirs is done with multi-point geostatistical simulation models or object based simulation models, their estimation is hampered because of the complex nature of the spatial distribution of the bodies that form the reservoir geology. The complicated part of the estimation process is when the geometry and the topology of the reservoir should be kept, such that the updated reservoirs to produce forecasted data in the closest proximity of the measurements. Traditionally, one could try to estimate the permeability field and afterwards to truncate the permeability in order to recreate the geology (the facies field). This approach is not suitable for complex geometrical structures (like the channels) because the AHM algorithms do not keep account for geometry. Jafarpour 2011 has proposed one solution, by using of a resampling procedure after an assimilation step of the ensemble Kalman filter (EnKF). The authors proposed a methodology called probability-conditioning method (PCM) in the context of the multipoint geostatistical simulation model for the simulation of the geology. Even if the permeability is updated, the resampling procedure ensures the repositioning of the channel geometry. After the assimilation of the production data, the updated reservoirs are geologically plausible but do not match the data well enough because after the last assimilation a resampling from the training image is

performed. This resampling introduces noise that cannot be removed without an accept-reject procedure. Sebacher et al. 2015 proposed a different solution, by parameterizing the facies fields and updating the parameter field. In this way, the facies fields are reconstructed at each assimilation time step. Even if the geometry is not preserved, the resampling is performed after a complete assimilation cycle, procedure necessary to repositions the channel geometry. In order to obtain updated reservoirs with high predictive capacity, a new assimilation cycle is performed. The theoretical framework that allows the re-assimilation of the data is the adaptive Gaussian mixture filter (IAGM, Stordal and Lorentzen 2014). The number of the iterations with the IAGM is user choice, but most of the experiments showed good results in two or three iterations. The parameterization introduced in Sebacher et al. 2015 is performed in a multidimensional real space, marginally linked with the normal score transform with the discrete space of the facies fields. In this study, we extend the parameterization from Sebacher et al 2015, by randomly sampling from a conditional Gaussian distribution. We also perform a comparison with the previous parameterization using the same iterative adaptive Gaussian mixture filter in three iterations. For the generation of the initial ensemble and of the reference, we have used a multipoint geostatistical simulation model named SNESIM (Strebelle 2002) in combination with a training image.

2. The facies parameterization using the normal score transform

We start by generating of an initial ensemble of *N* independent channelized reservoirs from the training image (Fig. 1), using a MPS methodology (here we have used SNESIM).

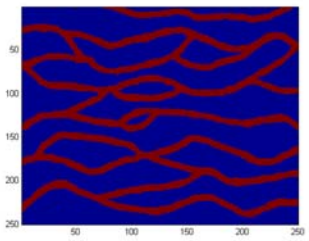

Figure 1: Training Image

At each grid cell *i*, of the reservoir domain, we calculate from the ensemble of facies fields, the probability of the channel occurrence using the relation:

$$prob_i = \frac{1}{N}\sum_{k=1}^{N} p_i^k, \; where \; p_i^k = \begin{cases} 1 & if \quad i \in channel \\ 0 & if \quad i \notin channel \end{cases} \tag{1}$$

Then, we define a random variable denoted *facies_distribution* that probabilistically defines the occurrence of the channel at that location:

$$facies\_distribution_i = \begin{pmatrix} channel & nonchannel \\ prob_i & 1 - prob_i \end{pmatrix} \tag{2}$$

This categorical random variables is linked with the standard normal distributed variables using the normal score transform (Fig. 2). The normal score transform defines a threshold on the real axis (denoted α_i) that delimitates the channel zone to the nonchannel zone.

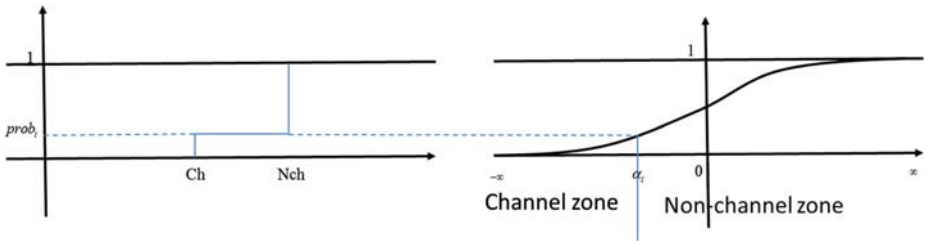

Figure 2: The normal score transform

We introduce the parameterization of the facies fields in the normalized space in two ways, corresponding to the experiments performed in this study. If we consider $X \propto N(0;1)$ then we define a parameter field on the reservoir domain as follows; for every facies field *k* of the initial ensemble of facies fields, we define a parameter field denoted θ^k at each grid cell *i* as:

$$(1)\quad \theta_i^k = \begin{cases} random\ sampling\ \ from\ (X \mid X \in channel) & if \quad i \in channel \\ random\ sampling\ \ from\ (X \mid X \in nonchannel) & if \quad i \in nonchannel \end{cases}$$

$$(2)\quad \theta_i^k = \begin{cases} E\ (X \mid X \in channel) & if \quad i \in channel \\ E\ (X \mid X \in nonchannel) & if \quad i \in nonchannel \end{cases} \qquad (3)$$

Consequently, we consider the first experiment the one where the parameterization is performed using the first relation and the second experiment where we are using the second relation. We mention that the second parameterization is introduced in Sebacher et al. 2015. From eq.3 one can observe that the first parameterization is much general that the second and the random sampling from the conditional distribution damages the possible two point relations inherited from the training image (Fig. 3).

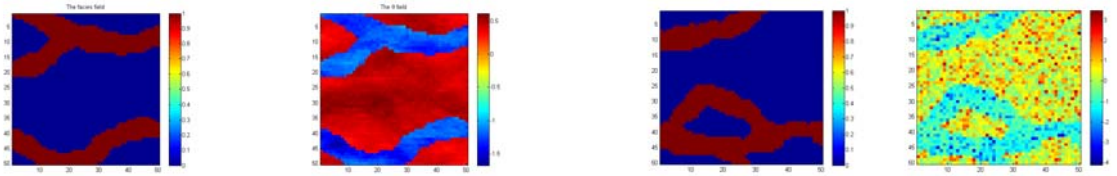

Figure 3: The parameter field in the second experiment (second picture), in the first experiment (third picture) and the associated facies fields (first and third picture)

3. The iterative adaptive Gaussian mixture filter (IAGM) implementation for facies update

The iterative adaptive Gaussian mixture filter is the iterative version of the adaptive Gaussian mixture filter (AGM). The AGM was introduced by Stordal in 2011 as a data assimilation methodology that provides approximate solutions for non-linear filtering problems. The AGM approximates the prior distribution with a Gaussian mixture of type:

$$p(x_t \mid d_{1:t-1}^{obs}) \propto \sum_{k=1}^{N} w_t^k N(x_t - x_t^k, C_t),\ where\ w_t^k\ is\ the\ weight\ of\ the\ particle\ x_t^k \qquad (4)$$

If we approximate in eq. 4 C_t with the covariance matrix of the particles calculated from the ensemble with respect to the weighted mean $\overline{x_t} = \sum_{k=1}^{N} w_t^k x_t^k$, then each particle x_t is updated as

$$\hat{x}_t^k = x_t^k + C_t H_t^T (H_t C_t H_t^T + h^{-2} R)^{-1} (d_t^{obs} - H_t x_t^k + \varepsilon_t^k) \quad (5)$$

where h is a positive sub unitary parameter that controls the update size, H is the observation operator and C is the covariance. At the initial moment, all the weights are equal with 1/N and are updated using the relations:

$$w_t^k = \alpha_t \overline{w}_t^k + (1-\alpha_t) N^{-1}$$

$$\overline{w}_t^k = \frac{\hat{w}_t^k}{\sum_{k=1}^{N} \hat{w}_t^k}$$

$$\hat{w}_t^k = \phi(d_k^{obs} - H_t x_t^{f,k}, h^2 H_t C_t H_t^T + R), \text{ where } \phi(x-\mu, P) \text{ represents a Gaussian density} \quad (6)$$

of mean μ and covariance P

$$\alpha_t = (N \sum_{k=1}^{N} (\overline{w}_t^k)^2)^{-1}$$

In this study we have used h=0.25. In our specific case, we define the state vector as:

$$x_t^k = \left[\theta^{k,T} \quad d_{sim}(\theta^k)^T \right]^T, k = 1, 2, \ldots, N \quad (7)$$

where, $d_{sim}(\theta^k)$ are the simulated measurements (the water and oil rates at the production wells and the bottom hole pressures at the injection wells).

After the assimilation of the production data, the values of the parameter field θ changes (eq.5). Comparing the updated values of the parameter field with the thresholds α (defined by the normal score transform) we are able to determine the updated facies fields and go to the next assimilation time. When a complete assimilation period is run (with AGM) the updated ensemble of facies fields match the observed data, but the channel connectivity is not completely obtained. In order to have plausible updated facies fields, we are using the iterative version of AGM, namely IAGM and sampling a new initial ensemble of facies fields from the same training image but conditioning on the weighted probability fields of the facies.

$$prob_i = \frac{1}{N} \sum_{j=1}^{N} w^k p_i^k, \text{ where } p_i^k = \begin{cases} 1 & \text{if} \quad i \in channel \\ 0 & \text{if} \quad i \notin channel \end{cases} \quad (8)$$

We apply the IAGM until the updated facies fields have geological plausible shapes and from our experience, this is gained after two or three iterations (in this study are made three iterations). Of course, we can perform many iterations but with small improvements. The forecast from time t to time t+1 is made with the GPRS reservoir simulator (Stanford University). All the geostatistical simulations are performed with SNESIM implemented in SGeMS.

4. Synthetic example

For the exemplification of the methodology, we present a synthetic case of a small-channelized 2D reservoir, with 50 grid cells in the *X* and *Y* direction. The dimension of each grid cell is set as 30 × 30 × 20 ft. The reservoir is designed as a 6-spot water flooding black oil model with three injection wells at the left side and three production wells at the right side (Fig. 4). The reservoir is initially filled with oil with a uniform saturation of 0.8 and a connate water saturation of 0.2

having a uniform pressure of 5000 psi in every grid cell. The producers work under constant bottom hole pressure (BHP) at 1500 psi and the injectors operate at 2500 stock tank barrels per day (STB/D) constrained by a maximum BHP of 10,000 psi. Water injection starts at the first day and continues for a period of 191 days. We assimilate data every 20th day between day 1 and day 191, resulting in ten assimilation steps.

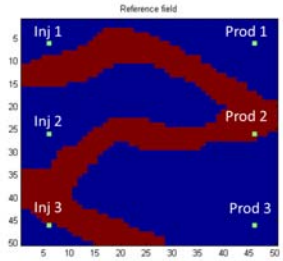

Figure 4: The reference field

The measurement errors of the production data are assumed Gaussian with 0 mean and standard deviations of 3 %. These values are used when generating the reference measurements and when adding noise to the production data in the analyzed step of the HM processes. The permeability values were set at 500 and 50 mD for the channel facies type and for the non-channel facies type, while the porosity of both facies types is set to 0.2 and assumed known

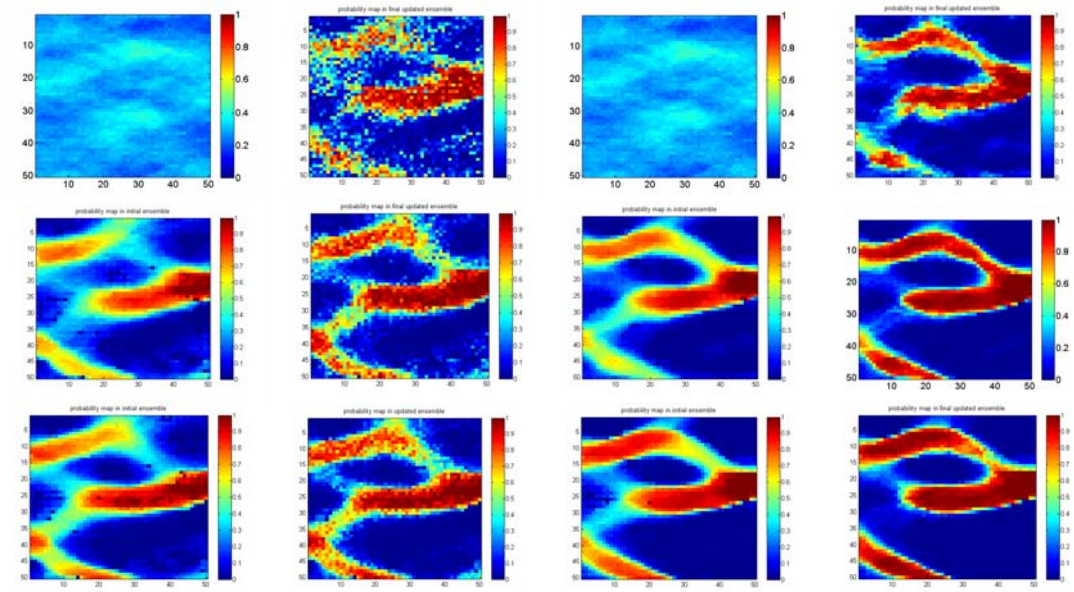

Figure 5: The probability fields of the channel

In Fig. 5 are presented the probability fields of the channel occurrence calculated from the initial ensemble (columns one and three) and from the updated ensemble (columns two and four). The first two columns are assigned to the first experiment whereas the last two columns are of the second. The lines of the figure correspond to the iterations performed with the IAGM. One can observe that both parameterizations are able to estimate the position of the channel but the first experiment was more close to it. Even if the estimation of the channel position is apparently

better, the geological plausibility is better preserved in the second experiment. One can observe this, comparing the evolution of the first ensemble member during the iterations in both experiments (Fig. 6).

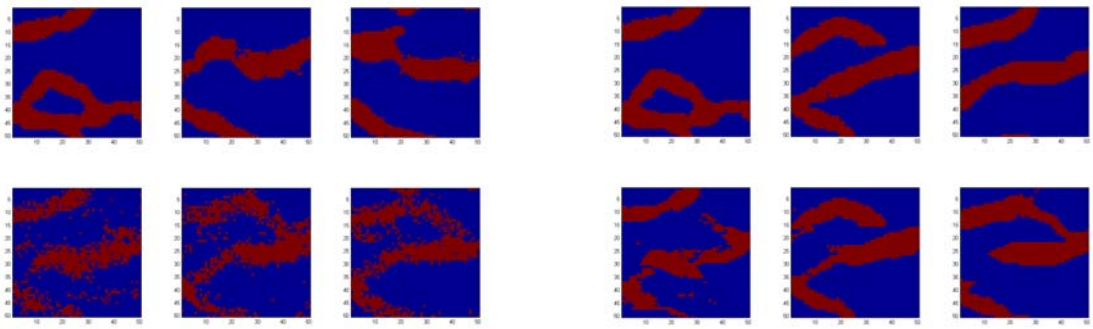

Figure 6: The first ensemble member in initial and updated ensemble

After the last iteration, the channel continuity is better preserved when for the facies parameterization is used the conditional mean (Fig 6, last row, picture six compared with picture three). Looking at the results obtained in the first experiment (Fig. second row, first three pictures), the channel facies does not shows like a channel and this is the due the noise introduced by the random sampling when parameterizing the facies field. It may be necessary more iteration to obtain updated facies fields with plausible geometry, but this was not done here. The data match in the first experiment can be visualized in Fig. 7 where the water rates at producers in the updated ensemble of the last iteration are shown.

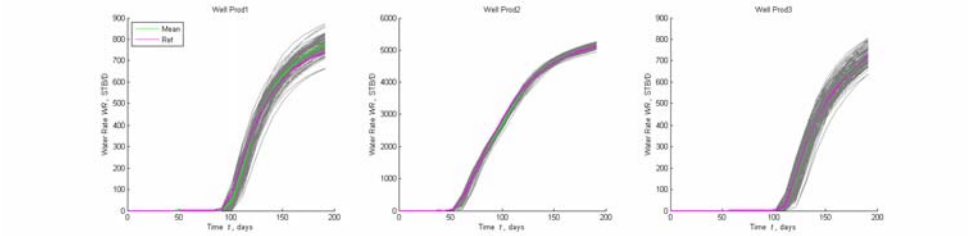

Figure 7: The water rates at the producers, last iteration first experiment.

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ABOUT METHODS OF SOLVING MULTI-OBJECTIVE FRACTIONAL PROGRAMMING PROBLEMS

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Abstract: The problem of optimizing one or several ratios of functions is called a fractional program. This paper presents solving methods for special classes of fractional programming problems, namely the sum and product of ratios problems.
Mathematics Subject Classification (2000): 90C05, 90C29, 90C32.
Key words: Global optimization, linear fractional functions, sum of ratios, product of ratios

1. Introduction

The problem of optimizing one or several ratios of functions is called a fractional program. Due to its importance in modeling various decision processes in management science, operational research, and economics, and also due to its frequent appearance in other problems that are not necessarily economical, such as information theory, numerical analysis, stochastic programming, decomposition algorithms for large linear systems, etc., the fractional programming method has received particular attention in the last four decades. Currently there is a growing interest in fractional programming. This paper considers special classes of fractional programming problems, namely the sum and product of ratios problems for which it presents solving methods [1],[2],[3],[5], [10], [11].

2. A brief description of the problem

Consider the multi-objective fractional programming problem

(P) $\max(\min)\{h(f_1(x)/g_1(x),..., f_m(x)/g_m(x)) \mid x \in D\}$

where D is a nonempty polytope in R^n, $f_1,..., f_m, g_1,..., g_m$ are linear affine functions on R^n such that $-\infty < a_i := \min_{x\in D} f_i(x)/g_i(x) < +\infty$, $i=1,\ldots,m$, while $h: R^m \to R$ is a continuous function, increasing on $R^m_{a+} := \left\{ y \in R^m \mid y_i \geq a_i, i = 1,..., m \right\}$, i.e. satisfying $a_i \leq y_i' \leq y_i, i = 1,..., m \Rightarrow h(y') \leq h(y)$.

In this paper we present important special cases of these problems that have been previously studied in the literature:

(1) $\max(\min)\left\{\min(\max)\left(f_1(x)/g_1(x),..., f_m(x)/g_m(x)\right) \mid x \in D\right\}$,

(2) $\max(\min)\left\{\sum_{i=1}^{m} f_i(x)/g_i(x) \mid x \in D\right\}$,

(3) $\max(\min)\left\{\prod_{i=1}^{m} f_i(x)/g_i(x) \mid x \in D\right\}$.

Consider the problem

(4) $\min \sum_{i=1}^{m} \frac{< n_i, y > +\alpha_i}{< d_i, y > +\beta_i}$, s.t. $y \in Y$

where $m \geq 2$, $Y \subseteq R^n$ is a nonempty, compact convex set and where, for each i=1,…,m, $[< d_i, y > +\beta_i]$ is a positive for all $y \in Y$.

3. Solving methods

In [1], Benson presented a method for constructing test problems of the form of problem (4). Given the nonempty, compact, convex set Y ,the method constructs a sum-of-ratios problem (4) that can be globally solved by convex programming and univariate search. The test problem construction method presented relies upon the following result.

Theorem

Let $i \in \{1,...,m\}$ and let $Z = \{y \in Y \mid a(< n_i, y > +\alpha_i) + b(< d_i, y > +\beta_i) = K\}$ where a, b, K are scalars and a, b are not both zero. Then, the function $f_i(y) = \frac{< n_i, y > +\alpha_i}{< d_i, y > +\beta_i}$ is a convex function on Z whenever $a \neq 0$ and $K/a \geq 0$.

In [2], Benson show that, by using suitable transformations, there is a potential to globally solve problem (4) by techniques that are well known or are similar to well-known techniques.

In [3], Benson presented a global optimization approach for generating efficient points for multi-objective concave fractional programming problems. The main work of the approach involves solving an instance of a concave multiplicative fractional program

$(\overline{W})$ $\quad \overline{w}^* = \max_{x \in X} \prod_{i=1}^{m} \frac{f_i(x)}{g_i(x)}$.

Problem $(\overline{W})$ is a global optimization problem for which, he developed and validate a branch and bound algorithm for globally solving problem $(\overline{W})$.

In [8],Phuong and Tuy presented an efficient unified method for solving a wide class of generalized linear fractional programming problems. This class includes such problems as: optimizing a pointwise maximum or pointwise minimum of a finite number of ratios of linear functions, optimizing a sum or product of such ratios over a polytope. Their approach is based on the theory of monotonic optimization.

In [6], Dür et al. consider the sum-of-ratios program

$\max \sum_{i=1}^{p} \frac{n_i(x)}{d_i(x)}$, s.t. $x \in P$, where $P = \{x \in R^n \mid g_k(x) \geq 0, k = 1,...,m\}$.

Let n_i, d_i, i=1,…,p, g_k ,k=1,…,m denote continuous real-valued functions on the n-dimensional Euclidean space R^n.We assume that P is a nonempty bounded set, $d_i(x) > 0$ on P ,i=1,…,p. They presented a general branch and bound algorithm which uses rectangular partitions in the Euclidian space of dimension p. Constrained maximization of a sum of p>1 ratios is a difficult nonconvex problem (even if all functions involved are linear) and it has many applications in management sciences. Theoretically, this algorithm is applicable under very general assumption. Practically, they give an efficient implementation for affine fractions. In their paper, the bounding procedures use dual constructions and the calculations of efficient points of a corresponding multiple-objective optimization problem.

In [7], Konno and Yamashita considered a special class of nonconvex minimization problems

$$\min \sum_{i=1}^{p} (d_i^T x + d_{i0})/(c_i^T x + c_{i0}),\ \text{s.t. } Ax = b,\ x \geq 0,$$

$$\min \prod_{i=1}^{p} (d_i^T x + d_{i0})/(c_i^T x + c_{i0}),\ \text{s.t. } Ax = b,\ x \geq 0,$$

where $c_i, d_i \in R^n, c_{i0}, d_{i0} \in R, i = 1,..., p, A \in R^{m \times n}, b \in R^m$.

They developed efficient deterministic algorithms for globally minimizing the sum and the product of several linear fractional functions over a polytope.

In [4], [5],Crouzeix,Ferland,Schaible presented an algorithm which combines the parametric approach and the Huard method of centers. A minimum of the problem is obtained by solving a sequence of unconstrained optimization problems.

In [9], Roubi consider the following generalized fractional programming problem

(5) $$\inf_{x \in X} \left\{ \max_{i \in I} \frac{f_i(x)}{g_i(x)} \right\}, I = \{1,..., m\},$$

$X = \{x \in R^n \mid h_j(x) \leq 0, 1 \leq j \leq p\}$ is a nonempty subset of R^n, f_i and g_i are continuous on X and satisfy $g_i(x) > 0$,for all $x \in X, i \in I$.

The Roubi's algorithm combines the parametric approach and the Huard method of centers. A solution of problem (5) is obtained by solving a sequence of unconstrained problem.

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TRANSMUTED GENERALIZED PARETO DISTRIBUTION

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Abstract: The generalized Pareto distribution was introduced by Pickands (1975) and it was used to model socio-economic phenomena, physical and biological processes, in reliability studies and the analysis of environmental extremes. Here we generalized this probability distribution using the quadratic rank transmutation map studied by Shaw et al (2009) [12]. Let a random variable X be, if c is the threshold or lower bound of X, then the distribution of X is the 3-parameter generalized Pareto distribution, given by

$$G(x,a,b,c)=\begin{cases}1-\left(1-\frac{a}{b}(x-c)\right)^{\frac{1}{a}} & for\ \ a\neq 0\\ 1-e^{-\frac{x-c}{b}} & for\ \ a=0\end{cases}$$

where c is a location parameter, b is a scale parameter, a is a shape parameter and G is the cumulative distribution function [13]. According to the quadratic rank transmuted map the cumulative distribution function of Transmuted Generalized Pareto Distribution can be expressed as [1]

$$F(x,a,b,c,\lambda)=(1+\lambda)G(x)-\lambda G^2(x).$$

Some mathematical properties of the new distribution are presented in this paper.
Mathematics Subject Classification (2010): 62P30, 62N02, 62N05
Keywords: transmuted probability distributions, quadratic rank transmutation map, distribution of order statistics, estimating of parameters, numerical simulation.

1. Introduction

In recent years there have been considerable efforts in finding statistical models, not necessarily symmetrical to represent real world phenomena. Given that many of these phenomena are not symmetrical, the efforts were directed towards skewed distributions from other popular distributions symmetrical or not. Asymmetrical patterns that express different degrees of asymmetry are a useful tool in modeling real world phenomena. Starting from a symmetrical distribution with cumulative distribution function $G(x)$ and probability density function $g(x)$, Azzalini [4] proposes asymmetric distribution whose probability density function is $f(x)=2g(x)G(\beta x)$, where β is the parameter the asymmetry. Shaw and Buckley [12] investigate a novel technique for introducing skewness or kourtosis into symetric or other distribution. Aryal and Tsokos [1] use quadratic rank transmutation map to generate a flexible family of probability distribution starting from extreme value distribution and generalize the two parameter Weibull distribution [2]. Merovci and Elbatal [4], [8], Merovci and Puka [9] generalize different kind of Lindley distribution and Pareto distribution using quadratic rank transmutation map obtaining new distributions with applications in

reliability. Elbatal et al. [5] consider like base distribution linear exponential distribution and by quadratic rank transmutation map obtain transmuted generalized linear exponential distribution which can use in modeling of life time data. Khan and King [10] introduce transmuted modified Weibull distribution as an important competitive model for life time distributions. The transmuted additive Weibull distribution introduced by Elbatal and Aryal [6] can be used to model lifetime data. The purpose of this paper is to investigate a probability distribution that can be obtained from an asymmetric distribution, namely generalized Pareto distribution and that can be used for modeling and analyzing real-world data.

2. Quadratic rank transmutation map

Definition 1. [1] *A functional composition of the cumulative distribution function of one probability distribution $F(x)$ with the inverse cumulative distribution function of another $G(x)$,*

$$\mathsf{R}_{GF}(u) = F\left(G^{-1}(u)\right) \tag{1}$$

is called the transmutation map, where G is considered as the base distribution and F as the modulated distribution.

Obviously, one can also define mutual representation

$$\mathsf{R}_{FG}(u) = G\left(F^{-1}(u)\right) \tag{2}$$

thus obtaining a pair of rank transmutation maps. Note that the inverse cumulative distribution function also known as quantile function is defined as

$$F^{-1}(y) = \inf_{x \in R}\left\{F(x) \geq y | y \in [0, 1]\right\} \tag{3}$$

The functions $\mathsf{R}_{FG}(u)$ and $\mathsf{R}_{GF}(u)$ both map the unit interval $I = [0, 1]$ into itself, and under suitable assumptions are mutual inverses and they satisfy $\mathsf{R}_{FG}(0) = 0$ and $\mathsf{R}_{GF}(0) = 1$.

Definition 2. [1] *A Quadratic Rank Transmutation Map, (QRTM), is defined as*

$$\mathsf{R}_{FG}(u)(u) = u + \lambda u(1-u)\ ,\ |\lambda| \leq 1 \tag{4}$$

from which it follows that the cdf's satisfy the relationship

$$F(x) = (1+\lambda)G(x) - \lambda(G(x))^2 \tag{5}$$

which on differentiation yields,

$$f(x) = g(x)[1 + \lambda - 2\lambda G(x)] \tag{6}$$

where $g(x)$ and $f(x)$ are the corresponding pdfs associated with cdf $G(x)$ and $F(x)$ respectively.

The effect of the QRTM is to introduce skew to a symmetric base distribution. There is no specific requirement that the base distribution F_1 be symmetric. However, if the F_1 distribution is symmetric about the origin, in the sense that $F_1(x) = 1 - F_1(x)$, we have the result that the distribution of the square of the transmuted random variable is identical to that of the distribution of the square of the original random variable. A consequence of this is that if the original distribution is symmetric, then the QRTM preserves all even moments [12].

3. Transmuted generalized Pareto distribution

Definition 3. [13] *Let a random variable X be defined as*

$$X = \frac{b}{a}\left(1 - e^{-aY}\right) \tag{7}$$

where a, b are parameters and $Y \sim> Exp(1)$, a random variable with the standard exponential distribution. If c is the threshold or lower bound of X, then the distribution of X is the 3-parameter generalized Pareto distribution, given by

$$G(x,a,b,c)=\begin{cases}1-\left(1-\dfrac{a}{b}(x-c)\right)^{\frac{1}{a}} & \{a<0 \text{ and } c\le x\} \text{ or } \left\{a>0 \text{ and } c\le x\le c+\dfrac{b}{a}\right\} \\ 1-e^{-\frac{x-c}{b}} & a=0 \text{ and } x\ge c\end{cases} \quad (8)$$

where c is a location parameter, b>0 is a scale parameter, a is a shape parameter.
The density probability function of random variable *X* is given by

$$g(x,a,b,c)=\begin{cases}\dfrac{1}{b}\left[1-\dfrac{a}{b}(x-c)\right]^{\frac{1}{a}-1} & \{a<0 \text{ and } c\le x\} \text{ or } \left\{a>0 \text{ and } c\le x\le c+\dfrac{b}{a}\right\} \\ \dfrac{1}{b}e^{-\frac{x-c}{b}}, & a=0 \text{ and } x\ge c\end{cases} \quad (9)$$

Definition 4. *A random variable X is said to have a transmuted generalized Pareto probability distribution with parameters* $a\in R, b>0,\ c>0$ *and* $|\lambda|\le 1$ *if its cdf is given by*

$$F(x,a,b,c,\lambda)=\begin{cases}\left\{1-\left[1-\dfrac{a}{b}(x-c)\right]^{\frac{1}{a}}\right\}\left\{1+\lambda\left[1-\dfrac{a}{b}(x-c)\right]^{\frac{1}{a}}\right\}, \\ \text{for } \{a<0 \text{ and } c\le x\} \text{ or } \left\{a>0 \text{ and } c\le x\le c+\dfrac{b}{a}\right\} \\ \left(1-e^{-\frac{x-c}{b}}\right)\left(1+\lambda e^{-\frac{x-c}{b}}\right), \qquad a=0 \text{ and } x\end{cases} \quad (10)$$

where λ *is the transmuted parameter.*
The density probability function of random variable *X* is given by

$$f(x,a,b,c,\lambda)=\begin{cases}\dfrac{1}{b}\left[1-\dfrac{a}{b}(x-c)\right]^{\frac{1}{a}-1}\left\{1-\lambda+2\lambda\left[1-\dfrac{a}{b}(x-c)\right]^{\frac{1}{a}}\right\}, & x>c, a\ne 0 \\ \dfrac{1}{b}e^{-\frac{x-c}{b}}\left(1-\lambda+2\lambda e^{-\frac{x-c}{b}}\right), & x>c, a=0\end{cases} \quad (11)$$

Note that the generalized Pareto distribution is a special case for $\lambda=0$ of the transmuted generalized Pareto distribution.

4. Statistic properties

4.1. Quantiles

The quantile x_q of transmuted generalized Pareto distribution is the real solution of the equation $F(x,a,b,c,\lambda)=q$. The median can be obtained by setting $q=0.5$.

$$X_q=c+\frac{b}{a}\left\{1-\left[\frac{\lambda-1+\sqrt{(1-\lambda)^2-4\lambda q+4\lambda}}{2\lambda}\right]^a\right\}, \quad \text{for } a\ne 0 \quad (12)$$

$$X_q = c - b\ln\left[\frac{\lambda - 1 + \sqrt{(1-\lambda)^2 - 4\lambda q + 4\lambda}}{2\lambda}\right],\ for\ a = 0\ ,\ 0 < q < 1 \qquad (13)$$

4.2. Random number generation

Using the method of inversion we can generate random numbers from a transmuted generalized Pareto probability distribution replacing in equations (12) and (13) q by $U \sim > \mathrm{U}(0,1)$.

4.3. Moments

The moments of a transmuted generalized Pareto distributed random variable X is given by the following proposition.

Proposition 1. *The n-moment $E[X^n]$ of transmuted generalized Pareto distributed random variable X are given as*

$$E[X^n] = \begin{cases} \sum_{k=0}^{n}\binom{n}{k}c^{n-k}\left(\frac{b}{a}\right)^k \sum_{i=0}^{k}\binom{k}{i}(-1)^{i+1}\left(\frac{2\lambda}{i\cdot a+2}\left[1-\frac{a}{b}(x-c)\right]+\frac{1-\lambda}{i\cdot a+1}\right)\times & \\ \times\left[1-\frac{a}{b}(x-c)\right]^{i\cdot a+1}\Big|_c^M & a \neq 0 \\ -b\cdot x^n e^{-\frac{x-c}{b}}\Big|_c^M + n\cdot b\cdot E[X^{n-1}],\ n \geq 1 & a = 0 \end{cases} \qquad (14)$$

where $M = \infty\ if\ a \leq 0\ or\ M = c + \frac{b}{a}\ if\ a > 0$

5. Estimation of parameters

In the rest of paper we consider the parameter $a \neq 0$.

5.1. Method of moments (MOM

Let $X_1, X_2, ..., X_n$ be a sample of random variable X, to determine the values for vector θ we must solve the system

$$E[X] = \overline{x}\ , E[X^2] = \overline{x^2}\ ,\ E[X^3] = \overline{x^3}\ ,\ E[X^4] = \overline{x^4} \qquad (15)$$

where $E[X^i]$, $i = \overline{1,4}$ are given by Proposition 1 and $\overline{x^i}$, $i = \overline{1,4}$ are empirical moments of order i.

5. 2. Method of probability-weighted moments (PWM

Consider the rth probability-weighted moments W_r given by

$$Wr = E\left[x(F)(1-F)^r\right] \qquad (16)$$

where $r = \overline{0,3}$. Solving the system (20) we obtain $\hat{\theta} = (\hat{a}, \hat{b}, \hat{c}, \hat{\lambda})$ a estimating vector for $\theta = (a, b, c, \lambda)$.

5.3. Method of maximum likelihood estimation (MLE)

We consider the likelihood function

$$L(X;\theta) = \prod_{i=1}^{N}\frac{1}{b}\left(1-\frac{a(X_i - c)}{b}\right)^{\frac{1}{a}-1}\left[1-\lambda+2\lambda\left(1-\frac{a}{b}(X_i - c)\right)^{\frac{1}{a}}\right] \qquad (17)$$

and its logarithm

$$\mathsf{L}(X;\theta) = \ln\big(L(X;\theta)\big) \tag{18}$$

To obtain the $\hat{\theta} = (\hat{a},\hat{b},\hat{c},\hat{\lambda})$ a estimating MLE vector for $\theta = (a,b,c,\lambda)$ one must solve the system

$$\frac{\partial \mathsf{L}(X;a,b,c,\lambda)}{\partial a} = 0 \,,\; \frac{\partial \mathsf{L}(X;a,b,c,\lambda)}{\partial b} = 0 \,,\; \frac{\partial \mathsf{L}(X;a,b,c,\lambda)}{\partial \lambda} = 0 \tag{19}$$

A maximum likelihood estimator cannot be obtained for *c*, because the likelihood function is unbounded with respect to *c*. Since *c* is the lower bound of the random variable *X,* we may use the constraint $c \le min\{X\}$, the lowest sample value. Clearly, the likelihood function is maximum with respect to *c* when $c = min\{X\}$. Applying the usual large sample approximation, the maximum likelihood estimators of $\theta = (a,b,c,\lambda)$ can be treated as being approximately multivariate normal with mean $\theta = (a,b,c,\lambda)$ and variance-covariance matrix equal to the inverse of the expected information matrix. That is,

$$\sqrt{n}\big(\hat{\theta} - \theta\big) \sim> N\big(0, I^{-1}(\hat{\theta})\big) \tag{20}$$

where $I^{-1}(\hat{\theta})$ is the variance-covariance matrix of the unknown parameters $\theta = (a,b,\lambda)$, i.e.

$$I_{11} = \frac{\partial^2 L}{\partial a^2} \,,\; I_{12} = \frac{\partial^2 L}{\partial a \partial b} \,,\; I_{13} = \frac{\partial^2 L}{\partial a \partial \lambda} \,,\; I_{22} = \frac{\partial^2 L}{\partial b^2} \,,\; I_{23} = \frac{\partial^2 L}{\partial b \partial \lambda} \,,\; I_{33} = \frac{\partial^2 L}{\partial \lambda^2} \tag{21}$$

Approximate 100(1-α)% two sided confidence intervals for a and α are, respectively, given by

$$\hat{a} \pm Z_{\frac{\alpha}{2}} \sqrt{I_{11}^{-1}\big(\hat{\theta}\big)} \,,\; \hat{b} \pm Z_{\frac{\alpha}{2}} \sqrt{I_{22}^{-1}\big(\hat{\theta}\big)} \quad \hat{\lambda} \pm Z_{\frac{\alpha}{2}} \sqrt{I_{33}^{-1}\big(\hat{\theta}\big)} \tag{22}$$

where $Z_{\frac{\alpha}{2}}$ is the upper α th percentiles of the standard normal distribution. We can easily compute the Hessian matrix and its inverse and hence the values of the standard error and asymptotic confidence intervals using a mathematical software.

6. Order statistics

In statistics, the *k*th order statistic of a statistical sample is equal to its *k*th smallest value. Together with rank statistics, order statistics are among the most fundamental tools in non-parametric statistics and inference. We know that if $X_{(1)} \le X_{(2)} \le ... \le X_{(n)}$ denotes the order statistics of a random sample $X_1, X_{(2)},...,X_{(n)}$ from a continuous population with cdf $F(x)$ and pdf $f(x)$ then the pdf of $X_{(j)}$ is given by

$$f_{X_{(j)}}(x) = \frac{n!}{(j-1)!(n-j)!} f(x)\big[F(x)\big]^{j-1}\big[1-F(x)\big]^{n-j} \tag{23}$$

It is useful in applications the pdf of largest order statistic $X_{(n)}$, given by

$$\begin{aligned} f_{X_{(n)}}(x) = n\left\{\frac{1}{b}\left(1-\frac{a}{b}(x-c)\right)^{\frac{1}{a}-1}\left[1-\lambda+2\lambda\left(1-\frac{a}{b}(x-c)\right)^{\frac{1}{a}}\right]\right\}\times \\ \times\left\{\left[1-\left(1-\frac{a}{b}(x-c)\right)^{\frac{1}{a}}\right]\left[1+\lambda\left(1-\frac{a}{b}(x-c)\right)^{\frac{1}{a}}\right]\right\}^{n-1} \end{aligned} \tag{24}$$

and also the pdf of smallest order statistic $X_{(1)}$, given by

$$f_{X_{(1)}}(x) = n\left\{1-\left[1-\left(1-\frac{a}{b}(x-c)\right)^{\frac{1}{a}}\right]\left[1+\lambda\left(1-\frac{a}{b}(x-c)\right)^{\frac{1}{a}}\right]\right\}^{n-1} \tag{25}$$

6. Conlusions

In the present paper, we have introduced a new generalization of the Generalized Pareto distribution called the Transmuted Generalized Pareto Distribution. The 3-parameter GPD is embedded in the proposed distribution. Some mathematical properties along with estimation issues are addressed.

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HOW TO TEACH ELEMENTARY GEOMETRIC PROPERTIES OF PLANE CURVES USING MATHCAD

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Abstract: The regular curves in the Euclidian plane admit parametric representations, and their elementary geometric properties and various entities associated to them, such as curvature, radius of curvature, osculating circle and arc length can be given using the differential and integral calculus from MathCad. In this paper we used MathCad because it is easy to be learned and can be studied simultaneously with differential geometry.
Mathematics Subject Classification (2010): 51M04, 14J29.
Key words: plane curve; tangent line; normal line; curvature; osculating circle; MathCad

1. Introduction

The purpose of this paper is the study of elementary geometric properties of plane curves using a modern software, MathCad. The plane and space curves are an important part of many engineering disciplines and it is well known that to teach Differential Geometry of curves in a traditional manner for engineering students is not so easy for the professor, because this requires that the students to be acquainted with some very useful results of the differential and integral calculus and to have a deeper knowledge of geometry and linear algebra.

In this work we are interested to show how MathCad can be used as an alternative teaching and learning tool for an important chapter of Differential Geometry, "Plane Curves".

Why we want to use MathCad?

Unlike many other mathematical software, we choose MathCad because it has a "user-friendly interface" based on "what-you-see-is-what-you-get" feature, in which it is possible to combine standard math equations, graphs and text regions. Using a software tool like MathCad the students learn to draw plane curves and all the other geometric objects related to them (points, straight lines, circles etc.), can be able to compute some numerical entities associated to the curves (curvature, radius of curvature, coordinates of curvature center etc.), can easily see the connections between plane curves and their equations and have a better understanding of mathematical concepts.

Section 2 of this paper is dedicated to several basic concepts about plane curves. In Section 3 we consider an example of a regular plane curve, namely cycloid, for which we present two MathCad worksheets about how to plot the tangent line, the normal line and the osculating circle.

2. Some Background on Plane Curve Theory

In this section we want to remind some basic notions about plane curves theory. In what follows, we recall terminology and several notations from the specialized literature, which will be further needed. We assume that the reader is familiar with some basic concepts, such as vector function, continuity, differentiability, function of class C^k, $k \geq 0$, integer, and has some experience working with these.

We let **R** denotes the field of real numbers and respectively, $\mathbf{R}^2$ denotes 2-dimensional Euclidean vector space, equipped with canonical orthonormal basis $\{\vec{i},\ \vec{j}\}$.

What means a curve in plane?

Intuitively, a simple way of thinking of a curve is as the trajectory described by the movement of a particle in plane: at the time t, the particle is at the point in the plane whose coordinates are $\big(x(t), y(t)\big)$. This suggests the following formal definition:

We regard a (*parametrized*) *plane curve* as the image of a vector-valued function

$$\vec{r}: I \subset \mathbf{R} \to \mathbf{R}^2,\ \vec{r}(t) = x(t)\,\vec{i} + y(t)\,\vec{j},\ t \in I,$$

or, in short,

$$\vec{r}(t) = \big(x(t), y(t)\big),\ t \in I,$$

which is one-to-one and (infinitely) continuously differentiable on the non-empty interval I.

A parametrized curve $\vec{r}: I \to \mathbf{R}^2$ is said to be *regular* (*smooth*) if all its points are regular, i.e., the points for which $\left\|\vec{r}\,'(t)\right\| \neq 0$, for all $t \in I$, where $\vec{r}\,'(t) = x'(t)\,\vec{i} + y'(t)\,\vec{j}$ is called the *tangent* (*velocity*) *vector* to the curve. Thus the velocity of a regular curve never vanishes. For the study of curves, it is essential that the curve to be regular and hence, for now on, we will work only with regular curves that have twice continuously differentiable equations.

Remark 2.1. If a plane curve resides in the Oxy plane, then the plane curve can be described in terms of a parameter (parametric equations) or using algebraic equations:

- *Parametric equations*: $\begin{cases} x = x(t) \\ y = y(t) \end{cases},\ t \in I,\ x,\ y: I \to \mathbf{R}$ of class C^1 (t is *parameter*)

or, equivalent,

- *Vector equation*: $\vec{r}(t) = x(t)\,\vec{i} + y(t)\,\vec{j},\ t \in I,\ x,\ y: I \to \mathbf{R}$ of class C^1.

1) Circle with center (x_0, y_0) and radius R: $\begin{cases} x = x_0 + R \cdot \cos t \\ y = y_0 + R \cdot \sin t \end{cases},\ t \in [0,\ 2\pi)$.

2) Cycloid: $\begin{cases} x = a\,(t - \sin t) \\ y = a\,(1 - \cos t) \end{cases},\ t \in \mathbf{R},\ a > 0$.

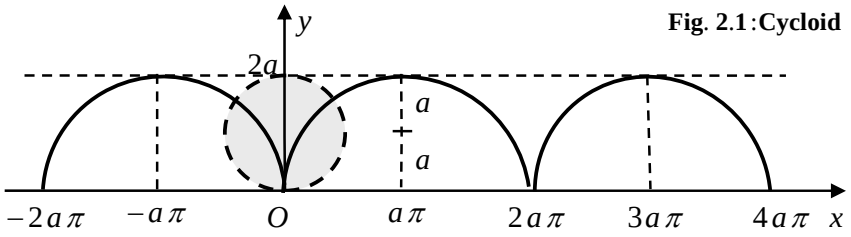


Fig. 2.1: Cycloid

- *Explicit equation*: $y = f(x),\ x \in I,\ f: I \to \mathbf{R}$ of class C^1.

Observation 2.1. An explicitly defined curve: $y = f(x),\ x \in I$, can be parametrized by the vector function: $\vec{r}: I \to \mathbf{R}^2,\ \vec{r}(t) = \big(t, f(t)\big),\ t \in I$.

1) Sinusoid (sine wave): $y = \sin x,\ x \in \mathbf{R}$.

2) Catenary: $y = a \cosh\left(\frac{x}{a}\right),\ x \in \mathbf{R},\ a > 0$.

- *Implicit equation*: $F(x, y) = 0,\ (x, y) \in D \subset \mathbf{R}^2$ domain, $F: D \to \mathbf{R}$ of class C^1.

Observation 2.2. It may or may not be possible to parametrize an implicitly defined curve. However, if $M_0(x_0, y_0)$ is an arbitrary point of the curve (i.e., $F(x_0, y_0) = 0$) and $\operatorname{grad} F(x_0, y_0) \neq \vec{0}$ (i.e., either $\frac{\partial F}{\partial x}(x_0, y_0) \neq 0$, or $\frac{\partial F}{\partial y}(x_0, y_0) \neq 0$), then, by the Implicit Function Theorem, the implicitly defined curve $F(x, y) = 0$ admits a local parametrization near M_0. Consequently, $F(x, y) = 0$ defines y as a differentiable function of x in some open interval $I \subset \mathbb{R}$ containing x_0, i.e., $y(x_0) = y_0$ and $F(x, y(x)) = 0$, for all $x \in I$, hence the vector function $\vec{r}(t) = (t,\ y(t))$, $t \in I$, is a parametrization of the curve.

Actually, the situation described in the previously observation ensures only the existence of a local parametrization for implicitly defined curves, but it gives no clue about how to find explicit local parametrizations.

1) Circle with the origin as its center and radius R: $x^2 + y^2 - R^2 = 0$, $R > 0$.

2) Folium of Descartes: $x^3 + y^3 - 3\,a\,x\,y = 0$, $a > 0$.

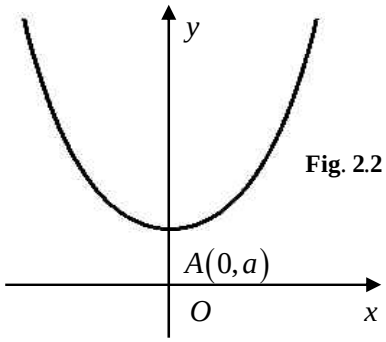
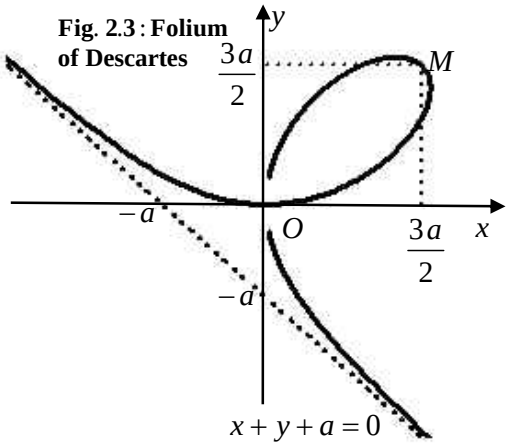


Fig. 2.2 : Catenary

Fig. 2.3 : Folium of Descartes



• *Polar equation*: $\rho = \rho(\theta)$, $\theta \in [\theta_1, \theta_2] \subset \mathbf{R}$, $\rho : [\theta_1, \theta_2] \to \mathbf{R}_+$ of class C^1.

1) Cardioid: $\rho = 2\,a\,(1 - \cos\theta)$, $\theta \in [0,\ 2\pi]$, $a > 0$.

2) Spiral of Archimedes: $\rho = a \cdot \theta$, $\theta \in [0, \infty)$, $a > 0$.

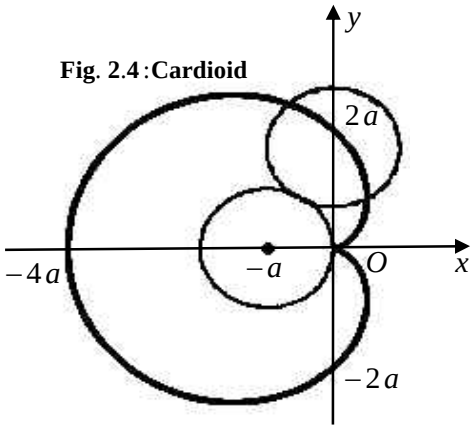


Fig. 2.4 : Cardioid

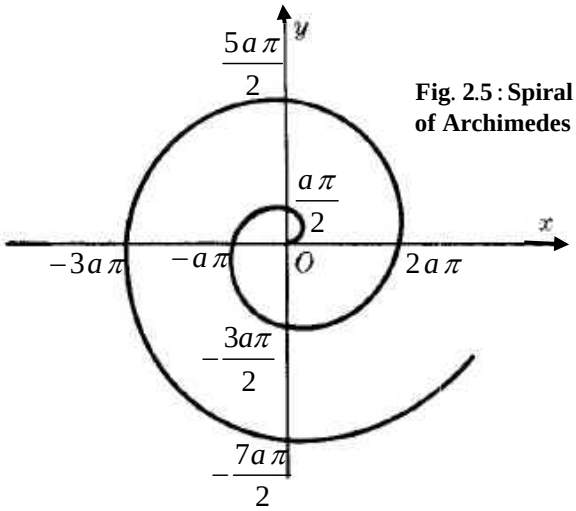


Fig. 2.5 : Spiral of Archimedes

Observation 2.3. A curve defined in polar coordinates as: $\rho = \rho(\theta)$, $\theta \in [\theta_1, \theta_2]$, has the parametric equations given by: $x = \rho(\theta)\cos\theta$, $y = \rho(\theta)\sin\theta$, $\theta \in [\theta_1, \theta_2]$.

2.1. Tangent line and normal line

The tangent line is the one that "approximates the best" a curve at a given point. For instance, if we consider an example of someone who goes around the Earth along the Equator, we notice that this person has at any moment the impression that goes along a straight line, namely the *tangent line*.

For an arbitrary regular parametrized curve by $\vec{r}(t) = x(t)\,\vec{i} + y(t)\,\vec{j}$, the tangent line is directed along by the velocity vector $\vec{r}\,'(t) = x'(t)\,\vec{i} + y'(t)\,\vec{j}$. Furthermore, the *normal line* to the curve is orthogonal to the tangent line, therefore its direction coincides with that of the *normal vector* $\vec{n}(t) = -y'(t)\,\vec{i} + x'(t)\,\vec{j}$. To write the equations of these straight lines we use their vector form. Based on what we said previously, the vector equation of the tangent line to the curve at the point $M_0(x_0, y_0)$ is

$$\overrightarrow{\mathrm{Tg}}(t) = \vec{r}(t_0) + t\,\vec{r}\,'(t_0),\ t \in \mathbf{R}$$

where $x_0 = x(t_0)$, $y_0 = y(t_0)$ and $\vec{r}(t_0) = x(t_0)\,\vec{i} + y(t_0)\,\vec{j}$ is the position vector of M_0.

The vector equation of the normal line through the point $M_0(x_0, y_0)$ of the curve is:

$$\overrightarrow{\mathrm{N}}(t) = \vec{r}(t_0) + t\,\vec{n}(t_0),\ t \in \mathbf{R}.$$

2.2. Curvature and radius of curvature

Now we present some useful facts concerning the curvature of a plane curve which probably will contribute to a better understanding of its geometrical meaning.

What is the curvature of a curve?

To answer this, for instance, it is natural to consider that a straight line is not curved at all; its curvature has to be identically zero. A circle with smaller radius R is more "curved" than a circle with larger one (for instance, if we travel again around the Earth along the Equator, we do not notice that we travel along a circle, but rather along a straight line, simply because the radius of the Earth at the Equator is huge); therefore, the curvature of a circle is defined to be the reciprocal of the radius: $K = \frac{1}{R}$. As such, circles and straight lines have constant curvatures. Regular curves that are not (parts of) circles or straight lines will have a curvature varying from point to point.

Curvature measures how “curved” is a curve at a certain point, hence it is a measure of how much the curve deviates from a straight line. In other words, the curvature of a curve at a point is a measure of how much the change in a curve at a point is changing.

Given a regular parametrically defined plane curve, $\vec{r}(t) = x(t)\,\vec{i} + y(t)\,\vec{j}$, the curvature can be written in the form

$$K(t) = \frac{x'(t)\,y''(t) - x''(t)\,y'(t)}{\left(x'(t)^2 + y'(t)^2\right)^{\frac{3}{2}}}.$$

The reciprocal of the curvature, $R(t) = \frac{1}{|K(t)|}$, is called the *radius of curvature*.

2.3. Osculating circle

Given a point on a plane curve, which is the circle going through it and giving "the best approximation" of the curve at the point?

Finally, instead of straight lines, we will be trying to approximate a plane curve by a circle. A better approximation of a plane curve than by straight lines can be obtained by circles, namely osculating circles. At every regular point M_0 on a plane curve, we ask for the circle passes through M_0 to be characterized by the following properties: the tangent lines to the curve and to the circle at M_0 to be the same, its curvature to be equal to that of the curve at M_0; this approximating circle is called the *osculating circle* to the curve at the point M_0.

The osculating circle to a plane curve with regular parametrization, $\vec{r}(t)=x(t)\,\vec{i}+y(t)\,\vec{j}$, at a given point $M_0(t=t_0)$ is the one on the concave side of the curve. More precisely, this circle has the radius $R(t_0)=\dfrac{1}{|K(t_0)|}$ (*radius of curvature*) and its center (*curvature center*) lies on the normal line through M_0 having the coordinates:

$$(\alpha(t_0),\beta(t_0))=(x(t_0),y(t_0))+\frac{1}{K(t_0)\left\|\vec{r}\,'(t_0)\right\|}\cdot(-y'(t_0),x'(t_0)).$$

3. A Case Study in MathCad: the Cycloid

Example 3.1. A MathCad worksheet for plotting the tangent line and normal line to the cycloid at the point $M_0\left(t_0=\pi/2\right)$

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The vector equation of the cycloid can be given in MathCad by: $r(t,a):=\begin{pmatrix} a\cdot(t-\sin(t)) \\ a\cdot(1-\cos(t)) \end{pmatrix}$

The value of parameter t corresponding to the point M_0 is: $t0:=\dfrac{\pi}{2}$

The Cartesian coordinates of M_0 are computed symbolically: $M0:=r(t_0,a)$

The tangent vector to the cycloid is: $r'(t,a):=\begin{pmatrix} \dfrac{d}{dt}r(t,a)_1 \\ \dfrac{d}{dt}r(t,a)_2 \end{pmatrix}$ simplify $\rightarrow\begin{pmatrix} -a\cdot(\cos(t)-1) \\ a\cdot\sin(t) \end{pmatrix}$

The tangent vector to the cycloid at M_0 is: $r'(t0,a)$ simplify $\rightarrow\begin{pmatrix} a \\ a \end{pmatrix}$

The normal vector to the cycloid is: $n(t,a):=\begin{pmatrix} -\dfrac{d}{dt}r(t,a)_2 \\ \dfrac{d}{dt}r(t,a)_1 \end{pmatrix}$ simplify $\rightarrow\begin{pmatrix} -a\cdot\sin(t) \\ -a\cdot(\cos(t)-1) \end{pmatrix}$

The normal vector to the cycloid at M_0 is: $n(t0,a)$ simplify $\rightarrow\begin{pmatrix} -a \\ a \end{pmatrix}$

The vector equations of the tangent line, respectively of the normal line to the cycloid at the point M_0, denoted by T and N, are given below:

$$T(t,a):=r(t0,a)+t\cdot r'(t0,a)\ \text{simplify}\rightarrow\begin{pmatrix} \dfrac{a\cdot(\pi+2\cdot t-2)}{2} \\ a\cdot(t+1) \end{pmatrix}$$

$$N(t,a) := r(\mathrm{t0},a) + t\cdot n(\mathrm{t0},a)\ \text{simplify} \rightarrow \begin{pmatrix} \dfrac{a\cdot(\pi - 2\cdot t - 2)}{2} \\ a\cdot(t+1) \end{pmatrix}$$

Example 3.2. A MathCad worksheet for plotting the osculating circle to cycloid at M_0

The second derivative of the vector equation of the cycloid at the point M_0 is given below:

$$r''(t,a) := \begin{pmatrix} \dfrac{d^2}{dt^2} r(t,a)_1 \\ \dfrac{d^2}{dt^2} r(t,a)_2 \end{pmatrix} \text{simplify} \rightarrow \begin{pmatrix} a\cdot\sin(t) \\ a\cdot\cos(t) \end{pmatrix} \quad \text{and} \quad r''(\mathrm{t0},a)\ \text{simplify} \rightarrow \begin{pmatrix} a \\ 0 \end{pmatrix}$$

The curvature of the cycloid at an arbitrary point has the expression:

$$K(t,a) := \frac{r'(t,a)_1\cdot r''(t,a)_2 - r''(t,a)_1\cdot r'(t,a)_2}{\left[\left(r'(t,a)_1\right)^2 + \left(r'(t,a)_2\right)^2\right]^{\frac{3}{2}}}\ \text{simplify} \rightarrow -\frac{1}{4\cdot\sqrt{a^2\cdot\sin\left(\frac{t}{2}\right)^2}}$$

The curvature and the radius of curvature of the cycloid at the point M_0 have the values:

$$K(\mathrm{t0},a)\ \text{simplify} \rightarrow -\frac{1}{2\cdot\sqrt{2}\cdot a} \qquad \text{and} \qquad R(a) := \frac{1}{\left|K(\mathrm{t0},a)\right|}\ \text{simplify} \rightarrow 2\cdot\sqrt{2}\cdot a$$

The Cartesian coordinates of the curvature center $C(a)$ of the cycloid at the point M_0 are:

$$C(a)) := r(\mathrm{t0},a) + \frac{1}{K(\mathrm{t0},a)\cdot\left|r'(\mathrm{t0},a)\right|}\cdot n(\mathrm{t0},a)\ \text{simplify} \rightarrow \begin{pmatrix} \dfrac{a\cdot(\pi+2)}{2} \\ -a \end{pmatrix}$$

To vector equation of the osculating circle passing through M_0 to the cycloid can be written as the form: $CO(t,a) := C(a) + R(\mathrm{t0},a)\cdot(\cos(t)\ \ \sin(t))^T$

Finally, to draw the first loop of cycloid we consider that $a := 1$ and $t := 0, 0.01..2\cdot\pi$

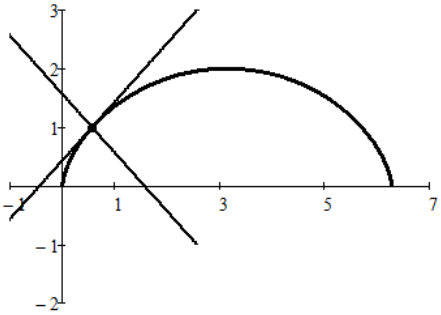

Fig. 3.1: Tangent and normal lines to cycloid

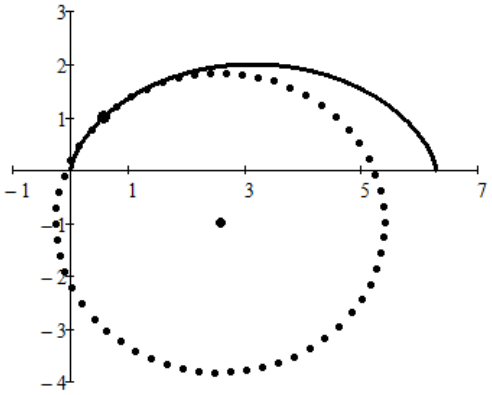

Fig. 3.2: Osculating circle to cycloid

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ON RESTRICTIONS AND QUOTIENTS OF DECOMPOSABLE OPERATORS IN BANACH SPACES

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Abstract: In this paper some results concerning the restrictions and quotients of decomposable operators are presented.
Let X be a Banach space, let T be a decomposable operator and let Y be a spectral maximal space of T. We will table some properties of $T|Y$, the restriction operator of T to Y, respectively of $\dot{T}$, the quotient operator induced by T on the quotient space $\dot{X} = X/Y$.
Mathematics Subject Classification (2010): 47B47, 47B40.
Key words: decomposable (S-decomposable); spectral maximal space; T-absorbing subspace; restriction and quotient of an operator; totally disconnected set.

1. Preliminaries

The aim of the present work is to give several results of the theory of decomposable (S-decomposable) operators in Banach spaces in a systematic way. The restrictions and the quotients of (strongly) decomposable operators with respect to invariant subspaces are presented here and it is shown that they are also (strongly) decomposable (respectively, S-decomposable) operators.
Section 1 is dedicated to the preliminaries. Let us briefly recall the terminology and the notations that are used in the sequel.
We shall denote by X a (complex) Banach space, by $\mathbf{B}(X)$ the Banach algebra of all linear bounded operators on X and by $\mathbf{C}$ the complex plane. If Y is a linear closed subspace of X invariant to an operator $T \in \mathbf{B}(X)$, then $T|Y$ is the restriction operator of T to Y and $\dot{T}$ is the quotient operator induced by T on the quotient space $\dot{X} = X/Y$. For $T \in \mathbf{B}(X)$, we also consider $\rho(T)$ to be the resolvent set of T and $\sigma(T) = \mathbf{C} \setminus \rho(T)$ to be the spectrum of T.

Definition 1.1. ([6], [7]) An operator $T \in \mathbf{B}(X)$ has the *single-valued extension property* if for any analytic function $f : D \to X$ ($D \subset \mathbf{C}$ open set), with $(\lambda I - T) f(\lambda) \equiv 0$, it follows that $f(\lambda) \equiv 0$.

If $T \in \mathbf{B}(X)$ has the single-valued extension property, then for any $x \in X$, $\rho_T(x)$ denotes the maximal domain of existence of the X-valued analytic function $\lambda \to x_T(\lambda)$ which

verifies $(\lambda I - T)x_T(\lambda) \equiv x$; the open set $\rho_T(x)$ is the *local resolvent set of* x *with respect to* T and $\sigma_T(x) = \complement\rho_T(x) = \mathbf{C} \setminus \rho_T(x)$ is the *local spectrum of* x *with respect to* T. We also have $\rho(T) \subset \rho_T(x)$, $\sigma_T(x) \subset \sigma(T)$ and we denote by

$$X_T(F) = \{x \in X;\ \sigma_T(x) \subset F,\ F \subset \mathbf{C}\}.$$

Definition 1.2. ([6], [8]) A closed linear subspace Y of X is a *spectral maximal space* of $T \in \mathbf{B}(X)$ if Y is invariant to T and if Z is another closed linear subspace of X, also invariant to T, the inclusion $\sigma(T|Z) \subseteq \sigma(T|Y)$ implies $Z \subseteq Y$.

Definition 1.3. ([12], [13]) An open set $\Omega \subset \mathbf{C}$ is said to be *of analytic uniqueness* of an operator $T \in \mathbf{B}(X)$ if for any open set $D \subseteq \Omega$ and any analytic function $f : D \to X$ verifying the equation $(\lambda I - T) f(\lambda) \equiv 0$, it follows that $f(\lambda) \equiv 0$ in D.

On account of the above definition, we deduce that for any $T \in \mathbf{B}(X)$, there is a unique maximal open set Ω_T of analytic uniqueness ([12]).

We shall denote by $S_T = \complement\Omega_T = \mathbf{C} \setminus \Omega_T$ and call it the *analytic residuum* of $T \in \mathbf{B}(X)$; T has the single-valued extension property if and only if $S_T = \varnothing$ ([12]); if $\operatorname{Int} S_T = \varnothing$, then $S_T = \varnothing$.

Definition 1.4. ([12]) Let $T \in \mathbf{B}(X)$. A subspace Y of X is said to be T*-absorbing* if for any $x \in Y$, the equation $(\lambda I - T)y = x$ has solutions y only in Y, for any $\lambda \in \sigma(T|Y)$.

We recall that any spectral maximal space of T is a T-absorbing subspace ([12]).

A subset of the complex plane $\mathbf{C}$ is said to be *of dimension* 0 (*totally disconnected*) if the connected component of each point is the set consisting of the point itself.

In Section 2, we refer to restrictions and quotients of (strongly) decomposable operators with respect to an invariant subspace; several results were obtained by Apostol in [1] and [2], and by Bacalu in [5]. Furthermore, Bacalu has shown that the quotient operator $\dot{T}$ (respectively, the restriction operator $T|Y$) of a decomposable operator T is decomposable if the intersection $\sigma(T|Y) \bigcap \sigma(\dot{T})$ is totally disconnected (respectively, if Y is T-absorbing subspace and $\sigma(T|Y) \bigcap \sigma(\dot{T})$ is totally disconnected).

2. Restrictions and quotients of decomposable operators

Definition 2.1. ([6], [8]) An operator $T \in \mathbf{B}(X)$ is called *decomposable* if for any finite open covering $\{G_i\}_{i=1}^n$ of the spectrum $\sigma(T)$, there is a system $\{Y_i\}_{i=1}^n$ of spectral maximal spaces of T such that following two conditions are verified:

1) $\sigma(T|Y_i) \subseteq G_i$, for all $i = 1, 2, ..., n$

2) $\sum_{i=1}^{n} Y_i = X$

(i.e. every $x \in X$ is of the form $x = y_1 + y_2 + ... + y_n$, with $y_i \in Y_i$, $i = 1, 2, ..., n$).

$T \in \mathbf{B}(X)$ is a *strongly decomposable operator* if for any finite system $\{G_i\}_{i=1}^n$ of open sets covering $\sigma(T)$, there is a system $\{Y_i\}_{i=1}^n$ of spectral maximal spaces of T such that:

3) $\sigma(T|Y_i)\subseteq G_i$, for all $i=1,2,...,n$

4) $\sum_{i=1}^{n}(Y_i\cap Y)=Y$, for any spectral maximal space Y of T ([1]).

One can easily see that if an operator is decomposable, than it has the single-valued extension property ([6]).

Let us mention that a strongly decomposable operator is a decomposable operator ([1]).

Lemma 2.2. ([2]) If $T\in\mathbf{B}(X)$ has the single-valued extension property and Y is a spectral maximal space of T, then $X_T(F)\cap Y=Y_{T|Y}(F)$, for any $F\subseteq\mathbf{C}$ closed.

Lemma 2.3. ([1]. [2]) If $T\in\mathbf{B}(X)$ is a decomposable operator and Y is a spectral maximal space of T, then $\sigma(\dot{T})=\overline{\sigma(T)\setminus\sigma(T|Y)}$.

Theorem 2.4. ([8]) Let $T\in\mathbf{B}(X)$ be decomposable and let $F\subseteq\sigma(T)$ be a closed subset. Then $X_T(F)$ is a spectral maximal space of T and $\sigma(T|X_T(F))\subseteq F$. Conversely, if Y is a spectral maximal space of T, then $Y=X_T(\sigma(T|Y))$.

Proposition 2.5. ([1], [2]) Let $T\in\mathbf{B}(X)$ be a strongly decomposable operator and let $Y\subseteq X$ be a spectral maximal space of T. Then the restriction $T|Y$ is strongly decomposable.

Theorem 2.6. ([1], [2]) Let $T\in\mathbf{B}(X)$ be an operator. Then T is strongly decomposable if and only if the restriction $T|Y$ is decomposable, for any spectral maximal space Y of T.

Proof. "$\Rightarrow$" Assume first that T is strongly decomposable. According to Proposition 2.5, $T|Y$ is strongly decomposable, therefore $T|Y$ is decomposable.

"$\Leftarrow$" Conversely, if $T|Y$ is decomposable for any spectral maximal space Y of T, then $T=T|X$ is decomposable (taking $Y=X$), hence T has the single-valued extension property. Let $\{G_i\}_{i=1}^{n}$ and $\{H_i\}_{i=1}^{n}$ be two finite open coverings of $\sigma(T)$ with $\overline{H_i}\subseteq G_i$, for all $i=1,2,...,n$. From Theorem 2.4, we obtain directly the following result: $Y_i=X_T(\overline{H_i})$ is a spectral maximal space of T such that $\sigma(T|Y_i)\subseteq\overline{H_i}\subseteq G_i$, $i=1,2,...,n$. Moreover, using once more Theorem 2.4, we have $Y=X_T(\sigma(T|Y))$ (where Y is a given spectral maximal space of T). By Lemma 2.2, it follows that

$$\sum_{i=1}^{n}(Y_i\cap Y)=\sum_{i=1}^{n}\left(X_T(\overline{H_i})\cap Y\right)=\sum_{i=1}^{n}\left(X_T(\overline{H_i})\cap Y\right)=\sum_{i=1}^{n}Y_{T|Y}(\overline{H_i})=Y$$

therefore T verifies Definition 2.1, hence T is strongly decomposable.

Proposition 2.7. ([2]) It $T\in\mathbf{B}(X)$ is a (strongly) decomposable operator and $P\in\mathbf{B}(X)$ is a projector with $TP=PT$, then $T|PX$ is a (strongly) decomposable operator.

Theorem 2.8. ([1], [2]) Let $T\in\mathbf{B}(X)$ be a strongly decomposable operator and let $Y\subset X$ be a spectral maximal space of T. Then the quotient operator $\dot{T}$ is strongly decomposable.

Definition 2.9. ([12]) A family $G_S\cup\{G_i\}_{i=1}^{n}$ of open sets of $\mathbf{C}$ is called an *S-covering* of a compact set $F\subset\mathbf{C}$ if $G_S\cup\left(\bigcup_{i=1}^{n}G_i\right)\supseteq F\cup S$ and $\overline{G_i}\cap S=\varnothing$, $i=1,2,...,n$, where $S\subset\mathbf{C}$ is a compact fixed set.

Definition 2.10. ([4]) An operator $T \in \mathbf{B}(X)$ is called S*-decomposable* ($S \subset \mathbf{C}$ compact) if for any finite open S-covering $G_S \cup \{G_i\}_{i=1}^n$ of $\sigma(T)$, there is a system $Y_S \cup \{Y_i\}_{i=1}^n$ of spectral maximal spaces of T such that:

(i) $\sigma(T|Y_S) \subseteq G_S$, $\sigma(T|Y_i) \subseteq G_i$, for all $i = 1,2,...,n$

(ii) $Y_S + \sum_{i=1}^{n} Y_i = X$.

When the condition (ii) is replaced by

(ii') $Y_S \cap Y + \sum_{i=1}^{n} (Y_i \cap Y) = Y$, for any spectral maximal space Y of T

then $T \in \mathbf{B}(X)$ is said to be *strongly* S*-decomposable*.

If in the definition of S-decomposability, the space Y_S is not necessarily a spectral maximal space of T and $\sigma(T|Y_S) \subseteq \widetilde{G_S}$, then we say that $T \in D_S(X)$ ($\widetilde{A}$ means $\mathbf{C} \setminus D_\infty$, where D_∞ is the unbounded component of $\mathbf{C} \setminus A$, for any bounded set $A \subseteq \mathbf{C}$).

Lemma 2.11. ([5]) If $T \in \mathbf{B}(X)$ has the single-valued extension property (i.e. $S_T = \varnothing$) and $\operatorname{Int}\left(\sigma(T|Y) \cap \sigma(\dot{T})\right) = \varnothing$, where Y is an invariant subspace to T, then the quotient $\dot{T}$ has also the single-valued extension property (i.e. $S_{\dot{T}} = \varnothing$).

Theorem 2.12. ([4], [5]) Let $T \in \mathbf{B}(X)$ be a (strongly) decomposable operator and let Y be an invariant subspace to T. Then the quotient operator $\dot{T}$ is (strongly) S-decomposable, where $S = \sigma(T|Y) \cap \sigma(\dot{T})$.

Theorem 2.13. ([4], [5]) Let $T \in \mathbf{B}(X)$ be a decomposable operator and let Y be an invariant subspace to T such that $S_{\dot{T}} = \varnothing$ and $\sigma(T|Y) \setminus \sigma(\dot{T}) \neq \varnothing$. Then the restriction operator $T|Y \in D_S(Y)$, where $S = \sigma(T|Y) \cap \sigma(\dot{T})$.

Corollary 2.14. ([4], [5]) If $T \in \mathbf{B}(X)$ is decomposable and Y is a spectral maximal space of T, then both operators $T|Y$ and $\dot{T}$ are S-decomposable, where $S = \partial\sigma(T|Y) \cap \sigma(\dot{T})$, $\operatorname{Int} S = \varnothing$ and $S_{\dot{T}} = \varnothing$ (∂A means the boundary of A, for any bounded set $A \subseteq \mathbf{C}$).

Proof. The assertion follows from Theorem 2.12, Theorem 2.13, by applying the relation

$$\sigma(\dot{T}) = \overline{\sigma(T) \setminus \sigma(T|Y)} \text{ (Lemma 2.3)}$$

and from Lemma 2.11.

Theorem 2.15. ([3]) Let $T \in \mathbf{B}(X)$ be decomposable, Y a spectral maximal space of T and $\dim\left(\partial\sigma(T|Y) \cap \sigma(\dot{T})\right) = 0$. Then both $T|Y$ and $\dot{T}$ are decomposable.

Proposition 2.16. ([5]) Let $T \in \mathbf{B}(X)$ be an S-decomposable operator such that $S_T = \varnothing$ and let S' be a separate part of S with $\dim S' = 0$. Then T is $S \setminus S'$-decomposable.

Theorem 2.17. ([4], [5]) Let $T \in \mathbf{B}(X)$ be a (strongly) S-decomposable operator such that $\dim S = 0$. Then T is (strongly) decomposable operator.

Proof. In Proposition 2.16, we choose $S' = S$, hence T is $\varnothing$-decomposable, meaning that T is decomposable.

Corollary 2.18. ([4], [5]) If $T \in \mathbf{B}(X)$ is a (strongly) decomposable operator and Y is an invariant subspace to T such that $\dim\left(\sigma(T|Y) \cap \sigma\left(\dot{T}\right)\right) = 0$, then $\dot{T}$ is (strongly) decomposable. Moreover, when Y is a spectral maximal space of T, then both $T|Y$ and $\dot{T}$ are decomposable operators.

Proof. The assertions follow from Theorem 2.12, Theorem 2.17 and Corollary 2.14.

Corollary 2.19. ([5]) Let $T \in \mathbf{B}(X)$ be (strongly) decomposable and let Y be an invariant subspace to T such that $\dim\left(\sigma(T|Y)\right) = 0$. Then $\dot{T}$ is (strongly) decomposable.

Proof. It easily follows from Corollary 2.18 since we have $\dim\left(\sigma(T|Y) \cap \sigma\left(\dot{T}\right)\right) = 0$.

Proposition 2.20. ([5]) Let $T \in \mathbf{B}(X)$ be decomposable and let Y be an invariant subspace to T such that $\dim\left(\sigma(T|Y) \cap \sigma\left(\dot{T}\right)\right) = 0$. Then for $T|Y$ the following property holds: for any finite open covering $\{G_i\}_{i=1}^n$ of $\sigma(T|Y)$ with simply connected sets, there is a system $\{Y_i\}_{i=1}^n$ of invariant subspaces to T such that $\sigma(T|Y_i) \subseteq G_i$, $i = 1, 2, ..., n$, and $\sum_{i=1}^{n} Y_i = Y$.

Corollary 2.21. ([5]) Let $T \in \mathbf{B}(X)$ be a decomposable operator and let Y be a T-absorbing invariant subspace to T such that $\dim\left(\sigma(T|Y) \cap \sigma\left(\dot{T}\right)\right) = 0$. Then the restriction $T|Y$ is a decomposable operator.

Proof. Apply Proposition 2.20.

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