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PREFACE
GAVRIIL PĂLTINEANU AT HIS 75: AN OUTSTANDING MATHEMATICIAN AND PROFESSOR

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Professor Gavriil Păltineanu was born on May 4th 1942 in Curtea de Argeş. He graduated as the first of his generation from "Vlaicu Vodă" high school in Curtea de Argeş in 1960.

Between 1960-1965, he attended the courses of the Faculty of Mathematics and Physics of "C. I. Parhon" University of Bucharest.

Attracted by both research and education, having completed his university studies, he took up teaching positions at the Institute of Constructions of Bucharest (UTCB) between 1966-1973 and 1979-2011, and also at the Military Technical Academy of Bucharest between 1976-1979, also working as a researcher at the Institute of Mathematics of the Romanian Academy (1973-1975), which granted him the title of *Honorary Member*, for his creative activity.

The title of *Professor Emeritus* of the Technical University of Civil Engineering Bucharest crowns the work and talent that Gavriil Păltineanu dedicated to education, sparing no effort.

In any given field, building one's personality is naturally based upon genetic endowment, work effort, and professional environment that the person grows and develops in, as well.

I will insist a bit on this last issue, trying to revive the academic atmosphere that reigned in the Faculty and Institute of Mathematics, in the years when the student Gavriil Păltineanu was trained as a future teacher and scholar.

At that time, in the Faculty of Mathematics and in the Institute of Mathematics of the Romanian Academy, there were two cultural age groups: the first was represented by senior Professors, such as Miron Nicolescu, Gheorghe Vrănceanu, Alexandru Ghika, Octav Onicescu, Gheorghe Mihoc, Victor Vâlcovici, Grigore Moisil, Caius Iacob,

Nicolae Teodorescu, Gheorghe Galbură, and the second was created, raised, and protected by the first, with a care and attention unrivalled ever since. What was the secret that kept united the seniors of Romanian mathematics in their attempt to raise a new generation at a time when, in other faculties of the same university, internal conflicts brought careers and scientific promises to a premature end?

Certainly, there were visionaries in other intellectual *millieux* as well, but they were only exceptions.

A plausible answer to the above-mentioned question might be found in the way of thinking of a mathematician, who masters the technique of longer syllogisms and prefers logical constructions and answers to hasty, emotional reactions.

The harsh reality – consisting of a Romania destroyed by war, by previous sterile political struggles, by poverty and famine following a terrible postwar drought – may have raised the question: What is to be done? And they did exactly what they could do better. They grouped around the great scholar Simion Stoilow - who died untimely (1961) - and started a work of reconstruction in mathematical research and higher education, by founding the Institute of Mathematics of the Romanian Academy (1948), and by generously sharing their knowledge in the lecture halls of Bucharest, Iaşi, Cluj and Timişoara universities, where they brought up the next generation of mathematicians, counting such names as Corneliu Constantinescu, Tudor Ganea, Romulus Cristescu, Solomon Marcus, Nicolae Dinculeanu, Cabiria Andreian Cazacu, Martin Jurchescu, Ionel Bucur, Dragoş Lazăr, Ştefan Gheorghiţă, Simona Pop, Todor Liviu, Gheorghe Ciucu, Ciprian Foiaş, Nicu Boboc, Gheorghe Gussi, Valentin Poenaru, Aurel Cornea, Paul Mustaţă, Costake Teleman, Silviu Teleman, Teodor Hangan, Marius Iosifescu, Ion Cuculescu, Aristide Halanay, Nicolae Radu.

Enthusiastic, gifted for scientific research, the components of this group - to which we could add some remarkable names in Iaşi, Cluj, and Timişoara - raised Romanian research in mathematics to the first places in the world, in the second half of the last century, taking the torch from their mentors. Moreover, the scientific seminars, initiated at the newborn Institute by Simion Stoilow, grew in number in this period, envolving, together with the members of the Institute, teachers and final-year students from the Faculty of Mathematics and Physics of the University of Bucharest.

In these scientific seminars, each component of this echelon would sow the seeds of scientific research in the younger generations, as well as the pride to witness surprising discoveries and to be part of such a conclave.

I have no knowledge of any mathematician, starting his research between the 1960-90s, who would not have spent his time of apprenticeship in such a workshop.

Such was the atmosphere prevalent in the mathematical community of Bucharest, in the student and apprentice years of the then-up-and-coming, now-celebrated, G. Păltineanu.

It goes without saying that, as a member of the Institute, he attended several scientific seminars: it was a privilege not to be missed. Among the seminars then chosen by G. Păltineanu, there was one conducted by his Professor, Romulus Cristescu, who coordinated his remarkable doctoral thesis, entitled "Density theorems in spaces of continuous function". I was honored to be part of the jury who decided to grant him the PhD in mathematics, together with my former teachers Cabiria Andreian Cazacu, Ion Colojoară, and our deeply regretted friend and collaborator Aurel Cornea.

When G. Păltineanu defended his thesis, I remember I made the remark that his work contained enough material for two doctoral researches. My remark was approved by all members of the jury.

As it is often the case, the field of research for a mathematician is the one in which he invested time, effort, intuition and imagination, in the effervescent period of preparation of his thesis, his first match in a grand gala. And I don't think it's completely wrong to say that, if you want to get a finer picture of a field, you'd better ask a final-year PhD candidate, rather than a specialist. Such was the case of G. Păltineanu, who took the opportunity to gain impressive knowledge, as proved by the results of his thesis and scientific publications subsequent to the thesis, including the monograph "Elements of Approximation of Continuous Functions", published in 1982 at Editura Academiei Române, a work widely appreciated by the specialists in the field.

A second domain he embraced - and he was not wrong in making this choice - is the theory of convexity; as a matter of fact, this choice and the other theme of the thesis are evenly balanced in the economy of his scientific work.

While the theory of approximation has a history of hundreds, maybe thousands of years, if we think of Archimedes, the theory of convexity is much more recent, and was founded by one of the most brilliant figures in the mathematics of the 20th century: Gustave Choquet.

I do not know what will remain after most of 20th or 21st c. mathematicians, but sure enough, at least through his theories of convexity and of capacity, this name will be long-lasting in history.

In these theories, sparked by a clear mind, we find entwined the unmistakable attributes of classical culture: simplicity, elegance, depth, and beauty. There were, and still are, many remarkable mathematicians who contributed to the enrichment of these theories. Among them, we find H. Bauer, E. Alfsen, Errett Bishop, Karel de Leeuw, Effros, Edward, L Loomis., G. Lions, Grassmann, P. A. Meyer, G. Mokobodzki, Skau, Wiles, T. Anderson, and Ellis.

Obviously, the list is incomplete and still very difficult to draw, as there are ”schools” that study convexity theory in France, Germany, Slovakia, Japan, and Romania.

I made this digression only to better emphasize Gustave Choquet's appreciation of G. Păltineanu's results in the theory of convexity, as written in his review of the monograph ”Order Structures in Functional Analysis” (Vol. 2, 1989, Editura Academiei Române, coordinated by Romulus Cristescu), which includes G. Păltineanu's work ”Lattice-Type Characterization Theorems of Choquet Simplexes”. In the letter that Choquet addressed to the authors, he said: "*J'ai été, bien sûr, particulièrement intéressé par la contribution de Gavriil Paltineanu à la théorie des simplexes, où sont présentées très légamment plusieurs beaux théorèmes*".

The amplitude of G. Păltineanu's scientific activity exceeds by far the list of 52 scientific papers published in scientific journals in Romania and abroad. His participation in 40 congresses, national and international workshops, as well as the publication of his presentations in the corresponding proceedings, draw a wider scope of his scientific production.

A special place in his scientific activity, dedicated to the mathematical community, have the dozens of scientific reports he elaborated for doctoral theses (at the

Institute of Mathematics of the Romanian Academy, at the faculties of Mathematics and Computer Science from Bucharest, Craiova, and Piteşti), the scientific reviews published in *Mathematical Reviews* or in Romanian journals, his work as a member of the editorial board of *Mathematical Reports*, a journal published by the Romanian Academy (2000-2013), and as a member of several committees of the same Academy.

There is only one epithet to summarize G. Păltineanu's work as a university teacher: exceptional! I guess, at first, what happened to him is somehow similar to what happened to many other students having come from "the provinces" to Bucharest. He chose the faculty of mathematics because he wanted to become a teacher. His later strong interest in scientific research didn't make him give up his dream as a child. And how could he give up this dream, when he could see, in the lecture halls of the faculty, great professors officiating like great actors, mastering the art of communication, such as Miron Nicolescu, Nicolae Dinculeanu, Grigore Moisil, Victor Vâlcovici, Alexandru Ghika, Solomon Marcus, Gheorghe Galbură, or the tumultuous Tudor Ganea?

Such teachers kept us tied with invisible threads within the few square meters in front of the blackboards, where we walked to and for hundreds of times, while we could see before our eyes unfolding the never-ending story of flawless reasoning.

I blame myself for not having witnessed (so far) any of the courses held by Professor G. Păltineanu, but I cannot imagine them other than his scientific communications at conferences: thoughtful, logical, clear, dense, still giving his audience time to breathe. I "spied" on him sometimes through an "agent" loyal to me, Georgiana, one of his students, who would attend his courses with the enthusiasm of a theatre-goer.

A brief look at his long list of educational publications gives us an idea about the interest shown by the author in teaching. Let us note that, soon after graduation, the first job he ever had was at the Institute of Civil Engineering, to which he returned after a six-year break, that he fruitfully spent to enrich his scientific knowledge at the Institute of Mathematics. He returned on a higher position, with a clearer perspective to the future, and a stronger hope for stability, so necessary for someone to reach a goal. Later events have shown that his intuition was right, because he found here an environment friendly to research and pure mathematics.

Professor Păltineanu regarded his job at the Institute of Civil Engineering with his proverbial seriosity. He understood that the math's lessons should play, here, a double role: to familiarize students with the logical reasoning, on the one hand, and to provide them useful tools for a career in engineering, on the other. This is why changes in the syllabi of various chapters of mathematics have been made, and also attempts to get a closer look into numerical methods, so useful in the everyday practice of engineering design and research. Many of the colleagues from the Department of Mathematics, led by Professor G. Păltineanu for about two decades, have turned their attention and efforts to acquire expertise in these directions. The first one to make these changes was the head of the department, together with those who developed the necessary skills in their student years, but it must be said that each component of the team took steps in this direction.

How they made it, we may deduce from the list of educational publications or monographs of Professor Păltineanu and his collaborators. His four courses of Numerical Analysis and Advanced Mathematics, his participation in scientific contracts, coordinated by Technical Departments and focused on their specific problems, show the true extent of his openness to new and multifold realities.

He elaborated an impressive number of didactic works, over 20, addressed to the students of all categories: from freshmen to master students. I was asked to make a presentation of one of these works, and I thought long and hard about the text. In mathematics, the classical things that make up the core of a syllabus do not change in time so easily as "reformists" pretend. What may change is the manner of presentation, the way to share insights gained from experience, the way to introduce specific issues arising from interdisciplinary contacts between mathematicians and other researchers or practitioners, who may have certain difficulties with mathematics.

The two traits of his personality – the rigor of the scientific researcher and the talent of the teacher, who uses the reader's intuition to hand on complex scientific information - are hand and glove in G. Păltineanu's writings, interweaving depth and clarity.

Many of his collaborators carry on his intellectual exercise and I am proud to count myself among those "contaminated" by his example; for this "aggression", I am grateful.

To illustrate the elegance and profundity of many mathematical ideas found in his scientific publications, I mention here the notion of *anti-algebraic subset* of a compact space X with respect to a space E of continuous on X functions, introduced by the author as a subset A of X for which the space E_A - the restriction of the space E to the set A - remains stable to the multiplication with a function defined on A only if this function is a constant. He proves the following beautiful theorem, which will endure in time also for its intrinsic potential:

The space X can be written as a partition $X=\bigcup_i A_i$ of maximal anti-algebraic sets A_i, two by two disjoint, and a continuous function on X is uniformly approximated on X by a sequence of functions from E if and only if the restriction of this function to each A_i has the same property of approximation with respect to the function space E / A_i.

This statement packs together most of the Stone-Weierstrass-type theorems, but also theorems of affine (harmonic) extensions of functions defined on Choquet's or Shilov's boundary of a compact convex**.**

I would rest my case by demanding for the great man we are celebrating today to be sentenced to unlimited inspiration.

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ON THE SECOND ABEL CONTINUITY THEOREM

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Abstract: It is well known that the uniform convergence of a sequence of functions preserve the continuity and just the differentiability in some supplementary conditions.
In this paper we prove the uniform convergence of a power series on some subsets of its convergence set and we derive as consequence the second Abel continuity theorem as well as the extension to the boundary of the formula of derivation term by term of a sum of power series.
Mathematics Subject Classification (2010): see http://www.ams.org/msc/
Key words: boundary, analytic functions, convergence.

1. Introduction

In this paper we show the uniform convergence theorem of a power series on a non – tangential set. As a consequence we give a proof for second Abel continuity theorem and we extend to the boundary the theorem of derivation term by term of a power series.

2. Preliminaries and first results

Let us consider a subset M of the set $\mathbb{C}$ of all complex numbers and $f:M\to X$ be a function defined on M with values in the complex Banach space X. We remember that the function f is *derivable* at a point $z_0\in M$ if there exists $x_0\in X$ such that the function $r:M\setminus\{z_0\}\to X$ given by

$$r(z)=\frac{f(z)-f(z_0)}{z-z_0}$$

has the limit x_0 at the point z_0 i.e.

$$\lim\frac{f(z)-f(z_0)}{z-z_0}=x_0.$$

The element x_0 is by definition the *derivative of* f *at the point* z_0 and we write $f'(z_0)=x_0$.

If a,b are two elements in $\mathbb{C}$ we denote by $[a,b]$ the convex covering of the set $\{a,b\}$ i.e.

$$[a,b]=\{a+t(b-a)\,|\,t\in\mathbb{R},\,0\le t\le 1\}.$$

Even if the following statement is well known we shall give some arguments for it.
Lagrange inequality. Let f *be a function which is derivable at any point* $c\in[a,b]$. *If we denote* $M:\sup\{\|f'(c)\|;c\in[a,b]\}$ *then we have*

$$\|f(b)-f(a)\| \le M\cdot|b-a|.$$

Indeed, there exists $\varphi: X\to\mathbb{R}$ a real continuous linear functional on X such that $\|\varphi\|=1$ and

$$\|f(b)-f(a)\| = \varphi(f(b)-f(a)) = \varphi(f(b))-\varphi(f(a)).$$

Let us consider the real function $\Psi:[0,1]\to\mathbb{R}$ given by

$$\Psi(t)=\varphi(f(a+t(b-a))).$$

The function Ψ is derivable on $[0,1]$ and for any $t_0\in[0,1]$ we have

$$\Psi'(t_0)=\lim_{t\to t_0}\frac{\varphi(f(a+t(b-a)))-\varphi(f(a+t_0(b-a)))}{t-t_0}=$$

$$=\lim_{t\to t_0}\varphi\left(\frac{f(a+t(b-a))-f(a+t_0(b-a))}{(t-t_0)(b-a)}\cdot(b-a)\right)=$$

$$=\varphi(f'(a+t_0(b-a))\cdot(b-a))\le\|\varphi\|\cdot\|f'(a+t_0(b-a))\cdot(b-a)\|;$$

$$\Psi'(t_0)\le\|\varphi\|\|f'(a+t_0(b-a))\|\cdot|b-a|\le M\cdot|b-a|.$$

Using Lagrange theorem there exists $t'\in[0,1]$ such that

$$\Psi(1)-\Psi(0)=\Psi'(t').$$

Hence we have

$$\|f(b)-f(a)\|=\varphi(f(b))-\varphi(f(a))=\Psi(1)-\Psi(0)=\Psi'(t')\le M|b-a|.$$

Notation. If h is a function defined on a non empty set A with values in X we denote by $\|h\|$ the *uniform norm* of h i.e.

$$\|h\|=\sup\{\|h(a)\|; a\in A\}.$$

With this notation the above Lagrange inequality has the form

$$\|f(b)-f(a)\|\le\|f'\|\cdot|b-a|.$$

We remember that for a sequence $(f_n)_n$ of real functions defined on a closed interval $[a,b]\subset\mathbb{R}$ the following assertion holds.

If the function f_n is derivable on $[a,b]$ for any $n\in\mathbb{N}$, if the sequence $(f'_n)_n$ is uniformly convergent to a function $g:[a,b]\to\mathbb{R}$ and if there exists $t_0\in[a,b]$ such that the sequence $(f_n(t_0))_n$ is convergent then

1) *The sequence $(f_n)_n$ is uniformly convergent on $[a,b]$ to a finite function $f:[a,b]\to\mathbb{R}$*
2) *The function f is derivable on $[a,b]$ and*

$$f'(t)=g(t)=\lim_{n\to\infty}f_n'(t),\ \forall t\in[a,b].$$

The following assertion is a generalization of the above statement.

Theorem 1.1 *Let M be a bounded convex subset of $\mathbb{C}$, let X be a Banach space and for any $n\in\mathbb{N}$ let $f_n: M\to X$ be a derivable function such that the sequence $(f_n')_n$ is uniformly convergent. Then:*

1) *The sequence* $(f_n)_n$ *is uniformly convergent on* M *to a finite function* $f: M \to X$
2) *The function* f *is derivable on* M *and we have*

$$f'(z) = g(z) = \lim_{n\to\infty} f_n'(z),\ \forall z \in M\ .$$

The proof has a classical line, following the above Lagrange inequality.

3. Continuity and derivation at the boundary of the power series

We remember that the set of convergence S_c of a power series $\sum_n a_n z^n$ where $a_n \in X$ is given by

$$S_c = \left\{ z \in \mathbb{C} \middle| \sum a_n z^n \text{ is convergent} \right\}.$$

If we denote

$$R = \frac{1}{\limsup \sqrt[n]{\|a_n\|}}$$

with the conventions $\frac{1}{0} = \infty$, $\frac{1}{\infty} = 0$, the number $R \in [0, \infty]$, called the radius of convergence of the series, then we have

$$D_R \subset S_c \subset \overline{D_R}$$

where for any $r \in [0, \infty]$ we have noted

$$D_r = \{ z \in \mathbb{C} \| z| < r \},\ \overline{D}_r = \{ z \in \mathbb{C} \| z| \le r \}.$$

We shall denote also by $\Omega r, z_0$ the convex covering of the set $D_r \bigcup \{z_0\}$ where z_0 is a point in $\mathbb{C}$.
We remember the following Abel theorems.

1) If R is the radius of convergence of the starting series then for any $r < R$ the series is normally convergent on D_r.
2) If the series is convergent at a point z_0 of the boundary of the convergence set S_c i.e. $|z_0| = R$ then the sum function $s: S_c \to \mathbb{C}$,

$$s(z) = \sum a_n z^n$$

is continuous on the subset $\Omega r, z_0$ for any $r < R$ but generally it is not continuous on entirely convergence set S_c. We are interested in the derivability of this function at the boundary point z_0 and also on the calculus of the derivative at such a point.

We have examples where the function s is derivable at the point z_0 but the series $\sum n a_n z_0^{n-1}$ can not be used to calculate its derivative at the point z_0.
Nevertheless we have the following answer.

Theorem. *If the series* $\sum na_n z_0^{n-1}$ *is convergent, then the series* $\sum a_n z_0^n$ *is convergent, the function* s *is derivable at the boundary point* z_0 *and the formula of derivation term by term is still valid in this case i.e*

$$s'(z_0) = \sum na_n z_0^{n-1}.$$

The proof is based on Theorem 1.1 showing that the series $\sum na_n z_0^n$ and $\sum na_n z_0^{n-1}$ are uniformly convergent on $\Omega\, r, z_0$.

This last assertion has obtained using the following.

Geometrical lemma. *Let the triangle* ΔABC *be such that* $|AC| = b$, $|AB| = c$, $|BC| = a = \sqrt{b^2 + c^2}$ *. If we consider a point* P *in this triangle, i.e.* P *is an element of the convex covering of the set* $\{A, B, C\}$ *and* $P \neq B$ *then the following inequality holds*

$$\frac{|PB|}{|BC| - |PC|} \leq \sqrt{\frac{a+b}{a-b}}.$$

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A DISCUSSION ON QR DECOMPOSITION ALGORITHMS

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Abstract: In this paper we discuss QR decomposition algorithms for square nonsingular *n* by *n* matrices. We compare several variants based upon Gram-Schmidt orthonormalization process and Householder reflections and we propose ways to optimize the algorithms, reducing the number of floating point operations involved. The Mathcad code for the optimized algorithms is provided.
Mathematics Subject Classification (2010): 97N40, 65F05, 65F15, 65F25.
Key words: QR decomposition, QR algorithm, Householder reflection, Householder matrix, Gram-Schmidt process, Mathcad.

1. Introduction

In linear algebra, it is a well-known fact that any real square matrix A can be factored as a product of an orthogonal matrix Q (i.e., the columns of Q form an orthonormal set of vectors or, equivalently, $Q^TQ = I_n$) and an upper triangular matrix R. Moreover, if the initial matrix A is nonsingular, then the factorization is unique, if we require all the diagonal elements of R to be positive. In the case of a square matrix A with complex elements, a similar factorization is possible, with Q being a unitary matrix (i.e., $Q^*Q = I_n$, where Q^* denotes the conjugate transpose of Q). Such a factorization is usually called a *QR decomposition* of A.

The *QR* decomposition of a matrix *A* is the basis of many useful techniques in numerical linear algebra. Most notably, *QR* decompositions are involved in two important categories of problems:

(1) *Solving linear systems*, especially for the case of large or ill-conditioned systems.
If the system to be solved is written in matrix form as: $Ax = b$, with $A = \begin{bmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ a_{2,1} & a_{2,2} & \dots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \dots & a_{n,n} \end{bmatrix}$, $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, $b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$ and if $A = QR$ is a *QR* decomposition of *A*, then we have: $Ax = b \Leftrightarrow QRx = b \Leftrightarrow Rx = Q^Tb$ ($Rx = Q^*b$ in the complex case). Thus, we have now an upper triangular system, which can be solved by back substitution method in $O(n^2)$ floating point operations (*flops*).

(2) Eigenvalues and eigenvectors computation, by the *QR* algorithm, one of the most commonly used techniques for such problems.
The *QR* algorithm is very simple. Let $A = A_0$. For $k = 0,1,2,\cdots$, we repeat the steps:

$$Step1:\ A_k = Q_k R_k \ (QR \text{ decomposition})$$
$$Step2:\ A_{k+1} = R_k Q_k$$

If certain conditions are satisfied (see, e.g.,[1]), the sequence $(A_k)_{k\in N}$ converges to an upper triangular matrix (diagonal matrix, if *A* is symmetric) whose diagonal entries are the eigenvalues of *A*. However, the convergence may often be slow, and each iteration requires the computation of a *QR* decomposition. Even in its improved form, which includes the technique of Wilkinson shifts (see [2]), the *QR* algorithm still requires the computation of a number of *QR* decompositions.

From what we said until now, it is clear that it is very much desirable to find a numerically stable and optimized algorithm for performing *QR* decompositions. Thus, in this note, we compare a few algorithms for *QR* decomposition, point out their weaknesses and discuss possible optimizations in what concerns numerical stability (for some of them) and flops count (for others).

In the next section, we briefly present the theoretical background for *QR* decomposition algorithms based on Gram-Schmidt orthonormalization process. Then, we present the Mathcad code for two versions (one classic and one improved) of such algorithms, and we illustrate by an example the shortcomings of the Gram-Schmidt based approach. In Section 3 we focus on the technique of Householder reflections. We first recall the theoretical foundation and the corresponding algorithm for QR factorization. Then we discuss strengths and weaknesses of this approach, as well as a number of possibilities for optimization (depending also on the purpose of our *QR* decomposition). We also present the Mathcad code for an iterative and a recursive implementation of the algorithm. The final section is dedicated to conclusions.

Throughout the article, we discuss the algorithms for the case of matrices of real elements. By doing so, we avoid unnecessary complications without losing generality, since our conclusions hold in the complex case as well. The theory of *QR* decomposition of a matrix and its applications can be found in many textbooks of Linear Algebra or Numerical Analysis (e.g. [2], [3], [4],[1]). The Mathcad programs have been written using the version 14 of this software [5].

2. *QR* decomposition by Gram-Schmidt orthonormalization process

The classical Gram-Schmidt orthonormalization process is widely used, as the name implies, for converting an arbitrary basis in a linear space to an orthonormal one. However, it is not difficult to see that it can also be used for *QR* decomposition of a square nonsingular matrix, as we shall see bellow.

In the following, we shall denote $M = [m_1, m_2 \ldots, m_n]$ whenever *M* is a matrix with vectors $m_1, m_2 \ldots, m_n \in \mathbf{R}^n$ as columns. If *A* is a square nonsingular matrix of real elements and $A = [a_1, a_2 \ldots, a_n]$, then its columns $a_1, a_2 \ldots, a_n$ form a basis of $\mathbf{R}^n$. Using the Gram-Schmidt process, we convert it to an orthogonal basis by subtracting, from each a_k its orthogonal projection on the subspace spanned by $\{a_1, a_2 \ldots, a_{k-1}\}$. Indeed, let us denote by $proj_u v$ the projection of vector *v* along the direction of vector *u*, i.e., $proj_u v = \frac{\langle u, v\rangle}{\langle u, u\rangle} u$, where $\langle \cdot, \cdot \rangle$ is the inner product in $\mathbf{R}^n$. Then we define:

$$f_1 = a_1$$
$$f_2 = a_2 - proj_{f_1} a_2$$

$$f_3 = a_3 - proj_{f_1} a_3 - proj_{f_2} a_3$$
$$\vdots$$
$$f_n = a_n - \sum_{k=1}^{n-1} proj_{f_k} a_n$$

The set $\{f_1, f_2 \dots f_n\}$ is now an orthogonal basis of $\mathbf{R}^n$, which can be easily orthonormalized by putting $g_i = f_i / \|f_i\|, i = 1,2,\dots n$. Since $Q = [g_1, g_2 \dots g_n]$ is the matrix of an orthonormal basis, it follows that *Q* is orthogonal. Now let us denote by *R* the transition matrix from basis $\{g_1, g_2 \dots, g_n\}$ to basis $\{a_1, a_2 \dots, a_n\}$. It is well known that we can compute *R* by the formula $R = Q^{-1}A$, since Q and A are the matrices of the two bases, and it follows that $QR = A$. On the other hand, we can express the vectors $a_1, a_2 \dots, a_n$ in the orthonormal basis $\{g_1, g_2 \dots, g_n\}$ as follows (since the sets $\{a_1, a_2 \dots, a_k\}$ and $\{g_1, g_2 \dots, g_k\}$ span the same linear subspace, for each $k \in \{1,2,\dots,n\}$):

$$a_1 = \langle a_1, g_1 \rangle g_1$$
$$a_2 = \langle a_2, g_1 \rangle g_1 + \langle a_2, g_2 \rangle g_2$$
$$\vdots$$
$$a_n = \sum_{k=1}^{n} \langle a_n, g_k \rangle g_k$$

It follows that the transition matrix from basis $\{g_1, g_2 \dots, g_n\}$ to basis $\{a_1, a_2 \dots, a_n\}$ is

$$R = \begin{bmatrix} \langle a_1, g_1 \rangle & \langle a_2, g_1 \rangle & \cdots & \langle a_n, g_1 \rangle \\ 0 & \langle a_2, g_2 \rangle & \cdots & \langle a_n, g_2 \rangle \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \langle a_n, g_n \rangle \end{bmatrix}$$

, which is upper triangular, and has the property that $A = QR$, with *Q* orthogonal.

We show, in Figure 2.1, the Mathcad code for the classical Gram-Schmidt (CGS) process. The CGS function returns the orthogonal matrix *Q*, while the matrix *R* can be computed by putting $R = Q^T A$. The function $proj(u, v) = \frac{u \cdot v}{u \cdot u} u$ computes the projection of vector *v* along the direction of vector *u*.

```
CGS(A) := | n ← rows(A)
          | for i ∈ 2..n
          |   A^⟨i⟩ ← A^⟨i⟩ − Σ_{j=1}^{i−1} proj(A^⟨i⟩, A^⟨j⟩)
          | for i ∈ 1..n
          |   A^⟨i⟩ ← A^⟨i⟩ / |A^⟨i⟩|
          | A
```

Figure 2.1

```
MGS(A) := | n ← rows(A)
          | for i ∈ 2..n
          |   for j ∈ 1..i − 1
          |     A^⟨i⟩ ← A^⟨i⟩ − proj(A^⟨i⟩, A^⟨j⟩)
          | for i ∈ 1..n
          |   A^⟨i⟩ ← A^⟨i⟩ / |A^⟨i⟩|
          | A
```

Figure 2.2

The Gram-Schmidt process with its intuitive geometric significance may seem attractive. However, there are inherent numerical stability issues with it. The columns of *Q* matrix may not be exactly orthogonal, with more severe instability occurring especially in the case of

column vectors of *A* being close to linear dependent [2]. We present an example that illustrate this situation.

Example 2.1 Let *A* be the matrix in Figure 2.3. It's determinant is $\det(A) = 0.295$. As we can see in Figure 2.4, the Q = CGS(A) matrix produced by the CGS algorithm is not quite orthogonal, since $\mathrm{CGS}(A) \cdot \mathrm{CGS}(A)^T$ is a rather poor approximation of the identity matrix.

$$A = \begin{pmatrix} 10000 & 30000.01 & -70000.08 & -10000.08 & -90000.1 \\ 3900 & 11700.02 & -27300.03 & -3900.02 & -35100.04 \\ 25700 & 77099.99 & -179899.92 & -25699.89 & -231299.93 \\ 93200 & 279599.98 & -652399.91 & -93199.95 & -838799.84 \\ 12510 & 37530 & -87570.02 & -12510.045 & -112590.05 \end{pmatrix}$$

Figure 2.3

$$\mathrm{CGS(A)\cdot CGS(A)^T} = \begin{pmatrix} 1.029 & -0.001 & -0.018 & 0.009 & -0.05 \\ -0.001 & 0.996 & 0.01 & -0.005 & 0.016 \\ -0.018 & 0.01 & 0.968 & 0.012 & -0.013 \\ 0.009 & -0.005 & 0.012 & 0.995 & 0.007 \\ -0.05 & 0.016 & -0.013 & 0.007 & 1.013 \end{pmatrix}$$

Figure 2.4

Surprisingly enough, a simple modification to the CGS algorithm is able to much improve its results. The Mathcad code for the modified Gram-Schmidt (MGS) process is given in Figure 2.2. As it can be noticed, the difference between the CGS and the MGS algorithm consists of the fact that, while in the former case we subtract from vector a_k its projections on previously orthogonalized vectors $a_1, a_2 \ldots, a_{k-1}$, in the later case we have an iterative process in which we recalculate a_k at each step, after subtracting its projection on vector a_j with $j = 1,..,k-1$. This means that at each iteration we use an actualized vector a_k, which prevents the accumulation of errors, resulting in a much more stable algorithm.

In Figure 2.5 we show, for the matrix *A* given in Figure 2.3, the orthogonality verification for the Q =MGS(A) matrix produced by the MGS program. The improvement of the modified version vs. the classic one is clearly visible.

$$\mathrm{MGS(A)\cdot MGS(A)^T} = \begin{pmatrix} 1 & -6.16\times10^{-11} & 1.122\times10^{-10} & 6.343\times10^{-10} & -9.637\times10^{-11} \\ -6.16\times10^{-11} & 1 & -2.434\times10^{-10} & -7.941\times10^{-10} & -1.774\times10^{-10} \\ 1.122\times10^{-10} & -2.434\times10^{-10} & 1 & -4.038\times10^{-10} & -5.207\times10^{-10} \\ 6.343\times10^{-10} & -7.941\times10^{-10} & -4.038\times10^{-10} & 1 & -1.604\times10^{-9} \\ -9.637\times10^{-11} & -1.774\times10^{-10} & -5.207\times10^{-10} & -1.604\times10^{-9} & 1 \end{pmatrix}$$

Figure 2.5

3. Householder QR factorization

Householder QR factorization is based upon a type of orthogonal transformations known as Householder reflections. Such a transformation takes a vector $x \in \mathrm{R}^n$ and reflects it about a hyperplane chosen in such a way that the reflection is a multiple of e_1, hence a vector with all components zero, except for the first one (see Figure 3.1). We omit here, for brevity, the details of the computation of the Householder matrix *H* corresponding to this transformation, which can be found in many textbooks such as, e.g., [4], [2], and we state only the result that $H = I - 2hh^T$, where h is the unit vector in the direction orthogonal to the reflection hyperplane (see Figure 3.1). The vector *h* can be computed by the following formula:

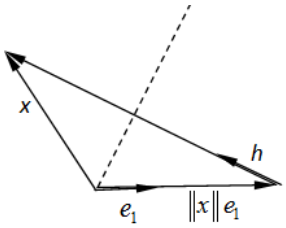

Figure 3.1

$$h = \frac{1}{\sqrt{2\|x\|(\|x\| + |x_1|)}} \underbrace{\begin{pmatrix} (|x_1| + \|x\|)\operatorname{sgn} x_1 & x_2 & \cdots & x_n \end{pmatrix}^T}_{u} \qquad (3.1)$$

where $\mathrm{sgn}(a) = 1$ if $a \geq 0$ and $\mathrm{sgn}(a) = -1$ if $a < 0$. It follows that we can now write

$H = I - \beta uu^T$, where $\beta = \left[\|x\|(\|x\| + |x_1|)\right]^{-1}$ and *u* is the vector such denoted in (3.1).

Let us now see how these transformations can be used for the purpose of *QR* factorization. Indeed, let *A* be a *n* by *n* real matrix $A = [a_1, a_2 \ldots, a_n]$ and let H_1 be the Householder matrix corresponding to the vector a_1. It follows that $H_1 A$ has the form: $H_1 A = \begin{pmatrix} \alpha_1 & \times & \cdots & \times \\ 0 & & & \\ \vdots & & A^{(2)} & \\ 0 & & & \end{pmatrix}$, where $A^{(2)}$ is a $(n-1)\times(n-1)$ matrix. Let us denote, for uniformity, $A = A^{(1)}$. We now repeat the procedure for $A^{(2)}$, finding a Householder matrix $\tilde{H}_2$ corresponding to its first column vector. Thus, $\tilde{H}_2 A^{(2)} = \begin{pmatrix} \alpha_2 & \times & \cdots & \times \\ 0 & & & \\ \vdots & & A^{(3)} & \\ 0 & & & \end{pmatrix}$. This process can be repeated $(n-1)$ times in an obvious way, until we reach a 1×1 matrix. Now if we put $H_j = \begin{pmatrix} I_{j-1} & 0 \\ 0 & \tilde{H}_j \end{pmatrix}$ for all $j = 2, \ldots n-1$, such that all H_j matrices are $n \times n$, it is easy to see that $H_{n-1} \ldots H_2 H_1 A$ will have zeros under the diagonal, thus being an upper triangular matrix which we'll denote by *R*. If we also denote $H = H_{n-1} \ldots H_2 H_1$, we find that *H* is a product of orthogonal matrices, hence orthogonal itself, and that $HA = R$. By putting now $Q = H^T$, it follows that $A = QR$, and we have the desired factorization.

The process described above suggests a recursive implementation, which is not only a natural approach, but also one that takes advantage of the decreasing size of matrices involved at each step (instead of multiplying $n \times n$ matrices that are filled with many zeros), thus significantly reducing the number of flops required. First we need some preparations, hence we begin by defining the **sgn** function as explained above and a **Border** function that increases the size of a matrix by adding a first row and a first column of zeros, with a 1 element on the diagonal (see Figure 3.2). We also define (Figure 3.3) a function **Hvect** that produces, for a given vector *x*, the coefficient β and the vector *u* that determine the Householder reflection $H = I - \beta uu^T$ corresponding to *x*. The recursive Mathcad program that performs the *QR* factorization is given in Figure 3.4, but it only returns the orthogonal matrix *Q*, while the

```
sgn(a) := | 1   if a ≥ 0
          | -1  otherwise

Border(M) ≡ | n ← cols(M)
            | N[1,1] ← 1
            | for i ∈ 1..n
            |   for j ∈ 1..n
            |     N[i+1,j+1] ← M[i,j]
            | N
```

Figure 3.2

```
Hvect(x) := | n ← last(x)
            | s ← |x|
            | error("Vector coloana nul")  if (s = 0)
            | β ← [s·(s + |x[1]|)]^-1
            | u[1] ← (|x[1]| + s)·sgn(x[1])
            | for i ∈ 2..n
            |   u[i] ← x[i]
            | (β)
            | (u)
```

Figure 3.3

upper triangular matrix *R* can be computed by $R = Q^T A$.

```
HQR(A) :=  n ← rows(A)
           return identity(n) if n = 1
           H ← identity(n)
           if n > 1
             (β; u) ← Hvect(A⟨1⟩)
             H ← H − u·(β·H^T·u)^T
             A ← A − u·(β·A^T·u)^T
             Q ← H·Border(HQR(submatrix(A, 2, n, 2, n)))
           Q
```

Figure 3.4

A few remarks are in order, concerning the Householder *QR* factorization and its implementation. First, let us notice that the Householder matrices are exact orthogonal matrices, which gives excellent numerical stability to this method. On the negative side, we must notice that at each step, a complete recalculation of the matrices is required, which results in increased time complexity of the algorithm. In order to alleviate this problem, one thing we can do is to consider the structure of the Householder reflection matrices, when performing matrix multiplication. Hence, we have:

Remark 3.1 Let us assume we have to compute *HA*, where $H = I - \beta uu^T$ and *A* are $n \times n$ matrices. Using regular matrix multiplication requires $O(n^3)$ flops. However, we can write: $HA = \left(I - \beta uu^T\right)A = A - u \cdot \left(\beta A^T u\right)^T$. Evaluating the leftmost expression requires only $O(n^2)$ flops, which is an order of magnitude less (see [2])! As it can be seen in Figure 3.4, our recursive program takes advantage of this observation when performing matrix multiplication of this type.

Let us now present an iterative version of the Householder *QR* decomposition program. An iterative algorithm benefits from avoiding the overhead of keeping track of the stack of all the nested function calls that appear in a recursive algorithm. However, it also lacks the advantage of multiplying matrices of smaller size, with each step. In order to compensate for this shortcoming, we need to carefully take advantage of the fact that $H_j = \begin{pmatrix} I_{j-1} & 0 \\ 0 & \tilde{H}_j \end{pmatrix}$ for all $j = 2, \ldots n-1$, when performing multiplication. A Mathcad program that implements such optimization is presented in Figure 3.7. The program computes both factors *Q* and *R* of the decomposition. The upper triangular matrix *R* overwrites the original matrix *A*, while the matrix *Q* is stored in a separate array. Let us discuss the ideas that allow the improvements presented in this code. In order to overwrite *A* with *R*, we multiply *A* successively to the left with $H_1, H_2 \ldots, H_{n-1}$, thus producing zeros on columns $1, 2, \ldots, n-1$, under the diagonal. At iteration *j*, we thus multiply a matrix *A* that has zeros on the first $j-1$ columns under the diagonal by $H_j = \begin{pmatrix} I_{j-1} & 0 \\ 0 & \tilde{H}_j \end{pmatrix}$, to the left. It follows that the first $j-1$ columns and $j-1$ lines of *A* will *remain unchanged*, while its submatrix consisting of the lines and columns from *j* to *n* is multiplied to the left by $\tilde{H}_j$. This observation is used in our program in order to replace $(n-1)$ multiplications of $n \times n$ matrices with multiplications of $(n-j+1) \times (n-j+1)$ matrices, for $j = 1, 2, \ldots, n-1$. But in order to reconstitute the full $n \times n$ matrix *A* after each step, we need a function to "paste" together the unchanged parts of *A* and the part that was modified by multiplication. This is done by the function **Aug** shown in Figure 3.5. It glues together three matrices U, V, W of matching sizes as shown in Figure 3.6.

$$\text{Aug}(U, V, W) := \text{augment}\left(W^T, \text{augment}(U, V)^T\right)^T$$

Figure 3.5

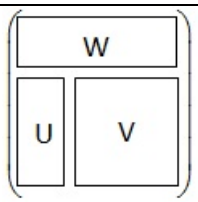

Figure 3.6

The other problem we need to address is the optimal computation of the orthogonal matrix Q. We recall that we have $Q^T = H_{n-1} \ldots H_2 H_1$, thus $Q = H_1{}^T H_2{}^T \ldots H_{n-1}{}^T$ and since the matrices H_j are symmetric for all $j = 1, 2, \ldots, n-1$, it follows that $Q = H_1 H_2 \ldots H_{n-1}$. It follows that, at step *j*, the current version of the matrix Q is multiplied to the right by $H_j = \begin{pmatrix} I_{j-1} & 0 \\ 0 & \tilde{H}_j \end{pmatrix}$. By doing so, the first $j-1$ columns of Q remain unmodified while the columns from *j* to *n* are multiplied (to the right) by $\tilde{H}_j$ - a fact that allows us to reduce the number of flops once again. Indeed, let us denote (as in the Mathcad code) by $Q2$ the matrix consisting of the last $(n - j + 1)$ columns of *Q*. Then we only need to compute $Q2 \cdot \tilde{H}_j$, instead of $Q \cdot H_j$. Moreover, the computation of $Q2 \cdot \tilde{H}_j$ can be optimized by an observation similar to Remark 3.1, except we are multiplying to the right instead of left. Indeed, we have: $Q2 \cdot \tilde{H}_j = Q2(I - \beta u u^T) = Q2 - \beta(Q2u)u^T$. By implementing these optimizations in the Mathcad code given in Figure 3.7, we succeed in keeping the number of flops for this algorithm of order $O(n^3)$.

```
HQR_it(A) :=
    n ← rows(A)
    Q ← identity(n)
    for j ∈ 1..n − 1
        (β)
        (u) ← Hvect(submatrix(A, j, n, j, j))
        Aj ← submatrix(A, j, n, j, n)
        if j = 1
            Q ← Q − u·(β·Q^T·u)^T
            A ← A − u·(β·A^T·u)^T
        otherwise
            Q2 ← submatrix(Q, 1, n, j, n)
            Q ← augment[submatrix(Q, 1, n, 1, j − 1), Q2 − β·(Q2·u)·u^T]
            A ← Aug[submatrix(A, j, n, 1, j − 1), Aj − u·(β·Aj^T·u)^T, submatrix(A, 1, j − 1, 1, n)]
    (Q  A)
```

Figure 3.7

Let us recall that one of the important scenarios for the use of *QR* decomposition was for solving linear systems $Ax = b$ by transforming them into $Rx = Q^T b$. In such a case (and in many others, as well), the explicit computation of the orthogonal matrix Q is not needed. Instead, we can overwrite A with R as in the program in Figure3.7. Then, in order to obtain the right-hand side term $Q^T b = H_{n-1} \ldots H_2 H_1 b$, we can multiply b to the left by H_j (as we do with A) at each iteration from $j = 1$ to $n - 1$. This is much "cheaper" then computing Q, since, at each step, b is just a vector, i.e., a $n \times 1$ matrix.

4. Conclusion

The *QR* decomposition is a widely used technique, e.g. in solving linear systems and in finding eigenvalues of a matrix. Its appeal stems from the fact that orthogonal transformations have excellent numerical stability properties and do not alter the conditioning of the original problem. We have seen that one of the methods for obtaining such a decomposition, namely the Gram-Schmidt process, can lead to matrices that are not exactly orthogonal, especially in the case of nearly linear dependent columns of the given matrix. If this method is chosen, then it is very important to mitigate such problems by using the modified Gram-Schmidt process.

However, a more advisable option would probably be to use the Householder *QR* method which offers exact orthogonal matrices and thus a much better numerical stability. Its main drawback is the number of flops required, but this can be greatly improved by cleverly taking advantage of the special structure of the Householder reflection matrices and even more so, if we take into account the possibility, in many cases, to avoid the actual computation of the orthogonal matrix *Q*.

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THE ESTIMATION OF QUANTILES OF THE KOLMOGOROV-SMIRNOV DISTRIBUTION BY THE MONTE CARLO METHOD

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Abstract: In this paper we use the Monte Carlo method to estimate the quantiles of the Kolmogorov-Smirnov distribution. In fact, we generate first a big number of sets of *n* random variables independent and identically distributed.
For each generated set we compute the D_n statistics for the Kolmogorov-Smirnov test. The quantile of the error ε is the above estimated statistics from the position $(1-\varepsilon)\cdot m$, where *m* is the above number of generated sets.
We will compare the obtained quantiles with the corresponding quantiles from statistical tables for several distributions: uniform discrete, binomial, geometric, Poisson, uniform continuous, normal and exponential.
Mathematics Subject Classification (2010): 62G10, 62Q05, 11K45.
Key words: Kolmogorov-Smirnov, quantiles, Monte Carlo.

1. Introduction

The Kolmogorov-Smirnov good-of-fit test tests if the random variable X has the cumulative distribution F. For this a sample $X_1, X_2,..., X_n$. The statistics D_n is [1,5,4]

$$D_n = \max_j \left| F^*\left(X_j\right) - F\left(X_j\right) \right|, \tag{1}$$

where F is the theoretical cdf, and F^* is the empirical cdf. We know [1,5] that

$$\lim_{n\to\infty} P\left(D_n \le \frac{\lambda}{\sqrt{n}} \right) = K\left(\lambda\right) = \sum_{k=-\infty}^{\infty} \left(-1\right)^k \cdot e^{-2k^2\lambda^2} . \tag{1'}$$

The formula (1') is proved in [3], using the law of large numbers. It is proved also the formula

$$\lim_{n\to\infty} P\left(D_{mn} \le \frac{\lambda}{\sqrt{N}} \right) = K\left(\lambda\right) = \sum_{k=-\infty}^{\infty} \left(-1\right)^k \cdot e^{-2k^2\lambda^2} , \tag{2}$$

where $m, n \to \infty$ such that

$$\begin{cases} \lim\limits_{m,n\to\infty} \dfrac{m}{n} = r < \infty \\ N = \dfrac{m\cdot n}{m+n} \end{cases} . \tag{2'}$$

The formula (2) is used to test if two samples of sizes *m*, respectively *n* have the same distribution (the Kolmogorov-Smirnov homogeneity test). The first condition in (2') is necessary in order not to have one of the two samples too large with respect the other. D_{mn} is computed as D_n in (1), but both distributions are empirical. The above Kolmogorov-Smirnov homogeneity test has been used together with other homogeneity tests in [2] to study the homogeneity of Lévy processes.
In [5] the Kolmogorov-Smirnov test is used to study the particular distribution on key

molecular properties that are related to drug likeness.
In [4] there is presented also a version of the Kolmogorov-Smirnov test with estimation of parameters (we know that for the Kolmogorov-Smirnov test the tested distribution is without parameters to estimate). It is the Lillefors test for normality, where for theoretical normal distribution we estimate the expectation μ and the variance S^2, the last estimator being computed using $n-1$ degrees of freedom (unbiased). The two tests, the Shapiro-Wilk test and the Anderson-Darling test are used in [4] for simulated samples in order to compare the powers of the tests. The conclusion obtained is that it depends on the considered distribution: $Beta(2,2)$, $Laplace(0,1)$, $Gamma(4,5)$ or $Gamma(1,5)$.

2. The methodology

For computing the quantile we choose first a distribution for the random variable *X*. For instance, the random variable $X:\begin{pmatrix}1 & 2 & 3 & 4 & 5 & 6\\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6}\end{pmatrix}$, i.e. the discrete uniform random variable with six possibilities. In our $C++$ header statist.h the function "cuantdmcd(n,tip,k,par)" return the value of D_n if $tip=\overline{0,4}$ is the type of the discrete random variable *X*, namely uniform with *k* possibilities, Bernoulli with the probability of success par, $Bin(k,par)$, geometrical of parameter par, or $Po(1)$. The same thing does the function "cuantdmcc(n,tip,nrint)" if the distribution is continuous, for uniform on $(0,1)$, standard normal distribution or standard exponential distribution $(tip=\overline{5,7})$, and *nrint* is the number of intervals having the same length for which we compute D_n using the middles of these intervals. In both cases we generate *n* random variables of the type tip, and we compute D_n according (1). For estimating quantiles we generate in our $C++$ program "cuantd.cpp" a big number $nrsim$ (for instance nrsim=10000) sets of *n* random variables of type tip. For each set we compute the corresponding statistics D_n, and finally we order the computed statistics increasing. The quantile of error ε is the value on position $nrsim(1-\varepsilon)$.

In fact in our main $C++$ program *n* is the maximum value of *n*, and we obtain a similar statistical table for D_n, with *n* from one to $n\max$ (in our case $n\max=500$). We compute for each pair (n,ε) the relative error, i.e.

$$e_{relative}=\left|\frac{quantile_{estimated}-quantile_{table}}{quantile_{table}}\right|. \qquad (3)$$

We consider for computation of the theoretical and empirical cdf when we have an n-size sample the *k* possible values in the case of uniform with *k* possibilities or $Bin(k,par)$, and two possibilities in the case of Bernoulli distribution. In the cases of unbounded distribution (geometric or Poisson), we consider only the values between zero and maximum (between the *n* values) simulated value.
In the following graphics we present the graphics of the relative errors for some discrete distributions for $\varepsilon=0.2$, $\varepsilon=0.15$, $\varepsilon=0.1$, $\varepsilon=0.05$ and $\varepsilon=0.01$.

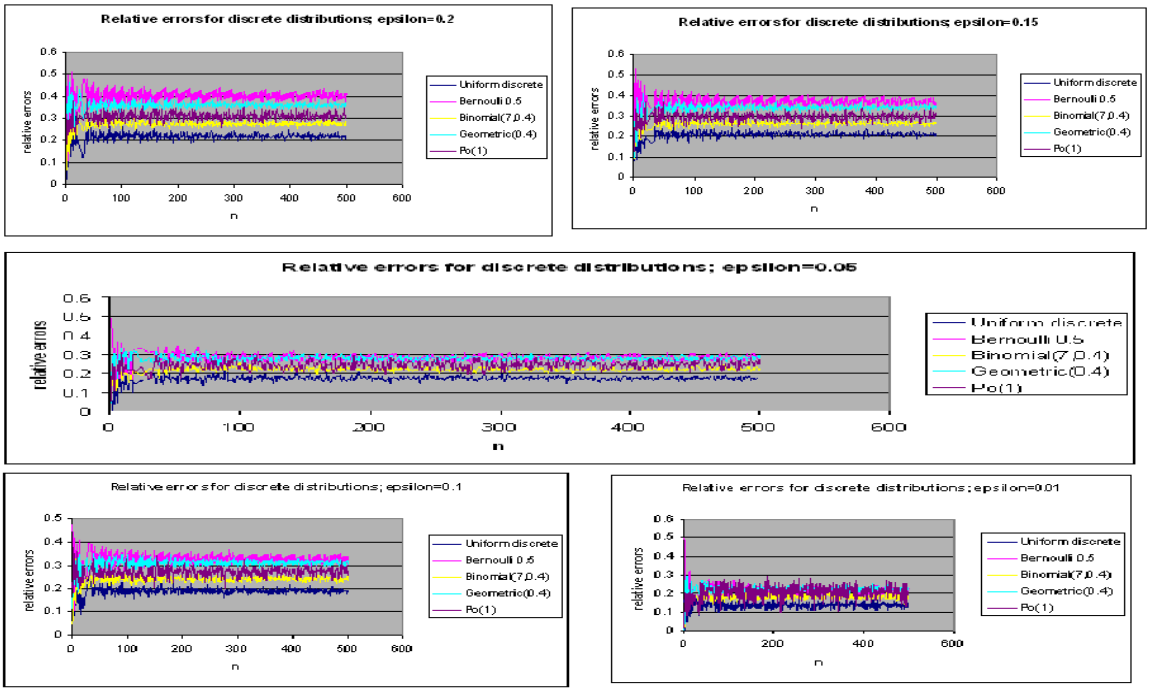

Fig. 1: The relative error in the case of discrete distribution.

The discrete distributions used in the above case are uniform discrete with 6 possibilities (throwing a dice), Bernoulli with p=0.5 (throwing a coin), $Bin(7,0.4)$, geometrical with the success probability p=0.4, and $Po(1)$. For the same errors we present next the graphics of the relative error for some continuous distributions, namely uniform on $[0,1]$, $N(0,1)$ and $Exp(1)$.

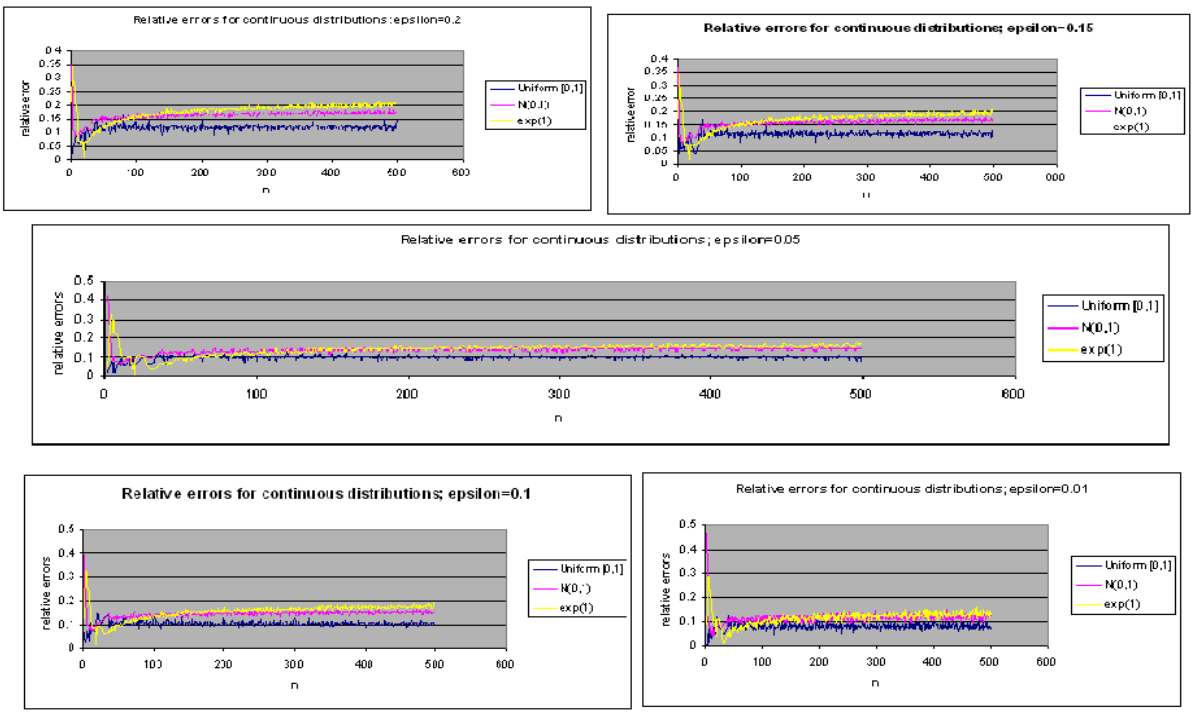

Fig. 2: The relative error in the case of continuous distribution.

3. Studying λ

In this section we consider the formula (1'). From here we derive the formula of quantile for large n

$$D_{n;\varepsilon} = \frac{\lambda}{\sqrt{n}} \tag{3}$$

such that $K(\lambda) = 1 - \varepsilon$. In statistical tables for $D_{n;\varepsilon}$ large n means over 35, and the available values of λ for ε being 0.2, 0.15, 0.1, 0.05 and 0.01 are 1.07, 1.14, 1.22, 1.36, respectively 1.63. For the values of quantiles obtained by Monte Carlo methods, we multiply the values for $n > 35$ by $\sqrt{n}$ for comparison.

In the following graphics we present the values of λ for the same discrete and continuous distributions and values of ε.

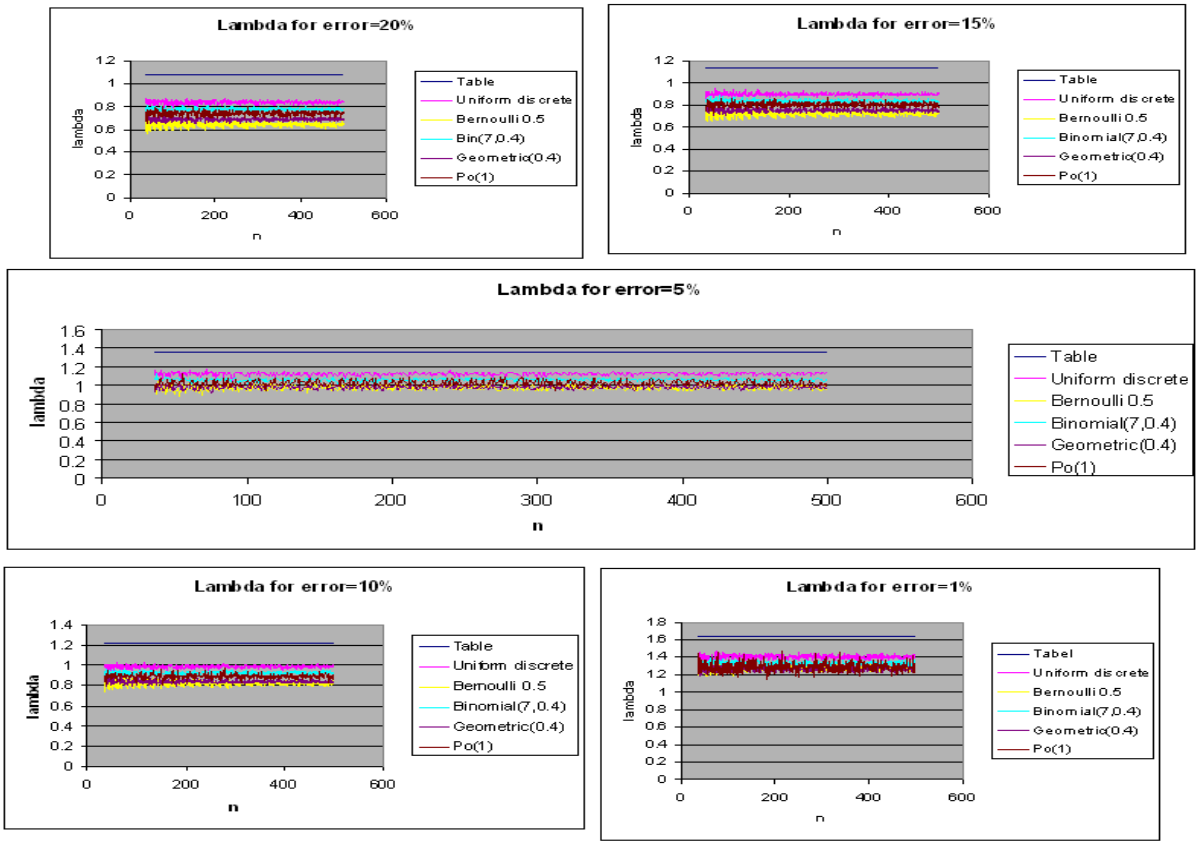

Fig. 3: The value of λ in the case of discrete distribution.

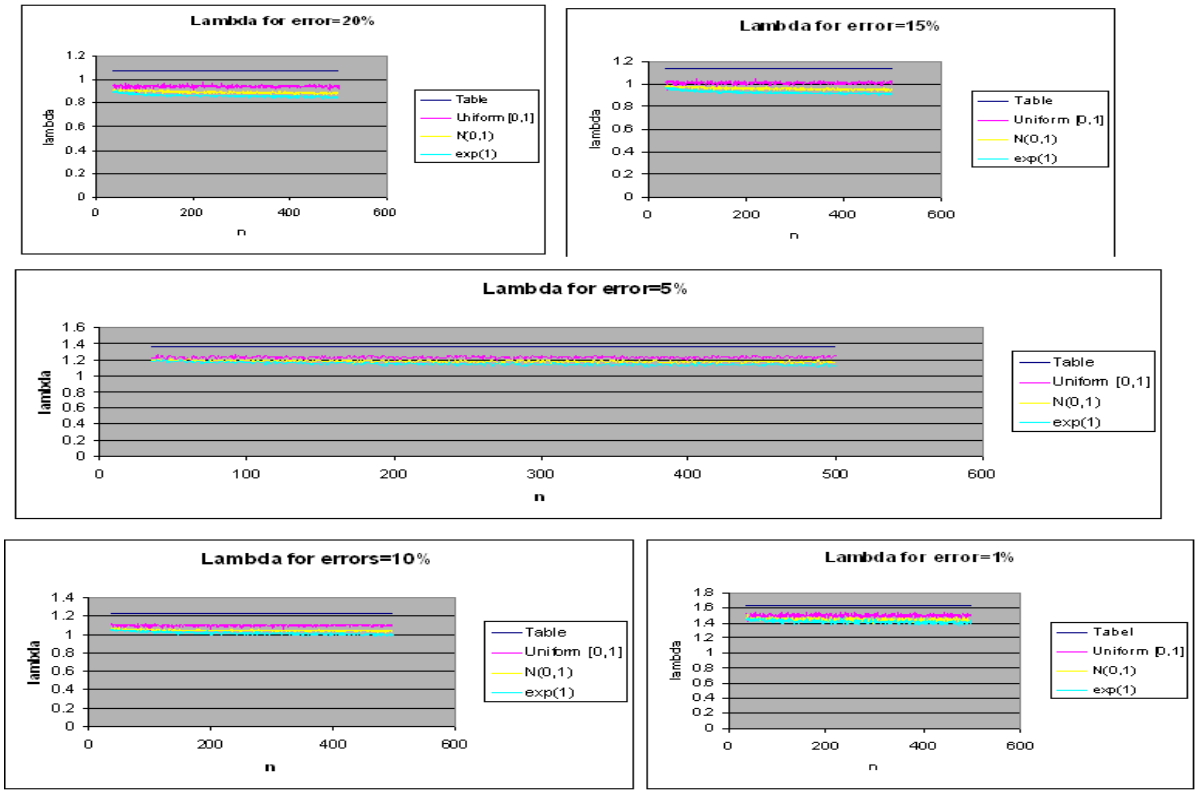

Fig. 4: The value of λ in the case of continuous distribution.

4. Conclusions

In the present paper we have considered $\varepsilon = 0.2$, $\varepsilon = 0.15$, $\varepsilon = 0.1$, $\varepsilon = 0.05$ or $\varepsilon = 0.01$, because these errors are available in the statistical table for D_n.

Because in the continuous uniform, Bernoulli, binomial or geometric we have not a standardized distribution as in the Poisson case $\left(Po(1)\right)$, uniform continuous (on $(0,1)$), normal $\left(N(0,1)\right)$ or exponential $\left(E\,\mathrm{xp}(1)\right)$, we need the parameters k and the probability of success par in our $C++$ header and main program. An open problem is how the relative error becomes lower/ upper varying these parameters. A $C++$ library called "DirectedDiversity" has been built for the algorithms used in [5].

We notice that the continuous distributions (with 20 values for computing the empirical and theoretical cdf) are better than the discrete ones. It is interesting that in both cases the simplest distributions, namely the uniform discrete/ continuous are the best. Comparison of power for some good-of-fit tests, including the Kolmogorov-Smirnov test is made in [4], as we have mentioned.

Methods to simulate the random variables in order to estimate D_n by Monte Carlo methods are found in [6]. We use in this paper the inverse method to simulate uniform discrete random variable with six possibilities (throwing a dice) or Bernoulli distribution, and for the variables related to the Bernoulli distribution (binomial and geometric) we apply the method of counting events. To generate a $Bin(k,p)$ distribution we generate k Bernoulli variables and we count the values equal to one. In the same manner we count the values one until the first value zero in the case of geometric distribution. For standard normal distribution we use the Box-Muler method, and for the exponential distribution we use the mixture-rejection method.

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MONTE CARLO METHOD AND LINEAR SYSTEMS

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Abstract: This paper presents an approach to obtain a component of the solution vector of linear equations system using the Monte Carlo Method. For this, one consider a homogeneous, finite, discrete, absorbing and first degree Markov chain. Starting from a state of this chain, which represents the range of solution component, one generates a chain trajectory. Such N trajectories are generated and for each trajectory one calculates a statistics.The mean of the N statistics represents an approximation of the desired solution component. The corresponding algorithm is codded and applied to solve a numerical problem.
Mathematics Subject Classification (2010): 91G60, 65F10, 93C05
Key words: linear equation system, Monte Carlo Method, absorbing Markov Chain, transition probabilities matrix, uniform random number.

1. Introduction

There are practical problems when the mathematical associated model is a linear algebraic equations system and we are interested to find the k-range component of the system solution.

An engineering application for determining a component of the solution vector of a system of linear equations is represented by computation of the maximum lateral displacement at the upper extremity of a multi-story building. The structure is exposed to static loads applied conventionally in the horizontal direction. This maximum lateral displacement can be used in the preliminary estimation with the approximate simplified formulas of the fundamental translational vibration period of the analyzed structure [2].

By modeling we reach a system of linear equations of form (1).

$$Ax = b\ ,\ A \in M_{n,n}(R)\ ,\ x, b \in R^n \tag{1}$$

There are well known direct numerical methods and indirect (iterative) numerical methods [4] to solve systems of linear equations.

Most frequently used direct methods are: the Gauss method, which involves bringing the system matrix A to a triangular form matrix; the Cholesky method, which decomposes the matrix A into the product of a triangular inferior matrix with a triangular upper matrix; the Householder method that consists in the decomposition of the non-singular matrix A in a product between an orthogonal matrix with a triangular upper matrix. Each of these methods requires the satisfaction of certain conditions by the matrix of system coefficients (1).
The iterative methods assume that the system (1) is brought to the form

$$x = b + Bx \text{ , } B = I_n - A \in M_{n,n}(R) \text{ , } x, b \in R^n \text{ , } \quad (2)$$
$$\text{where } I_n \text{ is unit matrix of order } n$$

with

$$\|B\| = \max_{1 \le i \le n} \sum_{j=1}^{n} |B_{i,j}| < 1 \quad (3)$$

to ensure the convergence of the string of successive approximations

$$x^{(k)} = Bx^{(k-1)} + b \text{ , } k \ge 1 \text{ , } x^{(0)} \in R^n \text{ given} \quad (4)$$

If the matrix A is strictly diagonally dominant

$$|A_{i,i}| > \sum_{\substack{j=1 \\ j \ne}}^{n} |A_{i,j}| \text{ , } (\forall) i = \overline{1,n} \quad (5)$$

it is considered

$$D \in M_{n,n}(R) \text{ , } cu \ D_{i,j} = \begin{cases} A_{i,i} & (\forall) i = \overline{1,n} \\ 0 & otherwise \end{cases} \text{ , } E = D - A \quad (6)$$

and

$$B = D^{-1}E \text{ , } b = D^{-1}b \quad (7)$$

The system (1) becomes of shape (2) and after a number of iterations, which depends on the accepted error, an approximate solution of the system (1) is obtained.

Usually the iterative Jacobi, Gauss-Seidel and relaxation methods are successfully applied.

In the following it will present a method that allows to determine approximately one component of the system solution (2), namely a method based on the Monte Carlo Method.

2. The Monte Carlo Method

The method is based on the numerical simulation of a set of trajectories of a first degree, finite and absorbing homogeneous Markov chain [4]. We recall some notions used in the following [3].

The set of random variables

$$\{X_t\}_{t \in T} \text{ , } T \subset R \text{ , } X_t \in S \quad (8)$$

is called a stochastic process with the t parameter; the index set T is usually the time, and S is the set of states of the process.

The set $\{(t, X_t)\}_{t \in T}$ is called *the trajectory of the stochastic process*.

If the set S is a discrete set, the process is a *process with discrete states*.

If the set $T = \{0, 1\ 2\ 3, ...\}$, the process is referred to as a *chain*.

The discrete process $\{X_t\}_{t \in T}$ is a *Markov process* if the next equation is verified.

$$P\left(X_{t_n} = x_n \middle| X_{t_{n-1}} = x_{n-1}, ..., X_{t_1} = x_1\right) = P\left(X_{t_n} = x_n \middle| X_{t_{n-1}} = x_{n-1}\right) \quad (9)$$

This is *the Markov property*, which means that the process has no memory (memoryless property).

The probability

$$P\left(X_{t_n} = x_n \middle| X_{t_{n-1}} = x_{n-1}\right)$$

known as the *one-step transition probability* is the probability of transitioning from one state to another in a single step.

If the sets T and S are finite sets the Markov chain is called *finite Markov chain*.

Any Markov chain is characterized by

- *The initial probability vector* $\pi = (\pi_1, ..., \pi_n)$ and
- *The probabilities transition matrix*

$$P_{i,j}(s,t) = P(X_t = j | X_s = i),\ i,\ j \in S,\ s < t,\ s,t \in T \tag{10}$$

If $t - s = 1$ then the matrix $\left(P_{i,j}(s, s+1)\right)_{\substack{i,j \in S \\ s \in T}}$ is called *one-step transition matrix*, and if the probabilities $P_{i,j}(s, s+1) = p_{i,j}$ do not depend on moment *s,* the chain is called *homogeneous*.

If a state *i* of the homogeneous chain has the property that there is another distinct state $j, j \neq i$, such that $p_{i,j} > 0$, then the state *i* is called the state of transition, and when $p_{i,i} = 1$ the state is called the absorbing state.

The numerical simulation of a Markov chain consists in the generation with the computer of some points of the chain trajectory, i.e., giving a value x_0 of the chain (that we assume is on a trajectory at the initial moment) determine the value x_1 on the same trajectory at time 1, and so on.

Algorithm to determine a solution component of a system of linear equations using the Monte Carlo Method

Let $x^* = (x_1^*,\ x_2^*,\ ..., x_{k-1}^*, x_k^*, x_{k+1}^*, ..., x_n^*)$ be the solution of the system (2). We are interested to determine the x_k^* component of the x^* vector solution.

For this we use the next algorithm.

P1. A discrete, finite, homogeneous and first-order absorbing Markov chain is associated with the problem (2) as follows:

- Matrix *B* is associated with a probability matrix $P = (p_{i,j})_{1 \leq i,j \leq n}$ that accomplishes the properties

$$p_{i,j} \geq 0,\ \sum_{j=1}^{n} p_{i,j} < 1,\ p_{i,j} \neq 0 \quad if \quad B_{i,j} \neq 0 \tag{11}$$

- The vector

$$\Pi = (\Pi_i)_{1 \leq i \leq n},\ \Pi_i = 1 - \sum_{j=1}^{n} p_{i,j},\ 1 \leq i \leq n \tag{12}$$

P2. The transition matrix of the Markov chain is built $P^* \in M_{n+1,n+1}([0,1])$

$$P^* = \begin{pmatrix} p_{1,1} & \cdots & p_{1,n} & \Pi_1 \\ p_{2,1} & \cdots & p_{2,n} & \Pi_2 \\ . & \cdots & . & . \\ p_{n,1} & \cdots & p_{n,n} & \Pi_n \\ 0 & \cdots & 0 & 1 \end{pmatrix} \tag{13}$$

P3. We define weights

$$w_{i,j} = \begin{cases} \dfrac{B_{i,j}}{p_{i,j}} & if\ p_{i,j} \neq 0 \\ 0 & otherwise \end{cases} \tag{14}$$

P4. A trajectory of the Markov chain is generated. This trajectory has the initial vector given by the *k*-line of the matrix *P* and the transition matrix *P**:

$$\gamma = (k, k_1, k_2, ..., k_m, k_{m+1})\ cu\ k_{m+1} = n+1 \tag{15}$$

i.e. the (n+1) state is the absorbing state of a previous chain.

Remark: The trajectory is finite from the chain construction, $p_{i_m,i_{m+1}} = \Pi_{i_m}$.

P5. Calculate the trajectory weight

$$W_m(\gamma) = w_{k,k_1} \cdot w_{k_1,k_2} \cdot ... w_{k_{m-1},k_m} \cdot \tag{16}$$

P6. Calculate the next statistics

$$X(\gamma) = W_m(\gamma) \cdot \frac{b_{k_m}}{\Pi_{k_m}} \tag{17}$$

P7. Calculate N trajectories and the mean

$$\tilde{x}_k^* = \frac{1}{N} \sum_i^N X(\gamma_i) \tag{18}$$

where $\tilde{x}_k^*$ is the approximant of the k component of the solution, and it is a primary estimator of the desired solution.

The next theorem can be demonstrated [5].

Theorem. *The statistics* $X(\gamma)$ *are a primary estimator for the* x_k^* *component of the* x^* *system solution* (2).

Algorithm for simulating a trajectory of a finite, homogeneous first-order Markov chain

P1. Input:

- the initial probability vector $\pi = (\pi_1, ..., \pi_n)$, given by the line corresponding to the component to be determined, $\pi = (P_{i0,j})_{1 \le j \le n}$,
- the probabilities transition matrix P^*,
- N- the length of the trajectory,
- The order i_0 of the desired solution component to be found;

P2. Calculate the finite distribution

$$f_i = \sum_{k=1}^{i} \pi_k \ , \ (\forall) i \in \overline{1,n} \tag{19}$$

P3. For each line i of the matrix P^* calculate the distribution function

$$F_{i,j} = \sum_{k=1}^{j} P_{i,k} \ , \ (\forall) i \in \overline{1,n} \ ; \tag{20}$$

P4. Take $i := i0$;

$tr_1 := i$;

P5. For $(\forall) k \in \overline{1,N}$

Generate $U \sim > \mathsf{U}(0,1)$;

j:=1;

while $U \ge F_{i,j}$ do $j := j+1$;

$tr_k := i$;

$i := j$;

P6. Stop!

3. Numerical application

Consider the linear system of equations (1) where

$$A=\begin{pmatrix} 4 & -1 & 0 & 0 & 0 \\ -1 & 3 & -1 & 0 & 0 \\ 0 & -1 & 5 & -1 & 0 \\ 0 & 0 & -1 & 3 & -1 \\ 0 & 0 & 0 & -1 & 6 \end{pmatrix}, b=\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \text{ having the solution } x^{*}=\begin{pmatrix} 0.403 \\ 0.613 \\ 0.436 \\ 0.566 \\ 0.261 \end{pmatrix}$$

The next transformations are considered to obtain the form (2)

$$\text{Let } V=\begin{pmatrix} 4 \\ 3 \\ 5 \\ 3 \\ 6 \end{pmatrix}=diag(A) \ , D=\begin{pmatrix} 4 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 6 \end{pmatrix}, \ E=D-A \ , \ B=D^{-1}(D-A)$$

$$B=\begin{pmatrix} 0 & 0.25 & 0 & 0 & 0 \\ 0.333 & 0 & 0.333 & 0 & 0 \\ 0 & 0.2 & 0 & 0.2 & 0 \\ 0 & 0 & 0.333 & 0 & 0.333 \\ 0 & 0 & 0 & 0.167 & 0 \end{pmatrix} \ , \ b=D^{-1}b=\begin{pmatrix} 0.25 \\ 0.333 \\ 0.2 \\ 0.333 \\ 0.167 \end{pmatrix}$$

The probabilities matrix *P*, associated with matrix *B* have for each line the binomial distribution with the parameters λ=(0.6 0.7 0.9 0.8 0.6) and the probabilities transition matrix becomes

$$P^{*}=\begin{pmatrix} 0 & 0.23 & 0 & 0 & 0 & 0.77 \\ 0.028 & 0 & 0.309 & 0 & 0 & 0.663 \\ 0 & 0.0081 & 0 & 0.328 & 0 & 0.664 \\ 0 & 0 & 0.205 & 0 & 0.328 & 0.468 \\ 0 & 0 & 0 & 0.259 & 0 & 0.741 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The corresponding weights

$$w=\begin{pmatrix} 0 & 1.085 & 0 & 0 & 0 \\ 11.758 & 0 & 1.08 & 0 & 0 \\ 0 & 24.691 & 0 & 0.61 & 0 \\ 0 & 0 & 1.628 & 0 & 1.017 \\ 0 & 0 & 0 & 0.643 & 0 \end{pmatrix}.$$

The cumulative distributions of each line is

$$F = \begin{pmatrix} 0 & 0.23 & 0.23 & 0.23 & 0.23 & 1 \\ 0.028 & 0.028 & 0.337 & 0.337 & 0.337 & 1 \\ 0 & 0.0081 & 0.0081 & 0.336 & 0.336 & 1 \\ 0 & 0 & 0.205 & 0.205 & 0.532 & 1 \\ 0 & 0 & 0 & 0.259 & 0.259 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Considering $i_0 = 3$, after 10000 generated trajectories, it results $\tilde{x}_3^* = 0.428$ which is a good approximation for the x_3^* component of the vector solution.

4. Conclusions

The presented method offers the possibility to determine a single component of the solution of a system of linear equations, which no other numerical method offers.
The numerical application comes to reinforce the aforementioned assertion for practical cases.

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ON THE SIMULTANEOUS EXTENSION OF TWO LATTICE OPERATORS

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Abstract: This paper is dedicated to providing some sufficient conditions for the simultaneously extension of two lattice operators to a new lattice operator. Generally to extend a single linear operator having some additional properties, with the preservation of these properties, is not a simple problem. For example it is known that there are some positive linear operators that cannot be extended to positive linear operators. So we understand why the problem of the extension of a lattice operator by preserving its *lattice* property was studied by many mathematicians: D. H. Fremlin (1974), W. A. J. Luxemburg and A. R. Schep (1974), Z. Lipecki (1979, 1980, 1982, 1985), A. W. Wickstead (1980), B. de Pagter (1981), E. R. Aron, A. Hager, J. J. Madden (1982), G. Buskes (1983, 1987), C. D. Aliprantis and O. Burkinshaw (1985), R-M. Dăneț (1987, 1993, 2001). More difficult becomes the problem when we want to extend simultaneously two lattice operators, keeping their defining property. This work continues an earlier research direction of the first author and is inspired by the results obtained by the authors in two previous papers. Note that all the vector spaces considered are over reals.
Mathematics Subject Classification (2010): 46A22, 47B60, 47B65.
Key words: lattice operators, extension of a lattice operator, sublinear operators, Hahn-Banach Theorem.

1. Introduction

The set of lattice operators is a special class of positive linear operators. For a better understanding we mention what mean these notions and others related to them.

Firstly we suppose that X and Y are two ***vector spaces*** and let $T: X \to Y$ and $S: X \to Y$ be two mappings. We say (see, for example, [4]) that

a) T is a *linear operator* if it has the following properties for all x, y in X and $\alpha \in \mathbb{R}$

1) $T(x+y) = T(x) + T(y)$ (i.e., T is *additive*),

2) $T(\alpha x) = \alpha T(x)$ (i.e., T is *homogeneous*).

b) S is a *sublinear operator* if it has the following properties for all x, y in X and $\alpha \geq 0$ in $\mathbb{R}$

1) $S(x+y) \leq S(x) + S(y)$ (i.e., S is *subadditive*),

2) $S(\alpha x) = \alpha S(x)$ (i.e., S is *positively homogeneous*).

Secondly we assume that E and F are two ***ordered vector spaces***, $T: E \to F$ is a linear operator and $S: E \to F$ is a mapping. We say that

a) T is a *positive operator* if $T(x) \geq 0$ for all $x \geq 0$ in E,

b) S is *monotone* (or, equivalently, *increasing*) if $S(x) \le S(y)$ for all $x \le y$ in E.

Now we suppose that E and F are two ***vector lattices*** and $T : E \to F$ is a linear operator. Obviously T preserves the algebraic operations. A lattice operator is a linear operator that preserves the lattice operations, too. More precisely we say that T is a *lattice operator* if for all x, y in E,

$$T(x \vee y) = T(x) \vee T(y) \tag{1}$$

or, equivalently,

$$T(x \wedge y) = T(x) \wedge T(y). \tag{2}$$

In [6] are given other properties equivalent with (1). Also are given few examples of lattice operators, some of them known from the literature.

In this paper we are interested on simultaneously extending of two lattice operators to a new lattice operator. Obviously the problem is part of a more large class, namely to extend some linear operators that have a common property (P), the extension being made with the preservation of this property (and of linearity). It is known that the problem is not trivial, even if, for example, we extend a single operator and the property (P) is the *positivity*; indeed, it is possible for the class $E(T)$, of all positive linear extensions of a positive linear operator T, to be empty. An example is [2, Example 1.29].

The classical result for the ***extension of linear operators*** is the well known Hahn-Banach Extension Theorem. We recall its vectorial form – see, for example, [2, Theorem 1.25]. *Let X be a vector space, $G \subseteq X$ a vector subspace and F a Dedekind complete vector lattice. Let also $S : X \to F$ be a sublinear operator and $T : G \to F$ a linear operator such that $T \le S$ on G (that is, $T(v) \le S(v)$, for all $v \in G$). Then there exists a linear operator $L : X \to F$ such that $L \le S$ on X and $L = T$ on G (that is, $L(x) \le S(x)$ for all $x \in X$ and $L(v) = T(v)$ for all $v \in G$).* We say that L is an *extension* of T.

Concerning the ***extension of a positive linear operator*** $T : G \to F$, with E a vector lattice, $G \subseteq E$ a vector sublattice and F a Dedekind complete vector lattice, in [2, Theorem 1.27] is shown that *T extends to a positive linear operator if and only if there exists a monotone sublinear operator $P : E \to F$ such that $T \le P$ on G*. Recall that if E is an ordered vector space, a vector subspace $G \subseteq E$ is called *majorizing* if for any $x \in E$ there exists $v \in G$ with $x \le v$ (or, equivalently, there exists $w \in G$ with $w \le x$). If moreover, F is a Dedekind complete vector lattice and $T : G \to F$ is a positive linear operator, then the mapping $\overline{T} : E \to F$, associated with T by the equality $\overline{T}(x) = \inf\limits_{\substack{v \in G \\ v \ge x}} T(v)$, is a well-defined monotone sublinear operator such that $T \le \overline{T}$ on G. Of course the classical result for the extension of positive linear operators is the following result known as the Kantorovich Theorem - see, for example, [2, Theorem 1.32]. *If E is an ordered vector space, $G \subseteq E$ a majorizing vector subspace, F a Dedekind complete vector lattice and $T : G \to F$ a positive linear operator, then T has a positive linear extension to all of E*. (For the proof, we apply the Hahn-Banach Extension Theorem for $P = \overline{T}$.)

In the sequel E and F will be two vector lattices and G, G_1, G_2 vector sublattices of E. Also $T_1 : G_1 \to F$, $T_2 : G_2 \to F$ will be two lattice operators such that $T_1 = T_2$ on $G_1 \cap G_2$. Sometimes by using an idea of R. Cristescu [5], who generalized a result of Z. Lipecki [11],

in order to give ***extension results for two lattice operators*** we will consider an additional set $M \subseteq E$ and a mapping $P : M \to F$ with some properties. We will use some previous results obtained in [9] (see also [7] and [8]) concerning the simultaneous extension of a family of positive linear operators.

2. Preliminaries

2.1. A description of the sublattice generated by a vector space

Let E be a vector lattice and $G \subseteq E$ a vector subspace. Denote $S(G)$ the vector sublattice generated by G. In [1], we meet descriptions of the *vector sublattice generated by a set* and *by a vector subspace*, respectively. To quote them we need some notation. For every nonempty subset A of a vector lattice E, the symbol $A^{\wedge}$ will denote the collection of all vectors that can be written as *infima* of *finite* subsets of A (i.e., $a \in A^{\wedge} \Leftrightarrow \exists\, a_1,...,a_n \in A : a = \bigwedge_{k=1}^{n} a_k$). Similarly, $A^{\vee}$ is the set consisting of all *suprema* of the *finite* subsets of A. Denote by $A^{\vee\wedge}$ the set $\left(A^{\vee}\right)^{\wedge}$ and by $A^{\wedge\vee}$ the set $\left(A^{\wedge}\right)^{\vee}$. It always turns out that $A^{\vee\wedge} = A^{\wedge\vee}$. This is a consequence of the following lemma, whose proof is based upon the distributive laws of the lattice operations in vector lattices.

Lemma A. (see [1]) *Let* $\left\{x_{ij} \,\middle|\, i \in I = \{1,2,...,n\},\ j \in J = \{1,2,...,m\}\right\}$ *be a finite subset of vectors in a vector lattice. Then* $\bigwedge_{\tau \in J^I} \bigvee_{i=1}^{n} x_{i,\tau(i)} = \bigvee_{i=1}^{n} \bigwedge_{j=1}^{m} x_{ij}$ *and* $\bigvee_{\tau \in J^I} \bigwedge_{j=1}^{m} x_{\tau(j),j} = \bigwedge_{i=1}^{n} \bigvee_{j=1}^{m} x_{ij}$, *where* J^I *denotes the* (*finite*) *collection of all mappings from* I *to* J.

If A is a nonempty subset of a vector lattice E, then the collection of all vector sublattices $H \subseteq E$ with $H \supseteq A$ is nonempty (since E is in this collection). The intersection of this collection of vector sublattices is the smallest vector sublattice that includes A. This vector sublattice is called the *vector sublattice* (or the *Riesz subspace*) *generated by* A. We will denote it by $S(A)$ ($R\left(A\right)$ in [1]). The following lemma leads to a description for $S(A)$.

Lemma B. (see [1]) *The vector sublattice* $S(G)$ *generated by a vector subspace* G *of a vector lattice* E *coincides with* $G^{\wedge\vee}$ *and also with* $G^{\vee\wedge}$. *That is,* $S(G) = G^{\wedge\vee} = G^{\vee\wedge}$.

Consequently, the vector sublattice generated by a nonempty subset A of a vector lattice E is $S(A) = \left(spanA\right)^{\wedge\vee}$. Now, using an idea of Z. Lipecki [11], we have another description for $S(G)$.

Proposition 1. $S(G) = M - M$ *with* $M = G^{\vee}\left(= \left\{ \bigvee_{i=1}^{n} v_i \,\middle|\, n \in \mathbb{N}^*, v_i \in G, i = \overline{1,n} \right\}\right)$.

2.2. Description of the sublattice generated by two vector sublattices and a vector

Proposition 2. *Let* E *be a vector lattice,* $G_1, G_2 \subseteq E$ *two vector sublattices and* $x_0 \in E \setminus span\left(G_1 \cup G_2\right)$. *Denote* $S\left(G_1 \cup G_2 \cup \{x_0\}\right)$ *the vector sublattice of* E *generated by* $G_1 \cup G_2 \cup \{x_0\}$. *Then* $S\left(G_1 \cup G_2 \cup \{x_0\}\right) = M - M$, *where*

$$M = \left\{ \bigvee_{i=\overline{1,n}} \left(v_i' + v_i'' + \alpha_i x_0\right) \,\middle|\, n \in \mathbb{N}^*,\ v_i' \in G_1,\ v_i'' \in G_2,\ \alpha_i \in \mathbb{R}_+,\ i = \overline{1,n} \right\}.$$

2.3. Vector subspaces included in a wedge

In [3] we find the following result.

Lemma C. (see [3, Lemma 1.4]) *Let E be a vector space and $M \subset E$ a nonempty convex subset. Then M is a wedge if and only if $M = G + K$, where $G \subseteq E$ is a vector subspace and K is a cone in E. Moreover, whenever M is a wedge, we have the following.*

1) *If we let $G = M \cap (-M)$ and $K = \{0\} \cup (M \setminus G)$, then G is a vector subspace and K is a cone satisfying $M = G + K$.*

2) *If $M = G + K$, where G is a vector subspace and K is a cone such that $G \cap K = \{0\}$, then $G = M \cap (-M)$.*

3) *The wedge $M = G + K$ is a cone, i.e., $G = \{0\}$, if and only if M contains no straight line passing through the origin.*

3. Main results

3.1. Common extensions for two positive linear operators

The following result is in the line of an extension theorem due to R. Cristescu [5], and is inspired by a result of Z. Lipecki [10].

Theorem 1. (see [9]) *Let E and F be two ordered vector spaces and $P : E \to F$ a monotone sublinear operator. Let $G_1, G_2 \subseteq E$ be two vector subspaces, $M \subset E$ an arbitrary nonempty set and $T_1 : G_1 \to F$ and $T_2 : G_2 \to F$ two positive linear operators such that $P = T_1$ on G_1 and $P = T_2$ on G_2. Denote $E_0 = span(G_1 \cup G_2 \cup M)$. Then the following are equivalent.*

i) *There exists a positive linear operator $L : E_0 \to F$ such that $L = T_1$ on G_1, $L = T_2$ on G_2 and $L = P$ on M.*

ii) $P\left(\sum_{j=1}^{n} z_j\right) = \sum_{j=1}^{n} P\left(z_j\right),\ \forall n \in \mathbb{N}^*, \forall z_1, z_2, ..., z_n \in M.$

(Note that the condition ii) determines L uniquely.)

Corollary 2. (see [5, Theorem 2]) *Let E and F be two ordered vector spaces and $P : E \to F$ a monotone sublinear operator. Let $G \subseteq E$ be a vector subspace, $M \subseteq E$ an arbitrary nonempty set and $T : G \to F$ a positive linear operator such that $P = T$ on G. Denote $E_0 = span(G \cup M)$. Suppose that:*

$$P\left(\sum_{j=1}^{n} z_j\right) = \sum_{j=1}^{n} P\left(z_j\right),\ \forall n \in \mathbb{N}^*, \text{ and } z_1, z_2, ..., z_n \in M.$$

Then there exists a positive linear operator $L : E_0 \to F$ such that $L = T$ on G and $L = P$ on M. This operator is unique.

Corollary 3. (see [10]) *Let E be an ordered vector space, F a Dedekind complete ordered vector space, $G \subseteq E$ a majorizing vector subspace, $M \subseteq E$ a nonempty set, $T : G \to F$ a positive linear operator and $P = \overline{T} : E \to F$ a sublinear operator (where $\overline{T}(x) = \inf\limits_{\substack{v \in G \\ v \geq x}} T(v)$).*

Denote $E_0 = span(G_1 \cup G_2 \cup M)$. Then the following are equivalent.

i) *T extends to a (unique) positive linear operator $L : E_0 \to F$.*

ii) $\overline{T}\left(\sum_{i=1}^{n} z_i\right) = \sum_{i=1}^{n} \overline{T}\left(z_i\right),\ \forall n \in \mathbb{N}^*,\ \forall z_1,\ z_2, ..., z_n \in M.$

The following consequence of Theorem 1 is in the line of Corollary 3 (the result of Z. Lipecki).

Corollary 4. (see [9]) *Let E be an ordered vector space and F an ordered complete vector lattice. Let $G_1, G_2 \subseteq E$ be two vector subspaces, one of them, say G_1, being majorizing. Let also $M \subseteq E$ be an arbitrary nonempty set and $T_1 : G_1 \to F$ and $T_2 : G_2 \to F$ two positive linear operators such that $\overline{T}_1 = T_2$ on G_2, where $\overline{T}_1(x) = \inf_{\substack{v_1 \in G \\ v_1 \geq x}} T_1(v_1),\ \forall x \in E$. Denote $E_0 = span(G_1 \cup G_2 \cup M)$. Then the following are equivalent.*

i) *There exists a positive linear operator $L : E_0 \to F$ such that $L = T_1$ on G_1, $L = T_2$ on G_2 and $L = \overline{T}_1$ on M.*

ii) $\overline{T}_1\left(\sum_{i=1}^{n} z_i\right) = \sum_{i=1}^{n} \overline{T}_1\left(z_i\right),\ \forall n \in \mathbb{N}^*,\ \forall z_1,\ z_2, ..., z_n \in M.$

The following result can be proved as a consequence of Theorem 1, but we also can give a simple direct proof (for ii)$\Rightarrow$i)).

Theorem 5. (see [9]) *Let E and F be two ordered vector spaces, $G_1, G_2 \subseteq E$ two vector subspaces, and $M \subset E$ a wedge such that $G_1, G_2 \subset M$. Let $T_1 : G_1 \to F$ and $T_2 : G_2 \to F$ be two positive linear operators and $P : M \to F$ a monotone positively homogeneous operator such that $P = T_1$ on G_1 and $P = T_2$ on G_2. Denote $E_0 = span(G_1 \cup G_2 \cup M)$. Then the following are equivalent.*

i) *There exists a positive linear operator $L : E_0 \to F$ such that $L = P$ on M (consequently, L is a common positive linear extension of T_1 and T_2).*

ii) *P is additive on M.*

Remark. $E_0 = M - M$, because $M \supseteq G_1$ and $M \supseteq G_2$.

3.2. Common extension for two lattice operators by using an additional set

Theorem 6. *Let E and F be two vector lattices, $G_1, G_2 \subseteq E$ two vector sublattices, and $M \subseteq E$ a wedge, closed under finite suprema, such that $M \supseteq G_1$ and $M \supseteq G_2$. Consider $E_0 = S(G_1 \cup G_2 \cup M)$ the sublattice generated by G_1, G_2 and M. Let $T_1 : G_1 \to F$ and $T_2 : G_2 \to F$ be two lattice operators and $P : M \to F$ such that*

1) $P(z_1 \vee z_2) = P(z_1) \vee P(z_2),\ \forall z_1,\ z_2 \in M,$

2) *$P = T_1$ on G_1 and $P = T_2$ on G_2,*

3) *P is positively homogeneous.*

Then the following are equivalent.

i) *There exists a lattice operator $L : E \to F$, such that $L = P$ on M (and consequently $L = T_1$ on G_1 and $L = T_2$ on G_2).*

ii) *P is additive on M.*

Remark. If we suppose that F is an Archimedean vector lattice, we can delete the hypothesis 3) because we use it only for ii)$\Rightarrow$i), to prove that L is (positively) homogeneous. But for F Archimedean, this follows from the additivity and the positivity of L.

3.3. Common extension for two lattice operators

The following result is based on an idea of Z. Lipecki (see [11, Corollary]) but it is a generalization of the Lipecki's result. The proof uses Theorem 6.

Theorem 7. *Let E be a vector lattice, $G_1, G_2 \subseteq E$ two vector sublattices and F a Dedekind complete vector lattice. Let $T_1 : G_1 \to F$ and $T_2 : G_2 \to F$ be two lattice operators and $P : E \to F$ such that*

1) $P(x_1 \vee x_2) = P(x_1) \vee P(x_2)$, $\forall x_1,\ x_2 \in E$,

2) $P(v_1 + v_2 + x) = T_1(v_1) + T_2(v_2) + P(x)$, $\forall\, v_1 \in G_1,\ v_2 \in G_2,\ x \in E$ (*hence $P = T_1$ on G_1 and $P = T_2$ on G_2*).

Then there exists a lattice operator $L : E \to F$, such that $L = T_1$ on G_1 and $L = T_2$ on G_2.

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SOLVING PARTIAL DIFFERENTIAL EQUATIONS, USING R

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Abstract: Partial differential equations (PDE) are describing most physical phenomena and have a wide range of applications in many branches of physics, engineering and other sciences. The *ReacTran* package provided by R language can be used to solve PDE of order less or equal than 2. For time-invariant problems all derivatives are approximated by finite differences, the resulting algebraic equations being solved with root-solving functions from package *rootSolve*. For PDEs which involve time as one of the independent variables only spatial derivatives are discretized, while the time derivative is left as a continuous function. The result is a system of ordinary differential equations (ODEs) in time that can be solved with the initial value problem solvers from package *deSolve*.
Mathematics Subject Classification (2010): 35A24; 65M06; 65M20; 68N15
Key words: Partial Differential Equations, R programming language, numerical methods, finite differences.

1. Introduction

First, we present a short comparison of the most popular packages that are typically used for data analysis. The choice of one of the packages is based on what people use or what they learn in school. Because the most mathematical models in physics, biology and many other applications are formulated in terms of differential equation (DE), the paper presents a few classical applications of PDE useful for engineering or science students.

2. Comparison of data analysis packages

In the field of Data Science there are programming languages as R, Python, MATLAB that provide the capabilities to perform data analytics operations or analytical solutions such as Excel, SPSS and SAS. There are numerous debates about which program is better, more advanced or faster. I will summarize the reasons why R is advantageous in most data analysis circumstances: it has powerful data manipulation capabilities; it reads any type of data; easier automation and faster computation; it supports larger data sets; it has advanced statistics capabilities; it runs on many platforms and anyone can contribute with packages to improve its functionality. For example, the R graphics programming is much easier than VBA graphics programming from Excel. The choosing of one programs depending on purpose: in order to test different algorithms R has a lot of packages and they are very easy to use; for general purpose, we can use Python and to prototype a new algorithm we can use MATLAB. In terms of speed MATLAB is slower than R and R is slower than Python. The programs SPSS and SAS are also prevalent because older universities established research groups that have used it for many years [6-8].
I recommend R for its statistical tools, because it is free, multiplatform and open source. R may require some programming skills but the availability of many packages (more than 10,000 R packages in 2017) minimizes this need.

Table 1: Comparison of data analysis packages

Programs	**R**	**SPSS**	**SAS**	**Excel**	**Phyton**	**Matlab**
Purpose and usability	Programming language At the beginning (1995) academics and research purposes Statistical computing Graphics programming language and environment Today R is a great tool for data analysis (commercial, engineering, statistics, etc.…) Library support (R has many packages that are not statistical) Visualization	SPSS is statistical package for social sciences SPSS has a user-friendly interface	SAS has a code for multiple purposes	Easy; Visual; Flexible Prototyping Quick analysis of small datasets Building small scale models/modifying them quickly Small scale data quality Excel-users have VBA which can be used to automate and reproduce analyses	General-purpose programming language	Good to develop new mathematical models quickly Elegant matrix support visualization Statistics Optimization Machine learning Signal and image processing
Open source?	Free/Open source/ No customer service support but R has the biggest online community	No/ does not have access to the source code	No	No	Yes	No
Ease of Learning	Steep learning curve but rapidly lowering by creating high quality tutorials	Easy to learn (similarities with Excel)	Not to hard: many tutorials	Easy to learn	Easy to learn	Easy to program very complex things using toolboxes
Graphical Capabilities	State-of-the-art graphics with packages such as ggplot2	SPSS has only the standard graphical options that accompany its statistical procedures	SAS has decent functional graphical capabilities Difficult customization on plots	Excel can quickly create a variety of data visualizations from tables	Options to use native libraries (matplotlib) or derived libraries	Two- and three-dimensional plots, data exploration, and visualization techniques
Typical users	Finance; Statistics; Engineering	Science	Business; Government	Business	Engineering	Engineering
Company using IT	Facebook, Google, Twitter, Microsoft, Uber, Airbnb, IBM, Ford, New York Times	HP, AIG, Google, Facebook, Twitter, Netflix Genpact, Dell Advanced Analytics, RBS	IBM, Credit Suisse, Canon, ABN-AMRO	Very popular	Google, Youtube Facebook Dropbox, Yahoo, NASA, IBM	Northrop Grumman, Vencore, Inc., Facebook, Ethertronics Inc

3. Partial differential equation

The paper presents a few examples of physical models using PDE: the heat equation for the evolution of the temperature distribution in a body and the wave equation for the motion of a wave front. In such cases we have to solve boundary value problems: we need to have not only the initial conditions, but also boundary conditions for the region in which the model applies.

As an example, considering the advection-diffusion equation:

$$\frac{\partial Y}{\partial t} = -v\frac{\partial Y}{\partial x} + D\frac{\partial^2 Y}{\partial x^2} \text{ on } \Omega \qquad (1)$$

with boundary conditions

$$\alpha \mathrm{Y} + \beta\frac{\partial Y}{\partial n} = g \text{ on } \partial\Omega \qquad (2)$$

where t denotes time, x is the spatial position, v is the advection rate, D is a positive constant known as the diffusion coefficient and $\mathrm{Y} = \mathrm{Y(x,t)}$ is the dependent variable, Ω is the model domain and $\partial\Omega$ is the boundary of domain Ω, α, β are constants (Dirichlet conditions if $\beta = 0$, Neumann condition if $\alpha = 0$ and if both $\alpha, \beta \neq 0$ we have a Robin problem) [1,2].

Numerical modelling of PDE is performed by finite difference method which is based on discretization of PDE which are included in the mathematical model.

The R package *ReacTran* offers grid generation routines, using functions *setup.grid.1D* and *setup.grid.2D* and the functions *tran.1D*, *tran.2D*, and *tran.3D* that implement finite difference approximations of the PDE (1), [3, 4].

For time-independent problems, usually all independent variables are discretized and all derivatives approximated by algebraic equations, that can be solved with root solving methods from package *rootSolve*. This technique applies to elliptic PDE.

For time-varying cases apply the method of lines: discretize the spatial coordinate(s) only, while time is left in continuous form. The PDE is translated into a large number of coupled ordinary differential equations (ODE) that can be solved with specially-designed functions from package *deSolve*. This applies to parabolic PDE such as the heat equation, and to hyperbolic PDE such as the wave equation [2-4].

Several kinds of boundary conditions are implemented, including specification of the boundary values or fluxes.

4. Application of PDE in R

4.1 The Heat Equation

Find the temperature $\mathrm{Y(x,t)}$ in a laterally insulated cooper bar 80 cm long if the initial temperature is $100 \cdot \sin(\pi x/80)°C$ and the ends are kept at $0°\mathrm{C}$.

We solve the one-dimensional heat equation

$$\frac{\partial Y}{\partial t} = D\frac{\partial^2 Y}{\partial x^2} \text{ for } x \in [0.80] \qquad (3)$$

with boundary conditions of Dirichlet type

$$\mathrm{Y(x=0,t)} = 0; \mathrm{Y(x=80,t)} = 0 \qquad (4)$$

and with sine-wave initial condition

$$\mathrm{Y(x, t=0)} = 100 \cdot \sin(\pi \cdot x/80) \qquad (5)$$

where D is a constant depending on physical data for cooper: density $\rho = 8.92 g/cm^3$, specific heat $\sigma = 0.092 cal/(g\ °C)$, thermal conductivity $K = 0.95 cal/(cm\,s\,°C)$.

Calculate $D = K/(\sigma\rho) = 1.158 cm^2/s$ and the analytical solution is

$$Y(x,t) = 100 \cdot \sin(\pi \cdot x/80) e^{-0.001785 \cdot t} \quad (6)$$

If the initial condition changes to

$$Y(x, t=0) = 100 \cdot \sin(3\pi \cdot x/80) \quad (7)$$

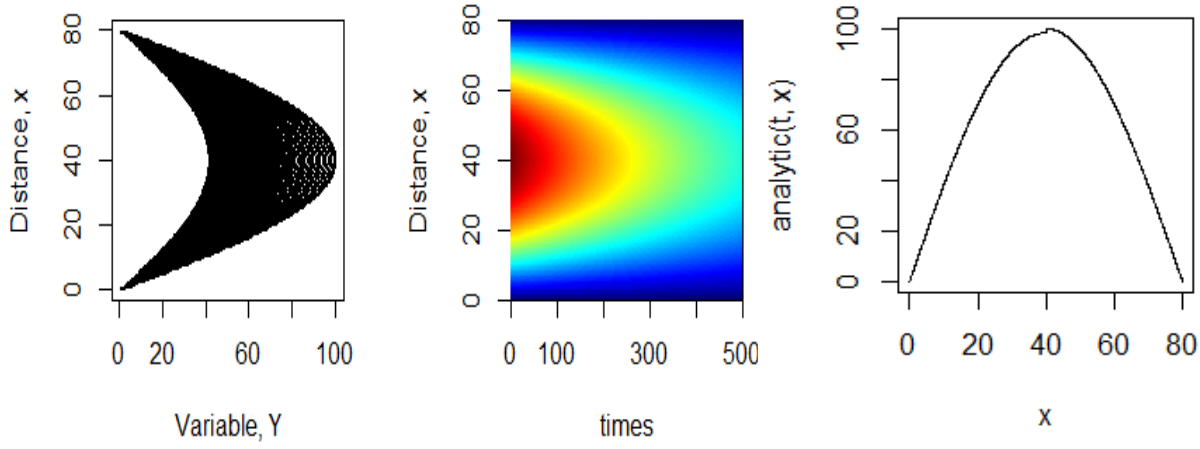
and the other data are as before the analytical solution is

$$Y(x,t) = 100 \cdot \sin(3\pi \cdot x/80) e^{-0.01607 \cdot t} \quad (8)$$

The maximum temperature in the bar drop to $50°C$ for first initial condition (5) in $t = 388s \approx 6.5\min$ and for second condition (7) in $t \approx 43s$ [1].

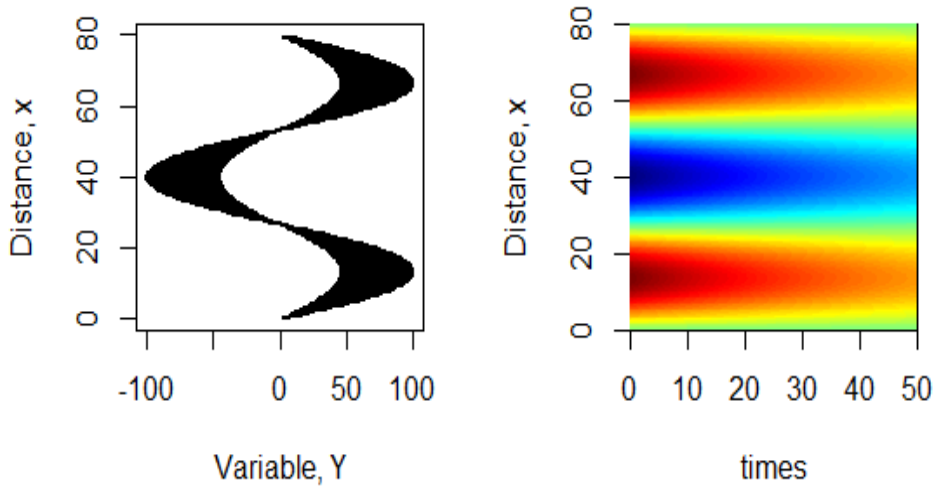
```
N <- 200
xgrid <- setup.grid.1D(x.up = 0, x.down = 80, N = N)
x <- xgrid$x.mid
D.coeff <- 1.158
                    Diffusion <- function (t, Y, parms){
  tran <- tran.1D(C = Y, C.up = 0, C.down = 0,
                  D = D.coeff, dx = xgrid)
  list(dY = tran$dC, flux.up = tran$flux.up,
       flux.down = tran$flux.down)
}
Yini <- 100*sin(pi*x/80)
times <- seq(from = 0, to = 10, by = 0.1)
print(system.time(
  out <- ode.1D(y = Yini, times = times, func = Diffusion,
                parms = NULL, dimens = N)
))
  par (mfrow=c(1, 2))
  plot(out[1, 2:(N+1)], x, type = "l", lwd = 2,
       xlab = "Variable, Y", ylab = "Distance, x")
  for (i in seq(2, length(times), by = 50))
    lines(out[i, 2:(N+1)], x)
  image(out, grid = x, mfrow = NULL, ylab = "Distance, x", main =
"Y")
  analytic <- function (t, x)
    100*sin(pi*x/80)*exp(-0.001785*times)
  curve(analytic(t,x), 0, 80, n=2001)
```

Fig.1. Heat equation code



a) b) c)

Fig. 2. The solution of the 1-D heat equation for initial condition (5): a) represented as time lines of profiles; b) represented as (time, distance) image; c) analytical solution



a) b)

Fig. 2. The solution of the 1-D heat equation for initial condition (7): a) represented as time lines of profiles; b) represented as (time, distance) image;

4.2 The Wave Equation

Consider the following the classical one-dimensional wave equation:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2} \\ \mathrm{u}(\mathrm{x},0) = f(x) \\ \dfrac{\partial \mathrm{u}}{\partial \mathrm{t}}(x,0) = g(x) \end{cases} \qquad (8)$$

For an infinite string [1,2], the boundary conditions impose a zero value at both ends, located at infinity:

$$\mathrm{u}(-\infty, \mathrm{t}) = \mathrm{u}(\infty, \mathrm{t}) = 0 \quad (9)$$

Using D'Alembert's formula, the function $\mathrm{u}(\mathrm{x}, \mathrm{t})$ is the unique solution of PDE (8):

$$u(x,t) = \frac{1}{2}(f(x-ct) + f(x+ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds \qquad (10)$$

In order to solve wave equation in R [2,3], the equation (8) is rewrite as two coupled PDE which are first order in time:

$$\begin{aligned} \frac{\partial \mathrm{u}}{\partial \mathrm{t}} &= v \\ \frac{\partial \mathrm{v}}{\partial \mathrm{t}} &= c^2 \frac{\partial^2 u}{\partial x^2} \end{aligned} \qquad (11)$$

As example, we take the initial conditions

$$\mathrm{u}(\mathrm{x},0) = e^{-\lambda x^2};\ \frac{\partial \mathrm{u}}{\partial \mathrm{t}}(x,0) = 0 \text{ for } c^2 = D = 1 \qquad (12)$$

$$\mathrm{u}(\mathrm{x},0) = 4\cdot\sin(\pi x) - \sin(2\pi x) - 3\sin(5\pi x);\ \frac{\partial \mathrm{u}}{\partial \mathrm{t}}(x,0) = 0 \text{ for } c^2 = D = 9 \qquad (13)$$

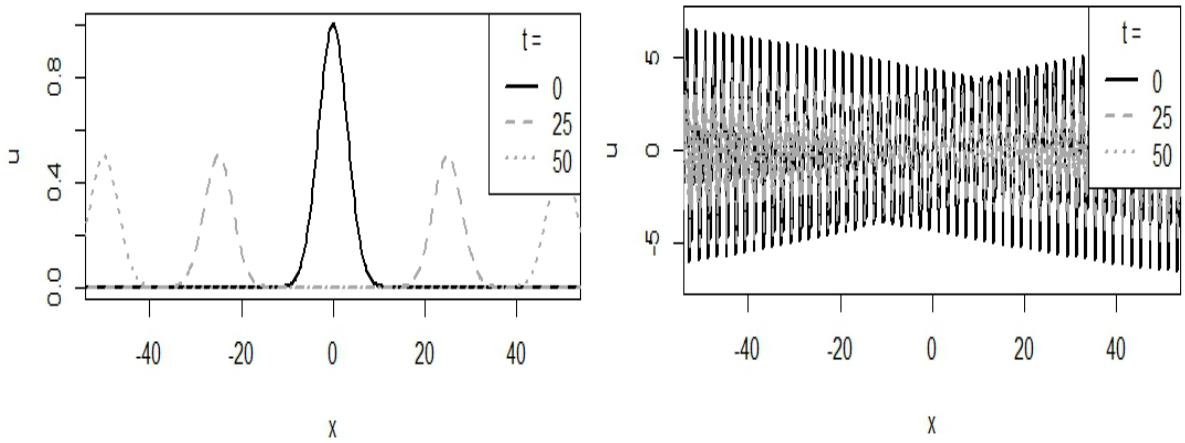


Fig. 2. The solution of the 1-D wave equation: a) for initial condition (12); b) for initial condition (13)

5. Conclusion

From the data analysis packages (R, MATLAB, Excel, SAS, SPSS, Python) R seems to be the outpacing closed open source alternatives because is suitable for solving a wide variety of practical and scientific problems. In Civil Engineering R is little-used but has huge potential for growth. Because R is suitable for mathematical computing and thanks to the differential equation solvers, R is also emerging as a powerful environment for dynamic simulations. The advantage of script-based software solutions is the flexibility. Python is easy and intuitive but it has separate libraries that don't always get along. R provides the potential for taking existing code and modifying it to your own needs, a common requirement in engineering applications of all types. Batch-processing and reproducibility are additional advantages.

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NEW ASPECTS OF THE LAPLACE AND INVERSE LAPLACE TRANSFORM

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Abstract: In our present paper we consider the problem of using the Laplace and inverse Laplace transform in connection with certain complete systems of orthogonal polynomials on the interval $[0,\infty)$.The results we obtain are strongly related to our previous papers [5],[6] concerning the generating function for exceptional and multi-indexed Laguerre polynomials [2],[3],[4],[8]. As possible applications of our results we can mention the fact that orthogonal polynomials are used in quadrature formula for finding the original function from its Laplace transform as well as in risk modeling.
Mathematics Subject Classification (2010): 33C45, 44A10
Key words: Laplace transform, exceptional Laguerre polynomials, generating functions, inverse of non- central chi square distribution

1. Introduction

The non central chi-square distribution plays a key role in many statistical applications, particularly in statistical finance. One of the main problems connected with tests using this distribution is the analysis with missing data. A possible approach for tackling with this problem is the use of exceptional orthogonal polynomials [5]. Another aspect of using such polynomials is their possible application to the study of the inverse of the non central chi-square distribution. The moments of the random variable $Y=\frac{1}{X}$, where $X\in\chi_{\alpha}^{2}(\delta)$, appear in the exact evaluation of the risk in some financial estimators. While there is a closed formula for computing the generating function along with the various moments for the random variable $X\in\chi_{\alpha}^{2}(\delta)$, no such formula exists for the generating function of the random variable $Y=\frac{1}{X}$. Our interest lies in the link between associated Laguerre polynomials and the moments of the random variable $X\in\chi_{\alpha}^{2}(\delta)$, in our quest of computing the moments of Y, again for the case of initially missing data. Thus, we are interested in Laguerre polynomials of negative argument and also of negative exponent and negative argument. This will be derived from a more general situation, since the exceptional Laguerre polynomials [2], [3], [4], [8], [9], present the advantage of including two type of Laguerre functions: $L_j^{-\alpha}(x^2)$ and $L_k^{\alpha}(-x^2)$. Our main interest will be in showing that they arise from the Laplace transform of certain functions. The principal argument in studying this problem lies in the use of orthogonal polynomials in quadrature formulae for the numerical inversion of the Laplace transform along with financial applications in risk modeling [7]. Moreover, the CEV stock

price model requires a non integer number of degrees of freedom, so in our study we shall take the parameter $\alpha > 0$.

Let us briefly recall that, for the random variable $X \in \chi^2_\alpha(\delta)$, with α degrees of freedom and non-centrality parameter δ, the density probability function is given by

$$f(x,\alpha,\delta) = \frac{e^{-\frac{\delta}{2}}}{2^{\frac{\alpha}{2}}} x^{\frac{\alpha}{2}-1} \sum_{j=0}^{\infty} e^{-\frac{x}{2}} \frac{1}{2^j \Gamma(\frac{\alpha}{2}+j)} (\frac{\delta}{2})^j \frac{x^j}{j!}$$

and the n-th moment can be computed as $m_n(X) = E(X^n) = \int_0^\infty x^n f(x)dx = 2^n n! L_n^{(\frac{\alpha}{2}-1)}(-\frac{\delta}{2})$.

Notice that, since the non-centrality parameter $\delta > 0$, we need Laguerre functions of type $L_k^{(\alpha)}(-x^2)$. A similar connection for moments of the random variable $Y = \frac{1}{X}$, for which the probability density function is given by

$$g(y,\alpha,\delta) = \frac{1}{y^2} f(\frac{1}{y},\alpha,\delta) = \frac{e^{-\frac{\delta}{2}}}{2^{\frac{\alpha}{2}}} y^{-(1+\frac{\alpha}{2})} \sum_{j=0}^{\infty} e^{-\frac{1}{2y}} \frac{1}{2^j \Gamma(\frac{\alpha}{2}+j)} (\frac{\delta}{2})^j \frac{y^{-j}}{j!}$$

is given by $m_n(Y) = E(Y^n) = \int_0^\infty y^n g(y,\alpha,\delta)dy = \frac{1}{2^n n!} e^{-\frac{\delta}{2}} [(\alpha+\delta-n) L_{n-\frac{\alpha}{2}+1}^{(\frac{\alpha}{2}-1)}(\frac{\delta}{2}) - \frac{\delta}{2} L_{n-\frac{\alpha}{2}+1}^{(\frac{\alpha}{2})}(\frac{\delta}{2})]$

and we shall need Laguerre functions of type $L_k^{(-\alpha)}(-x^2)$.

The following two remarks are necessary here.

Remark 1. Although in the definitions for exceptional and multi-indexed Laguerre polynomials from [4],[8],[9],[10], the authors consider parameters of the form $\alpha - \frac{1}{2}$, we shall take the parameter to be α, or, equivalently, we shall consider the starting point to be the random variable $X \in \chi^2_{2\alpha+2}(2\delta)$, with $\delta > 0$ and $\alpha > 0$.

Remark 2. The admissibility condition for the construction of type II exceptional Laguerre polynomials with missing data $D = \{d_1, d_2, .., d_M\}$ implies that $\alpha > \max_{1 \le j \le M} \{M, d_j + 1\}$ (see also [8],[9]), while the admissibility condition for type I exceptional Laguerre polynomials implies that the finite set $D = \{d_1, d_2, .., d_M\}$, arranged in natural order, is formed from subsets consistent with the Krein-Adler transform [2],[8], [9].

In the present paper we shall consider the connection with the Laplace transform only for exceptional Laguerre polynomials obtained from seed functions of the type $L_{d_j}^{\alpha}(-x^2)$, $1 \le j \le M$.

2. Laplace transform, Laguerre polynomials and the chi-square measure

The Laguerre polynomials appear as the principal factor in certain functions which serve as solutions to the Schrödinger equation. More precisely, from the following equation

$$y'' + (4n + 2 + 2\alpha - x^2 + \frac{1-\alpha^2}{4x^2}) y = 0$$

the solution is given by $\varphi_n(x) = e^{-\frac{x^2}{2}} x^{(\alpha+\frac{1}{2})} L_n^{\alpha}(x^2)$ [12]. By applying discrete symmetries to the original potential involved in the radial oscillator problem, the following equation was considered in [8], [9], [10]

$$y'' + (4n + 2 + 2\alpha + x^2 - \frac{1-\alpha^2}{4x^2})y = 0$$

with the solution given by $\phi_n(x) = e^{\frac{x^2}{2}} x^{(\alpha+\frac{1}{2})} L_n^{\alpha}(-x^2)$ and the equation

$$y'' + (4n + 2 - 2\alpha + x^2 - \frac{1-\alpha^2}{4x^2})y = 0$$

with the solution $\psi_n(x) = e^{-\frac{x^2}{2}} x^{(\frac{1}{2}-\alpha)} L_n^{-\alpha}(x^2)$. It is worth noticing that, since the results in [8] are stated in the variable $z = x^2$, the equations verified by the polynomial part of each solution are in fact

i) for $y(x) = L_n^{\alpha}(x^2)$, the equation is $xy'' + (1 + 2\alpha - 2x^2)y' + 4nxy = 0$

ii) for $y(x) = L_n^{\alpha}(-x^2)$, the equation is $xy'' + (1 + 2\alpha + 2x^2)y' - 4nxy = 0$.

For a random variable $X \in \chi_{\alpha}^2(\delta)$ with non central chi-square distribution, α degrees of freedom and non centrality parameter δ, we shall denote by $f(x, \alpha, \delta)$ the probability density function and by $\mu_{\alpha,\delta}$ the corresponding probability measure. The moments of the variable X can be expressed by means of the Laplace transform of the measure $\mu_{\alpha,\delta}$

$$m_j(X) = (-1)^j \lim_{t \to 0+} \mathcal{L}^{(j)}(\mu_{\alpha,\delta}) = (-1)^j \lim_{t \to 0+} \mathcal{L}^{(j)}(f(x, \alpha, \delta))$$

As a consequence, we have the following relation

$$2^j j! L_j^{(\alpha)}(-\delta) = (-1)^j \lim_{t \to 0+} \mathcal{L}^{(j)}(f(x, 2\alpha + 2, 2\delta))$$

In the above formula we can take into consideration one of the properties of the Laplace transform relative to the derivative of the image function, thus obtaining

$$L_j^{(\alpha)}(-\delta) = \frac{1}{2^j j!} \lim_{t \to 0+} \mathcal{L}(x^j f(x, 2\alpha + 2, 2\delta))$$

Therefore, we shall use the following notation: for a random variable $X \in \chi_{\alpha}^2(\delta)$, with probability density $f(x, \alpha, \delta)$, the function $f_j(x, \alpha, \delta)$ will denote the probability density of the variable X^j. Consequently, the former relation becomes

$$L_j^{(\alpha)}(-\delta) = \frac{1}{2^j j!} \lim_{t \to 0+} \mathcal{L}(f_j(x, 2\alpha + 2, 2\delta))$$

Since the Laplace transform of $f(x, 2\alpha + 2, 2\delta)$ is $(1 + 2t)^{-(\alpha+1)} \exp(-\frac{2\delta t}{1 + 2t})$, one can obtain $\mathcal{L}(f_j(x, 2\alpha + 2, 2\delta))$ by taking the corresponding derivative in t.

We shall consider now the exceptional orthogonal Laguerre polynomials introduced in [2], [3], [4], [8], [9] and studied also in [10]. Since we are interested in their connection with the chi-square distribution [5], [6], two separate cases arise, namely:

i) Case I. For a finite set of positive integers $D = \{d_1, d_2, .., d_M\}$ with d_j degrees of polynomial seeds, greater or equal to 1, let $l = \sum_{j=1}^{M} d_j - \frac{1}{2} M(M+1)$. Let $N_D = \mathbb{N} \setminus D$. Define the degree $n + l$ polynomial

$$L^{(\alpha)}_{\mathcal{D},n+l}(z) = W[L^{(\alpha)}_{d_1}(z),..,L^{(\alpha)}_{d_M}(z),L^{\alpha}_{n}(z)],\ n \in N_D,$$

where the variable $z = x^2$, the finite set $D = \{d_1, d_2,..,d_M\}$, arranged in natural order, is formed from subsets consistent with the Krein-Adler transform and the parameter $\alpha > -1$.
The derivatives in the above Wronskian are computed relative to the variable z. The polynomial $L^{(\alpha)}_{\mathcal{D},n+l}(z)$ differs from type I exceptional Laguerre polynomials given in [2] , [8] , [10], by a normalization constant.
ii) Case 2. For a finite set of positive integers $D = \{d_1, d_2,..,d_N\}$ with d_j degrees of polynomial seeds, greater or equal to 1, let $l = \sum_{j=1}^{N} d_j - \frac{1}{2}N(N+1)$. Define the following

$$L^{(\alpha)}_{\mathcal{D},n+l}(z) = z^{(\alpha+1)N} W[z^{(-\alpha)}L^{(-\alpha)}_{d_1}(z),..,z^{(-\alpha)}L^{(-\alpha)}_{d_N}(z),L^{\alpha}_{n}(z)]$$

where the variable $z = x^2$. The admissibility condition for the definition of the type II exceptional Laguerre polynomials implies that $\alpha > \max_{1\le j\le M}\{N, d_j + 1\}$.

Case 2 will be the starting point for computing the moments of the random variable $Y = \frac{1}{X}$.
In our present paper we shall deal only with case I polynomials.

By taking the argument of $L^{(\alpha)}_{\mathcal{D},n+l}(z) = W[L^{(\alpha)}_{d_1}(z),..,L^{(\alpha)}_{d_M}(z),L^{\alpha}_{n}(z)]$ to be $-\delta$, and replacing in the definition of the polynomial $L^{(\alpha)}_{\mathcal{D},n+l}(-\delta)$ each term by the corresponding Laplace transform, we obtain, for $1 \le j \le M$, that

$$L^{(\alpha)}_{\mathcal{D},n+l}(-\delta) =$$

$$= \lim_{t\to 0+} \begin{vmatrix} \frac{1}{2^{d_j} d_j!}\mathcal{L}(f_{d_j}(x,2\alpha+2,2\delta)) & .. & \frac{1}{2^{n} n!}\mathcal{L}(f_{n}(x,2\alpha+2,2\delta)) \\ .. & & \\ \frac{(-1)^M}{2^{d_j-M}(d_j-M)!}\mathcal{L}(f_{d_j-M}(x,2\alpha+2M+2,2\delta)) & .. & \frac{(-1)^M}{2^{n-M}(n-M)!}\mathcal{L}(f_{n-M}(x,2\alpha+2M+2,2\delta) \end{vmatrix}$$

Consider now the determinant

$$\begin{vmatrix} \frac{1}{2^{d_j} d_j!}\mathcal{L}(f_{d_j}(x,2\alpha+2,2\delta)) & .. & \frac{1}{2^{n} n!}\mathcal{L}(f_{n}(x,2\alpha+2,2\delta)) \\ .. & & \\ \frac{(-1)^M}{2^{d_j-M}(d_j-M)!}\mathcal{L}(f_{d_j-M}(x,2\alpha+2M+2,2\delta)) & .. & \frac{(-1)^M}{2^{n-M}(n-M)!}\mathcal{L}(f_{n-M}(x,2\alpha+2M+2,2\delta) \end{vmatrix}$$

Clearly each term in the development of the determinant is a product of Laplace transforms of probability density functions corresponding to independent random variables. Using again a property of the Laplace transforms, it will follow that each term is the Laplace transform of a convolution of probability density functions corresponding to independent random variables.

Since convolution corresponds to the sum of independent random variables, the previous argument shows that

Theorem. The exceptional orthogonal Laguerre polynomials $\{2^{n+l}(n+l)!L^{(\alpha)}_{\mathcal{D},n+l}(-\delta)\}_{n\in N_D}$ are the moments of a random variable which is a weighted sum of independent distributions of the type $\sum(\chi^2_{2\alpha+2s+2}(2\delta))^{d_j-s} + (\chi^2_{2\alpha+2r+2}(2\delta))^{n-r}$.

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SOME APPLICATIONS OF NEWTON INTERPOLATION SERIES

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Abstract: Results on Newton interpolating series at sequence *S* either purely periodic or having distinct elements are presented.
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Keywords: divided differences, Newton interpolating series, functional equations, analytic functions

1. Introduction

If $S=\{\alpha_k\}_{k\geq 1}$ is a sequence of elements from an interval $[a,b]$, consider the polynomials

$$u_0=1,\ u_k=\prod_{j=1}^{k}(X-\alpha_j),k\geq 1. \tag{1}$$

Then

$$\sum_{k=0}^{\infty}a_k u_k(x),\ a_k\in\mathbf{R}, \tag{2}$$

is called a *Newton interpolating series in one variable with real coefficients at* $\alpha_i, i=1,2,\ldots$.

As a generalization of Taylor series, Newton interpolating series are useful tools either in approximation and interpolation of functions (e. g. solutions of differential equations) or in number theory. This note presents some known results and also others unpublished.

We say that a function $f:[a,b]\to\mathbf{R}$ *can be represented as Newton interpolating series at* $\{\alpha_k\}_{k\geq 1}$ if there exists a series of the form (2) which converges uniformly to *f* on $[a,b]$. As in [2], for a sequence $S=\{\alpha_k\}_{k\geq 1}$, consider the set $I_S=\{i:\alpha_i\neq\alpha_j \text{ for all } j<i\}$. The sequence *S* is called *purely periodic* if there exists a positive integer *m* such that I_S is a finite set having *m* elements and for every positive integer *i* less than or equal to *m*, $\alpha_i=\alpha_{i+jm}$, $j=1,2,\ldots$.

In this paper we'll study the cases when either the terms of the sequence *S* are distinct or the sequence *S* is purely periodic.

2. Purely periodic sequence of interpolating points

Let $S=\{\alpha_k\}_{k\geq 1}$ be a purely periodic sequence from an interval $[a,b]$ such that $\alpha_1<\alpha_2<\ldots<\alpha_m$ and, for every $k\geq m$, $\alpha_k=\alpha_i$, where $i=r(k)$ is the remainder obtained after dividing *k* by *m*. Then by (1)

$$u_0=1,\ u_k=u_m^{q(k)}u_{r(k)},k\geq 1.$$

where $q(k)$ is quotient obtained after dividing *k* by *m*.

Choose $\alpha_i^{(1)}$, $i=0,1,...,m$, such $\alpha_0^{(1)} \le \alpha_1 < \alpha_1^{(1)} < \alpha_2 < ... < \alpha_{m-1}^{(1)} < \alpha_m \le \alpha_m^{(1)}$, where $\alpha_k^{(1)}$, $k=1,2,...,m-1$, are the roots of $u_m'(x)$. Let

$$D := \left\{ x \in \left[\alpha_0^{(1)}, \alpha_m^{(1)} \right] : \text{ the series } (2) \text{ converges} \right\}$$

be the *domain of convergence* of the series (2). Define *the left/right radius of convergence at* α_k, $k=1,2,...,m$, by the formulas

$$R_k^{(L)} := \sup_{x \in [\alpha_{k-1}^{(1)}, \alpha_k] \cap D} \{\alpha_k - x\}, \quad R_k^{(R)} := \sup_{x \in [\alpha_k, \alpha_k^{(1)}] \cap D} \{x - \alpha_k\}.$$

The following result holds.

Theorem 1. ([4]) *Given the series (2), define* $\rho := \limsup_{k\to\infty} \sqrt[k]{|a_k|}$. *Then*

(a) If $\rho < \left| u_m\left(\alpha_k^{(1)}\right) \right|^{1/m}$, $k=0,1,...,m-1$, *then the equation*

$$|u_m(x)| = \rho^m \tag{3}$$

has exactly one root $y_{k+1} \in (\alpha_k^{(1)}, \alpha_{k+1}]$ *and* $R_{k+1}^{(L)} = \alpha_{k+1} - y_{k+1}$;

(b) If $\rho < \left| u_m\left(\alpha_k^{(1)}\right) \right|^{1/m}$, $k=1,2,...,m$, *then the equation (3) has exactly one root* $z_k \in [\alpha_k, \alpha_k^{(1)})$ *and* $R_k^{(R)} = z_k - \alpha_k$;

(c) If $\rho \ge \left| u_m\left(\alpha_k^{(1)}\right) \right|^{1/m}$, $k=0,1,...,m-1$, *then* $R_{k+1}^{(L)} = \alpha_{k+1} - \alpha_k^{(1)}$, *and if* $\rho > \left| u_m\left(\alpha_k^{(1)}\right) \right|^{1/m}$, *then* $\alpha_k^{(1)} \in D$;

(d) If $\rho \ge \left| u_m\left(\alpha_k^{(1)}\right) \right|^{1/m}$, $k=1,2,...,m$, *then* $R_k^{(R)} = \alpha_k^{(1)} - \alpha_k$, *and if* $\rho > \left| u_m\left(\alpha_k^{(1)}\right) \right|^{1/m}$, *then* $\alpha_k^{(1)} \in D$.

For $x \in D$ we denote

$$S(x) := \sum_{k=0}^{\infty} a_k u_k(x).$$

From Theorem 1 it follows that $S(x)$ is a continuous function on *D*. Moreover by defining, in a usual way the *derivative series* of (2),

$$\sum_{k=0}^{\infty} a_k^{(1)} u_k(x), \tag{4}$$

that is by termwise differentiation of (2) and reordering the terms (see [4]), the derivative series converges on the interior of *D*, $S(x)$ is differentiable and the sum of the series (4) is equal to $S'(x)$.

We note that there are functions $f : [a,b] \to \mathbf{R}$ which can be represented as Newton interpolating series at $\{\alpha_k\}_{k\ge1}$ which are not differentiable at some the points of $[a,b]$. Thus

$$|x| = \sqrt{1-(1-x^2)} = 1 + \sum_{k=1}^{\infty} \frac{(-1)^{k+1} \prod_{i=1}^{k-2}(k+i)}{(k-1)!2^{2k-1}} (x-1)^k (x+1)^k$$

implies that $f(x) = |x|$, $x \in [-1,1]$, can be represented as Newton interpolating series at $\{\alpha_k\}_{k\geq 1}$, with $\alpha_k = (-1)^k$, but f is not differentiable at $x = 0$.

In [4] we used Newton interpolating series at a purely periodic sequence $\{\alpha_k\}_{k\geq 1}$ to approximate solution of multipoint boundary value problems for linear differential equations.

In this section we use these series to approximate the solution of a functional equation.

Consider the functional equation (see [6], p. 81)

$$f\left(\frac{x(x+1)}{2}\right) = \frac{f(x)}{2}. \tag{5}$$

We seek a solution $f(x)$ of (5) as the sum of a convergent Newton interpolation series

$$f(x) = a_0 + a_1 x + a_2 v(x) + a_3 x v(x) + a_4 v(x)^2 + a_5 x v(x)^2 + a_6 v(x)^3 + \dots, \tag{6}$$

where $v(x) = x(x+1)$ and $x \in [0,\gamma] \subset [0,1]$. Since $v(v(x)/2) = \frac{v(x)(v(x)+2)}{4}$, by (5) and (6), we find

$$a_0 + \frac{a_1}{2} v + \frac{a_2}{4} v(v+2) + \frac{a_3}{2\cdot 4} v^2 (v+2) + \frac{a_4}{4^2} v^2 (v+2)^2 + \frac{a_5}{2\cdot 4^2} v^3 (v+2)^2$$
$$+ \frac{a_6}{4^3} v^3 (v+2)^3 + \dots = \frac{1}{2}(a_0 + a_1 x + a_2 v + a_3 x v + a_4 v^2 + a_5 x v^2 + a_6 v^3 + \dots).$$

Hence we get $a_{2r+1} = 0$, $r = 0,1,\dots$, and

$$\frac{a_{2r}}{2} = \frac{a_{2r}}{4^r}\binom{r}{r} \cdot 2^r + \frac{a_{2r-2}}{4^{r-1}}\binom{r-1}{r-2} \cdot 2^{r-2} + \frac{a_{2r-4}}{4^{r-2}}\binom{r-2}{r-4} \cdot 2^{r-4} + \dots$$
$$+ \frac{a_{2r-2\left(r-\left[\frac{r+1}{2}\right]\right)}}{4^{r-\left[\frac{r+1}{2}\right]}} \binom{r-\left(r-\left[\frac{r+1}{2}\right]\right)}{r-2\left(r-\left[\frac{r+1}{2}\right]\right)} \cdot 2^{r-2\left(r-\left[\frac{r+1}{2}\right]\right)}.$$

Then it follows that

$$a_{2r} = \frac{1}{2^{r-1}-1}\left(\binom{r-1}{1} a_{2r-2} + \binom{r-2}{2} \cdot a_{2r-4} + \dots + \binom{\left[\frac{r+1}{2}\right]}{2\left[\frac{r+1}{2}\right] - r} \cdot a_{2\left[\frac{r+1}{2}\right]}\right). \tag{7}$$

Hence we get $a_4 = a_2$, $a_6 = \frac{2}{3}a_2, a_8 = \frac{3}{7}a_2, a_{10} = \frac{26}{105}a_2, a_{12} = \frac{94}{651}a_2, a_{14} = \frac{3292}{41013}a_2, \dots$.

Consider $a_2 > 0$ and $\gamma = \frac{6}{7}$. We'll show that, for every $k \geq 2$,

$$a_{2k} \leq \gamma^{k-2} a_2. \tag{8}$$

Indeed, for every $k = 2,3,...,8$, (8) holds. Now for a fixed $r \geq 2$, by induction on *r*, we suppose that (14) holds for every $k < r$ and we'll prove for it for $k = r$. Since, for every $t \in \left[1, r - \left[\frac{r+1}{2}\right]\right]$, $\binom{r-t}{t} \leq 2^t \binom{\left[\frac{r+1}{2}\right]}{t}$, by (7), for $a_2 > 0$, it follows that

$$a_{2r} \leq \frac{\gamma^{r-2} a_2}{2^{r-1}-1}\left(\binom{\left[\frac{r+1}{2}\right]}{1}\frac{2}{\gamma} + \binom{\left[\frac{r+1}{2}\right]}{2}\left(\frac{2}{\gamma}\right)^2 + ... + \binom{\left[\frac{r+1}{2}\right]}{\left[\frac{r+1}{2}\right]}\left(\frac{2}{\gamma}\right)^{\left[\frac{r+1}{2}\right]}\right).$$

Hence

$$a_{2r} \leq \frac{\gamma^{r-2} a_2}{2^{r-1}-1}\left(\left(1+\frac{2}{\gamma}\right)^{\left[\frac{r+1}{2}\right]} - 1\right). \tag{9}$$

Since $\left(1+\frac{2}{\gamma}\right)^{\left[\frac{r+1}{2}\right]} = \left(\frac{10}{3}\right)^{\left[\frac{r+1}{2}\right]} < 2^{r-1}$, for every $r \geq 8$, by (9) it follows (8). Then, for $x \in \left[0, \frac{-1+\sqrt{\frac{17}{3}}}{2}\right)$, $x(x+1) < 7/6$ and the series $a_0 + a_2 v(x) + a_4 v(x)^2 + a_6 v(x)^3 + ...$ converges uniformly to $f(x)$ on every interval $[c,d] \subset \left[0, \frac{-1+\sqrt{\frac{17}{3}}}{2}\right)$.

3. Sequence of distinct interpolating points

Let $S = \{\alpha_k\}_{k \geq 1}$ be the following sequence (see [3] and [5]) $\alpha_1 = 0$, $\alpha_2 = 1$, $\alpha_3 = \frac{1}{2}$, and for $k \geq 4$,

$$a_k = \frac{2s+1}{2^{m+1}}, \text{ where } k \in (2^m + 1, 2^{m+1} + 1],\ s = k - 2^m - 2. \tag{10}$$

The following result follows from [3].

Theorem 2. *Let* $f : [0,1] \to \mathbf{R}$ *be an infinitely differentiable function such that, for every non-negative integer s,*

$$\left\| f^{(s)}(x) \right\|_\infty \leq s! \left(\frac{e}{2}\right)^s o(\sqrt{s}). \tag{11}$$

Then f can be represented as Newton interpolating series at $S=\{\alpha_k\}_{k\geq 1}$, where the sequence S is defined in (10), the coefficients a_k are divided differences of f, that is

$$a_k=[f(\alpha_1), f(\alpha_2),..., f(\alpha_{k+1})], \tag{12}$$

and there exists a positive constant C such that, for every k,

$$\|u_k(x)\|_\infty \leq \frac{C}{\sqrt{k}}\left(\frac{2}{e}\right)^k. \tag{13}$$

In this section we'll show that Newton interpolating series are useful tools to extend analytic functions as a type of analytic continuation.

For a fixed $\beta>1$, consider the function

$$f(x)=\frac{1}{1+\beta x},\ x\in[0,1].$$

Let $\beta\in(1,e/2)$. Since f is an infinitely differentiable function and $f^{(s)}(x)=\frac{(-1)^s s!\beta^s}{(1+\beta x)^{s+1}}$, it follows that $\left\|f^{(s)}(x)\right\|_\infty \leq s!\beta^s$ and (11) holds. Hence, by (13), because $a_s=[f(\alpha_1),..., f(\alpha_{s+1})]=\frac{f^{(s)}(\xi_s)}{s!}$, with $\xi_s\in[0,1]$ (see [1]), we get $\|a_k u_k(x)\|_\infty \leq \frac{C}{\sqrt{k}}\left(\frac{2\beta}{e}\right)^k$. Thus, by Weierstrass M-test, the series (2), with a_k given by (12) converges uniformly to *f* on [0,1]. Since the Taylor series of *f* at $x_0=0$ diverges on $(1/\beta,1]$ it follows that Newton interpolating series gives a more general representation by extending on the right the interval $[0,1/\beta]$ to the interval [0,1].

We'll see that, by using exact expressions of divided differences, the domain of convergence of the Newton interpolating series extends on the right the domain of convergence of the Taylor Series also for some $\beta\geq e/2$. Indeed

$$[f(\alpha_1), f(\alpha_2)]=\frac{f(\alpha_2)-f(\alpha_1)}{\alpha_2-\alpha_1}=\frac{1}{\alpha_2-\alpha_1}\left(\frac{1}{1+\beta\alpha_2}-\frac{1}{1+\beta\alpha_1}\right)=-\frac{\beta}{(1+\beta\alpha_1)(1+\beta\alpha_2)},$$

$$[f(\alpha_1), f(\alpha_2), f(\alpha_3)]=\frac{[f(\alpha_2), f(\alpha_3)]-[f(\alpha_1), f(\alpha_2)]}{\alpha_3-\alpha_1}=\frac{\beta^2}{(1+\beta\alpha_1)(1+\beta\alpha_2)(1+\beta\alpha_3)},$$

and by recurrence

$$[f(\alpha_1), f(\alpha_2),..., f(\alpha_k)]=\frac{(-1)^{k-1}\beta^{k-1}}{\prod_{i=1}^{k}(1+\beta\alpha_i)}. \tag{14}$$

Let take $\beta=2$. If $k=2^m$, then by (12)-(14) we get

$$\|a_k u_k(x)\|_\infty \le \frac{C}{\sqrt{k}}\left(\frac{2}{e}\right)^k \frac{2^k}{\prod_{i=1}^{k+1}(1+2\alpha_i)} = \frac{C\cdot 2^{2^m-m/2}}{e^{2^m}\prod_{j=0}^{2^m}\left(1+\frac{2j}{2^m}\right)} = \frac{C\cdot 2^{2^m-m/2+(m-1)2^m}}{e^{2^m}\prod_{j=1}^{2^m}\left(2^{m-1}+j\right)}$$

$$= \frac{C\cdot 2^{m2^m-m/2}}{e^{2^m}\,\frac{\Gamma(3\cdot 2^{m-1}+1)}{\Gamma(2^{m-1}+1)}}.$$

Hence, by using Stirling formula, we obtain

$$\|a_k u_k(x)\|_\infty \le \frac{C_1\cdot 2^{m2^m-m/2}\sqrt{2\pi(2^{m-1}+1)}(2^{m-1}+1)^{2^{m-1}+1}}{\sqrt{2\pi(3\cdot 2^{m-1}+1)}(3\cdot 2^{m-1}+1)^{3\cdot 2^{m-1}+1}} = \frac{C_1 2^{m2^m-m/2}}{\left(3\cdot 2^{m-1}+1\right)^{2^m}}\sqrt{\frac{1+2^{1-m}}{3+2^{1-m}}}$$

$$\cdot\left(\frac{1+2^{1-m}}{3+2^{1-m}}\right)^{2^{m-1}+1} \le \frac{C_2 2^{m2^m-m/2}}{3^{2^m}2^{(m-1)2^m}}\cdot\frac{1}{2^{2^{m-1}+1}} \le \frac{C_2 2^{2^m-2^{m-1}}}{3^{2^m}2^{m/2}} = C_2\left(\frac{\sqrt{2}}{3}\right)^{2^m}\cdot\frac{1}{2^{m/2}},$$

where C_1 and C_2 are positive constants. Thus, for $k = 2^m$,

$$\|a_k u_k(x)\|_\infty \le C_2\left(\frac{\sqrt{2}}{3}\right)^k\cdot\frac{1}{\sqrt{k}}. \tag{15}$$

Now, if k is arbitrary, we choose m such that $k = 2^m + s$, with $s \in [0,2^m)$. Then, by (15), it follows that

$$\|a_k u_k(x)\|_\infty = \frac{2^k}{\prod_{i=1}^{k+1}(1+2\alpha_i)}\cdot\left\|u_{2^m}(x)\left(x-\frac{1}{2^{m+1}}\right)\ldots\left(x-\frac{2s-1}{2^{m+1}}\right)\right\|_\infty$$

$$\le C_2\left(\frac{\sqrt{2}}{3}\right)^{2^m}\cdot\frac{2^{2^m}\left\|\left(x-\frac{1}{2^{m+1}}\right)\ldots\left(x-\frac{2s-1}{2^{m+1}}\right)\right\|_\infty}{\prod_{r=1}^{s}\left(1+\frac{2r-1}{2^m}\right)} \le C_2\left(\frac{2\sqrt{2}}{3}\right)^{2^m} \le C_2\left(\frac{3-\varepsilon}{3}\right)^{2^{m+1}} \le C_2\left(\frac{3-\varepsilon}{3}\right)^k,$$

where $\varepsilon \in (0,3-\sqrt[4]{72})$. Hence that Newton interpolating series of f converges uniformly and the statement holds also for $\beta = 2$.

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STUDYING THE EQUIVALENCE BETWEEN A FEED-FORWARD NEURAL NETWORK AND A LINEAR REGRESSION MODEL

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Abstract: A two-layer feed-forward neural network with a specific activation function is identical to a linear regression model. The input neurons are equivalent to independent variables or regressors, while the output neuron is the dependent variable. The various weights of the network are equivalent to the estimated coefficients of a regression model. We use the following baselines: Multiple Linear Regression Model (MLRM) and a Multilayer Perceptron (MP). Some advanced neural network techniques are related to more complex statistical methods such as: kernel discriminant analysis, *k*-means cluster analysis, Principal Component Analysis(PCA). Some neural networks do not have any close parallel in statistics, such as: Kohonen's self-organizing maps, Fuzzy Gaussian Neural Network.
Mathematics Subject Classification (2010): 62H12, 68T10
Key words: Multiple linear regression model, Multilayer Perceptron, feed-forward neural network, activation function

1. Introduction

The Artificial Neural Networks (ANNs) are well-suited for a very broad class of nonlinear approximations and mappings. The ANN with nonlinear activation functions are more effective than linear regression models in dealing with nonlinear relationships.
A feed-forward neural network is a nonparametric statistical model for extracting nonlinear relations in the data, namely it is a useful statistical tool for nonparametric regression.
A feed-forward neural network with a specific activation function is identical to a linear regression model:

- the input neurons are equivalent to independent variables or regressors;
- the output neuron is the dependent variable;
- the various weights of the network are equivalent to the estimated coefficients of a regression model.

Some advanced neural network techniques are related to more complex statistical methods such as:

1) kernel discriminant analysis
2) *k*-means cluster analysis
3) Principal Component Analysis(PCA).

Some neural networks do not have any close parallel in statistics, such as:

1) Kohonen's self-organizing maps
2) Fuzzy Gaussian Neural Network.

The regression and correlation are related as the both of them are designed to extract relations between some variables.
In the case of a linear regression model, of the first order, "the slope of the regression line is the correlation coefficient times the ratio of the standard deviation of y to that of x."[1]

[1] Bingham, N.H. and Fry, J.M.: *Regression. Linear Models in Statistics*, Springer, 2010, New York.

We use the following baselines: Multiple Linear Regression Model (MLRM) and a Multilayer Perceptron (MP).

2. Multiple Linear Regression Model

Multiple Linear Regression Model (MLRM) is a method used to model multiple linear relationship between a dependent variable and more independent variables.
The measurements of the *i*-th regressor variable are now denoted with two indices, namely $x_{i1}, \ldots, x_{in}$.
In total, there are $k \times n$ data values. The data values for the "individual regressor variables are schematically represented as follows:"[2]

Table 1. The data values for the individual regressor variables system

variable	y	x_1	x_2	$\cdots$	x_k
observation *1*	y_1	x_{11}	x_{21}		x_{k1}
observation *2*	y_2	x_{12}	x_{22}		x_{k2}
$\vdots$		$\vdots$	$\vdots$		$\vdots$
observation *n*	y_n	x_{1n}	x_{2n}		x_{kn}

MLRM can be define as [11]:

$$y_j = \hat{y}_j + \varepsilon_j = \alpha_0 + \sum_{i=1}^{k} \alpha_i x_{ij} + \varepsilon_j, \quad (\forall)\, j = \overline{1,n},$$

namely each value y_j is to be approximated by the previous equation, where:

- k means the prediction order of the model;
- y_j represents the observed value of the predictand;
- n is the number of the multiple predictors x_{ij} for the response variable y_j;
- $\alpha_i, (\forall)\, i = \overline{1,n}$ are the regression parameters;
- α_0 representing a bias term, i.e. an intercept term (a regression constant);
- $\hat{y}_j$ is the predicted value of predictand, namely is the y_j predicted;
- ε_j is the prediction error (error term) in observed value y_j, namely:

$$\varepsilon_j = y_j - \hat{y}_j, (\forall)\, j = \overline{1,n}.$$

This type of MLRM can be also written in the vector form [11]:

$$y = X\alpha + \varepsilon,$$

where:

$$y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}, X = \begin{pmatrix} 1 & x_{11} & \cdots & x_{k1} \\ \cdots & \cdots & \cdots & \cdots \\ 1 & x_{1n} & \cdots & x_{kn} \end{pmatrix}, \quad \alpha = \begin{pmatrix} \alpha_0 \\ \vdots \\ \alpha_k \end{pmatrix}, \quad \varepsilon = \begin{pmatrix} \varepsilon_1 \\ \vdots \\ \varepsilon_n \end{pmatrix}.$$

By finding the optimal values of the parameters $\alpha_i, (\forall)\, i = \overline{1,k}$, the MLRM minimizes the Sum of Squared Errors (SSE):

$$SSE = \varepsilon^t \varepsilon = (y - X\alpha)^t (y - X\alpha),$$

[2] Oberguggenberger, M. and Ostermann, A.: *Analysis for Computer Scientists. Foundations, Methods, and Algorithms*, 2011, Springer-Verlag Berlin Heidelberg

where the superscript *t* denotes the transpose.
The vector $\hat{\alpha}$ of the optimal parameters is [11]:

$$\hat{\alpha} = \left(X^t X\right)^{-1} X^t y.$$

Afterwards, the MLRM model is estimated by least squares (which yields parameter estimates such that the sum of squares of errors is minimized), the resulting prediction equation is:

$$\hat{y}_j = \hat{\alpha}_0 + \sum_{i=1}^{k} \hat{\alpha}_i x_{ij},$$

where "^" denotes the estimated values.
A major problem with multiple regression consists in the large number of predictors that are available, although only a few of them are actually significant.

3. Multilayer Perceptron

A common neural network model configuration is to introduce a layer of hidden neurons between the input and output variables (also called neurons), see Fig. 1.

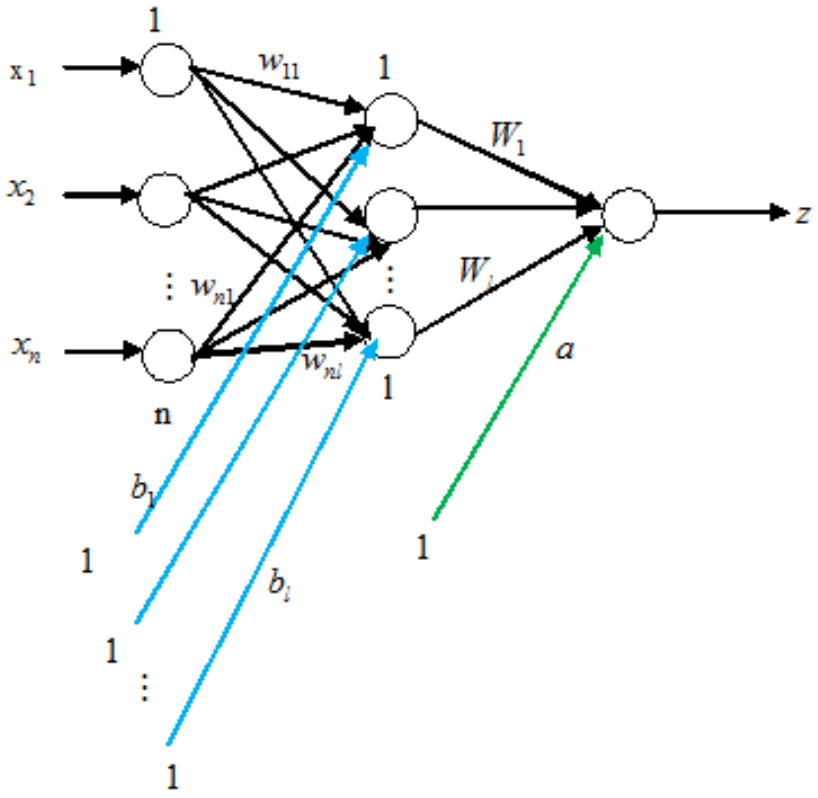

Fig. 1. Mathematical structure of a feed-forward neural network with one hidden layer

The output value of the *j*- th hidden neuron will be:

$$y_j = \tanh\left(\sum_{i=1}^{n} w_{ij} x_i + b_j\right), (\forall)\ j = \overline{1,l}$$

where:

- n is the number of the neurons from the input layer;
- l means the number of the neurons from the hidden layer;
- x_i is the i-th input;
- w_{ij} are the weight parameters, denoting the weight for the connection linking input i to the hidden unit j;
- b_j means the bias parameters for the hidden units.

The output neuron is given by:

$$z = \sum_{j=1}^{l} W_j y_j + a,$$

where:

- W_j are the associated weights to the output layer;
- y_j is the output value of the *j*- th hidden neuron;
- a is a bias term for the output unit.

The parameters b_j and a can also be regarded as the weights for constant inputs of value 1. The optimal parameters w_{ij}, W_j, b_j, a for the network are determined after the training algorithm (a back-propagation algorithm), which has the aim to adjust these parameters such that the cost function (it measures the mean square error between the model output *z* and the observed value z_{obs}) is minimized.

4. Experimental evaluation

The task that we are addressing is to compare the predictive ability of multiple regression and neural model, on by which a user's personality can be accurately predicted through the publicly available information on their Facebook profile.
We use a data set made available by [12], [3].
The personality test called "The Big Five" (the five factor model of personality) represents the most comprehensive, reliable and useful test of personality concepts. It has emerged as one of the most well-researched and well-regarded measures of personality structure in recent years. This test is used to study the relationship between real world social networks (Twitter or Facebook) and personality.
The Big Five traits are characterized [3] by the following:

- *Openness*: curious, intelligent, imaginative. High scorers tend to be artistic and sophisticated in taste and appreciate diverse views, ideas, and experiences.
- *Conscientiousness*: responsible, organized, persevering. Conscientious individuals are extremely reliable and tend to be high achievers, hard workers, and planners.
- *Extroversion*: outgoing, amicable, assertive. Friendly and energetic, extroverts draw inspiration from social situations.
- *Agreeableness*: cooperative, helpful, nurturing. People who score high in agreeableness are peace-keepers who are generally optimistic and trusting of others.
- *Neuroticism*: anxious, insecure, sensitive. Neurotics are moody, tense, and easily tipped into experiencing negative emotions.

We want to predict M components (M being the number of the neurons from the output layer of FGNN) for every vector in order to complete the behavior corresponding to a person.
For evaluation, we use the Normalized Root Mean Square Error (NRMSE) [1].
Following [14], the prediction is considered:

- ✓ **excellent** if NRMSE≤0.1;
- ✓ **good** if 0.1 < NRMSE≤0.2;
- ✓ **fair** if 0.2 < NRMSE≤0.3;
- ✓ **poor** if NRMSE > 0.3.

For significance testing we use the three models: Multiple Linear Regression Model (MLRM) and Multilayer Perceptron (MP). The two models: MLRM and MP have been evaluated using a corresponding test lot, having a number of vectors equal to that of the training lot.
Table 2 shows the normalized root mean square errors achieved on the personality trait prediction task, by a Multilayer Perceptron (MP), a Multiple Linear Regression Model (MLRM).

Table 2. Normalized root mean square errors for personality trait prediction with MP and MLRM.

	Performance on	
Approach	Training set	Test set
MP	**0.168**	**0.189**
MLRM	0.198	0.201

The comparison of MP and respectively MLRM marks both the competition *nonlinear* over *linear* and of *neural* over *statistical*, too.

5. Conclusions

A two-layer feed-forward neural network with a specific activation function is identical to a linear regression model. The input neurons are equivalent to independent variables or regressors, while the output neuron is the dependent variable. The various weights of the network are equivalent to the estimated coefficients of a regression model.

The ability to predict personality has implications in many areas:

- ✓ like other studies relating to personality and language we adopted the five factor model of personality, which describes the following traits on a continuous scale: *neuroticism*, *extraversion*, *openness*, *agreeableness* and *conscientiousness*;
- ✓ in justice as the **personality rights** are some non-patrimonial civil rights, being regulated in article 58 NCC (New Civil Code); the protection of human personality is regulated by the Constitution of the Romania and the NCC (see the article 252).

To emphasize the performances of our proposed approach for predicting personality we have compared it both with a neural method of regression (like MP) and with a nonneural approach (MLRM), too.

According with the NRMSE criterion, we have achieved that the prediction with MP is better than with MLRM both over the training lot and over the test lot, too.

A major problem with multiple regression consists in the large number of predictors that are available, although only a few of them are actually significant.

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ON THE CHARACTERIZATION OF COMPOSITE PLATES

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Abstract: Lamb waves are used for detecting the flaws/faults in structures, such as composite plates with defects. The Lamb wave particles move in 2D vibration modes, the direction of wave propagation and the normal direction of the plate. In order to excite Lamb waves in plates, transducers are mounted on their surfaces in order to emit different types of signals. Our paper is concerning to detecting of defects in a damaged plate. The location and size of defects can be diagnosed by analyzing the received signals. Given the wave frequency, thickness and mechanical properties of the material, the Local Interaction Simulation Approach (LISA) evaluates the interaction of Lamb waves with defects, giving information about integrity of the structure.
Mathematics Subject Classification (2010): 74Bxx, 74Jxx, 74Hxx
Key words: Lamb waves, plates, LISA

1. Introduction

Lamb waves are elastic waves whose particle motion lies in the plane that contains the direction of wave propagation and the direction perpendicular to the plate. The Lamb waves are the preferred tool to detect the defects in thin composite plates [1-3].

Lamb waves can travel over a long distance in fiber-reinforced composite plates and the modes can investigate the integrity of the plate by significant changes of the modal signature [4, 5]. Guo and Cawley [6] investigated the interaction between symmetric S0 mode Lamb waves and delamination at different interfaces in a composite plate through FEM analysis and experiment. The results show that delamination does not affect the S0 mode Lamb wave propagation when the delamination is located at the position under zero shear stress. Ramadas et al [7-9] studied the interaction between antisymmetric A0 mode Lamb waves and symmetric and asymmetrical delamination with 2D finite element models and air-coupled experimental technique and found that both symmetric and asymmetrical delamination affected A0 mode Lamb wave propagation. The Local Interaction Simulation Approach (LISA) [10, 11] provides reliable simulations for the motion of ultrasonic waves and pulses in complex media. In particular, the LISA method is currently employed for the study of Lamb waves propagation and flaws detection in composite plates [12, 13]. LISA visualizes the pulse evolving as it travels through the plate. As it propagates, the transmitted principal pulse is reduced by traveling into the material, and close to his arrival becomes a broad pulse because of multiply scattered energy which appears to diffuse through defects. LISA yield results

which agree well with the exact solution for the dispersion of harmonic waves propagating through the jointed medium.

2. Theoretical aspects

The starting point of the Lamb procedure is Cauchy's equation of motion

$$\iiint_V \rho_0 v(r,t)\mathrm{d}V = \iint_S n \cdot T(r,t)\mathrm{d}S + \iiint_V f(r,t)\mathrm{d}V\,, \tag{1}$$

and the temporal derivation of Hooke's law

$$\iiint_V \dot{T}(r,t)\mathrm{d}V = \iint_S \lambda In \times v(r,t) + \mu\{nv(r,t) + v(r,t)n\}\mathrm{d}S\,, \tag{2}$$

where $v(r,t)$ is the particle velocity vector, $T(r,t)$ the stress tensor, and $\lambda(r)$, $\mu(r)$ are the Lamé's constants, I denotes the unity tensor, n the outward normal unit vector on S, and $f(r,t)$ is a volume force density.

Note that both equations (1) and (2) are given in the integral form. Applying the LISA procedure, we obtain a discretized version of the equations which can be solved numerically by using an explicit time domain scheme on a staggered grid. In comparison with traditional finite difference formulations, LISA proves superiority in the treatment of boundary conditions and in questions concerning numerical stability.

Stress discontinuity at the interface between the matrix and the defect of location x_d can be written as

$$\sigma_{11}(x_d^-,t) = \sigma_{11}(x_d^+,t)\,, \sigma_{13}(x_d^-) = \sigma_{13}(x_d^+,t)\,. \tag{3}$$

The constitutive law associated to the damaged material is

$$\sigma_{ij}(x_d^-,t) = D_{ij}[u_1(x_d^+,t) - u_1(x_d^-,t)]\,, \qquad i,j = 1,3\,, \tag{4}$$

with $D_{11} = \lambda_0 + 2\mu_0$, $D_{13} = \mu_0$ and $D_{33} = \lambda_0$, where λ_0, μ_0 are the unknown elastic moduli of defects.

In this paper, a reinforced carbon fiber plate of dimensions $5 \times 50 \times 50$ cm, made of the epoxy matrix with embedded fibers in 4 plies, is considered. Each ply consists of two sets of mutually orthogonal and interwoven bundles of fibers.

In our simulation, both modes S_0 and A_0 Lamb are operating at the frequency 300 kHz and 100 kHz, respectively. In the first case, the specimen is excited mainly through vibrations parallel to the length of the specimen, while in the latter perpendicular vibrations prevail. Schematic picture of the cross section of the plate (T is transmitter, R is receiver) is presented in Fig. 1. In the bottom of the figure the vibrational modes of the transmitter are illustrated.

To generate Lamb waves into material we use the analytical Lamb displacement solutions for an elastic undamaged plate. We excite the portion of the plate surface corresponding to the transmitter with a sine function. The frequency and direction (parallel or perpendicular) of the injected signals are suitably chosen to select either the S_0 or the A_0 Lamb mode.

The symmetrical wave S_0 is symmetrical with respect to the middle of the plate, while the skew symmetric wave A_0 has skew symmetrical pattern. The stress-free condition leads to the dispersion relation which relates the wave number of individual waves to their respective frequency. The symmetric Lamb wave number k_x are defined by

$$C_s(k_x,\omega) = 4k_x^2 k_{lz} k_{tz} \tan(k_{lz}h) + (2k_x^2 - k_t^2)^2 \tan(k_{tz}h) = 0\,, \tag{5}$$

where

$$k_{lz} = \sqrt{k_l^2 - k_x^2} \ , \ k_{tz} = \sqrt{k_t^2 - k_x^2} \ . \tag{6}$$

In a similar way, the skew symmetric wave number k_x are the solutions of the following dispersion equation

$$C_a(k_x, \omega) = 4k_x^2 k_{lz} k_{tz} \tan(k_{tz} h) + (2k_x^2 - k_t^2)^2 \tan(k_{lz} h) = 0 \ . \tag{7}$$

The quantities (6) can be either real or imaginary, so that the two dispersion equations have ussualy complex wave numbers. In fact, a certain small number of real valued wave numbers exist for a given frequency, together with one or two purely imaginary wave numbers and an infinite number of complex wave numbers. The band gaps are determined from the dispersion curves. The nonzero purely imaginary wave numbers are related to the band gaps which are regions of frequencies where the amplitude of vibration are reduced after reflections. For the complex wave vectors, the waves propagate through the material but the magnitude of the waves decreases with increasing distance.

3. Results and discussions

The plate contains a defect of $5 \times 5 \times 5$ cm located to the distance of 25cm from the right end. The propagation of Lamb waves in a plate has a 2D symmetry if no defects are present, otherwise a 3D treatment is necessary. We construct a 2D mesh, simulating the flaw by means of an effective material with different properties then the matrix.
Reflections and transmissions at the interfaces between layers and between the layers and defects reproduce the signal reduction in amplitude.

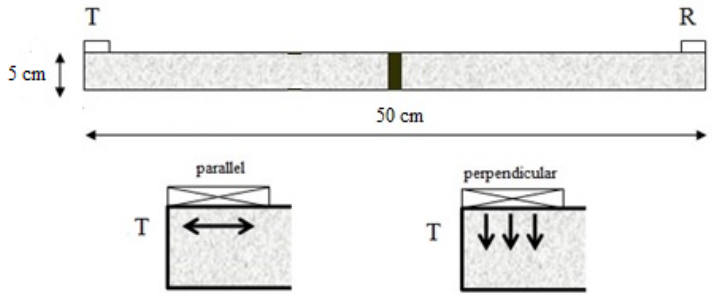

Fig. 1. Schematic picture of the cross section of the plate with two defects.

Some more details of the results are explained in the figure captions. The example include interaction of the longitudinal component of displacements with the defect (Fig. 2). Fig. 3 shows longitudinal component of displacement immediately after the incident wave has reached the defect. One can clearly observe a quasi-longitudinal and a flexural wave, both running in forward as well as in backward direction. The time signals show the calculated velocity components v_x (in relative units) detected at x =15 cm. The incident wave arrives at t ≈ 27 μs while the reflected waves can be observed at t ≈ 65 μs. Fig. 4 shows the shear component of displacement immediately after the incident wave has reached the defect. The velocity components v_x (in relative units) is calculated at x =15 cm. The incident wave arrives at t ≈ 27 μs while the reflected waves can be observed at t ≈ 80 μs.

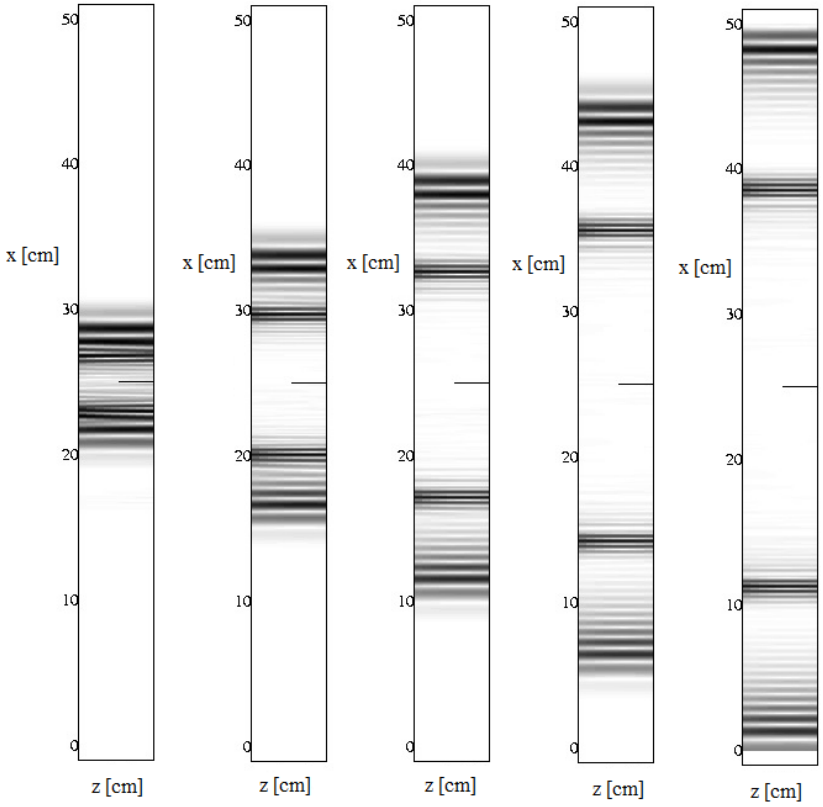

Fig.2. LISA simulation of interaction of longitudinal component of displacement (center frequency 300 kHz) with the defect.

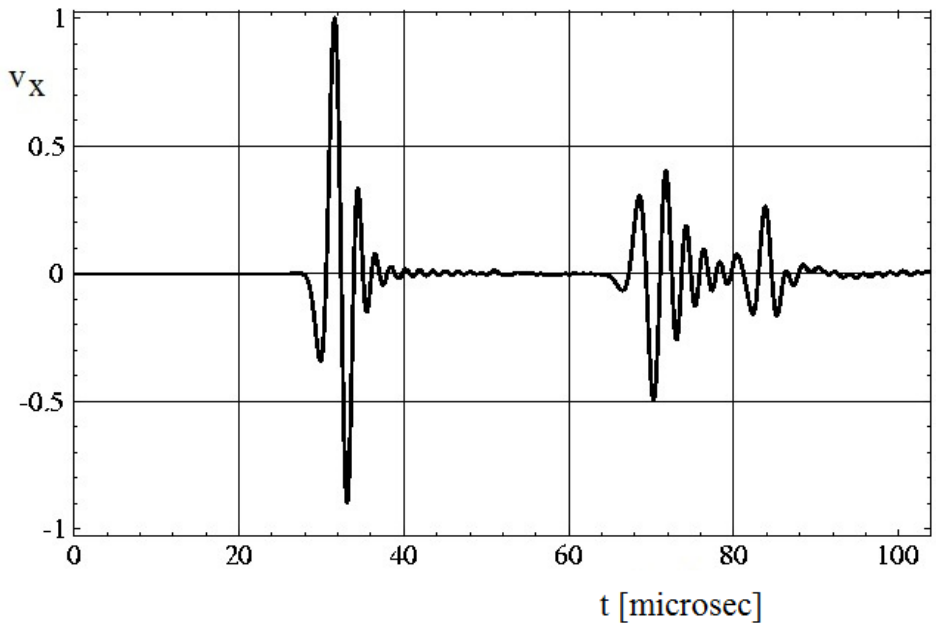

Fig. 3. Longitudinal component of displacement immediately after the incident wave has reached the defect.

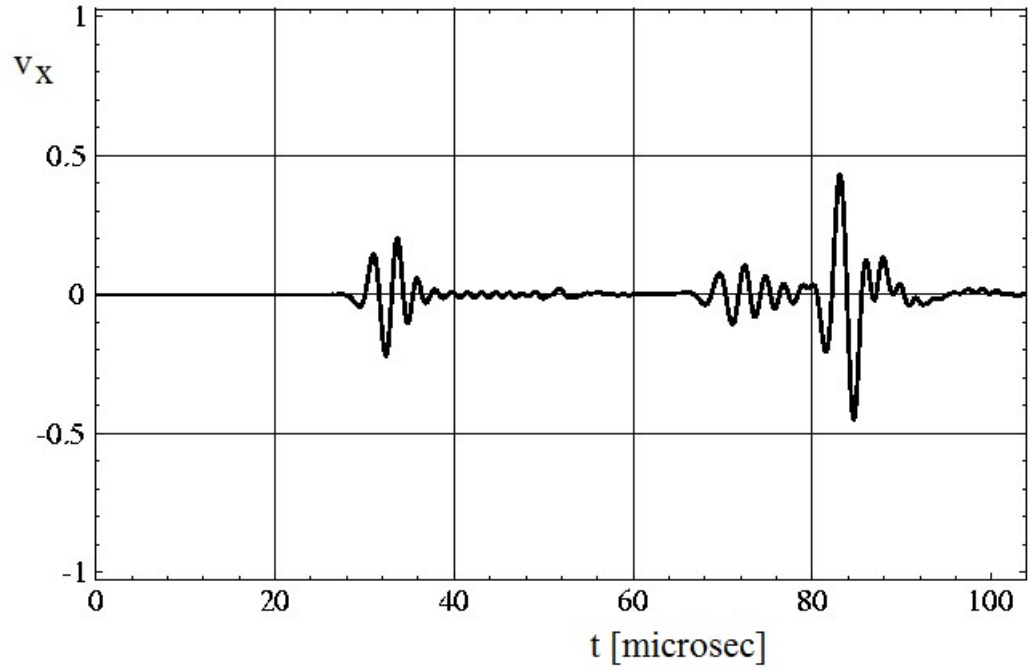

Fig. 4. Shear component of displacement immediately after the incident wave has reached the defect.

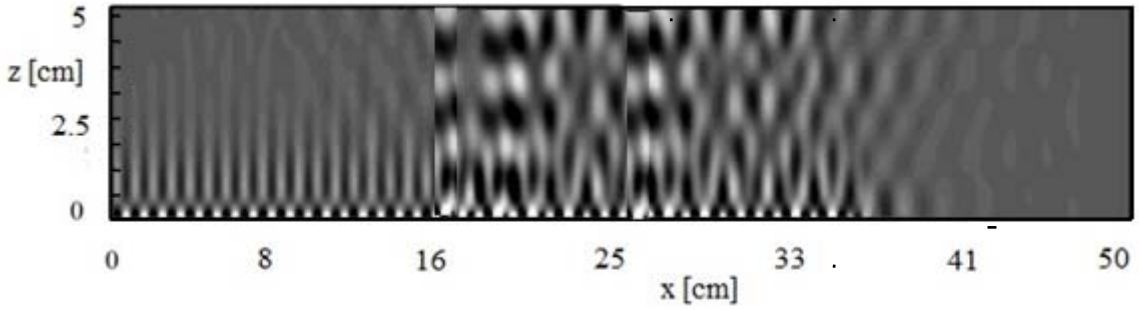

Fig. 5. Snapshot of the longitudinal component of displacement of a Lamb wave at $t = 250\mu s$.

Fig. 5 shows a snapshot of the longitudinal component of displacement of a Lamb wave at time $t = 250\mu s$. The anti-symmetric profile of the displacement reveals the presence of the A_0 and S_0 Lamb modes. The energy associated to A_0 wavelength peak is much greater than the one relative to the S_0, and $\lambda_{A0} > \lambda_{S0}$. A snapshot of the shear component of displacement of a Lamb wave at time $t = 250\mu s$ is presented in Fig. 6.

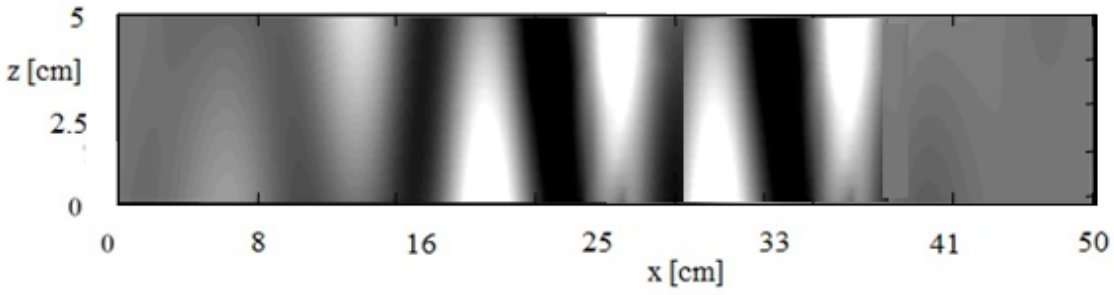

Fig. 6. Snapshot of the shear component of displacement of a Lamb wave at $t = 250\mu s$.

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A NUMERICAL METHOD FOR SOLVING BAGLEY – TORVIK EQUATION USING FRACTIONAL TAYLOR POLYNOMIALS

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Abstract: This paper describes a numerical method for solving fractional differential equations. The method uses fractional Taylor polynomials and is applied for the solution of Bagley – Torvik equation. The coefficients of the polynomial are easy to obtain by a recurrence relation. The efficiency of the method is proved by numerical examples.
Mathematics Subject Classification (2010): 26A33, 34A08, 65L05
Key words: Fractional differential equations, Bagley-Torvik equation, Fractional Taylor polynomials

1. Introduction

The fractional calculus, involving derivatives and integrals of non-integer order, is the natural generalization of the classical calculus, which during recent years became a tool for better modeling of processes in many fields of science and engineering such as viscoelasticity, heat conduction, electromagnetic waves, diffusion wave and control theory. The definitions of the fractional order derivative are not unique and there exist several definitions, including: Riemann-Liouville, Grünwald-Letnikov, Weyl, Riesz and the Caputo representation for fractional order derivative [1, 2].

The Bagley–Torvik equation, originally proposed in [3], is a class of linear fractional differential equations. This equation with 3/2-order derivative or 1/2-order derivative describes motion of real physical systems (an immersed plate in a Newtonian fluid and a gas in a fluid, respectively) and has been studied both analytically and numerically. Recently, considerable attention has been devoted to numerical solutions of the fractional Bagley-Torvik equation. For example, the generalized Taylor collocation method [4], the fractional Taylor method [5], [6], Chebyshev wavelet operational matrix [7], have been used to solve this fractional differential equation. In this study, a method using fractional Taylor polynomials is applied for the solution of Bagley-Torvik equation.

2. Riemann-Liouville fractional integral and Caputo derivative

Definition 1. A function $f:(0,\infty)\to\mathbb{R}$ is said to be of class C_μ ($\mu\in\mathbb{R}$) if there exist $p>\mu$ such that $f(t)=t^p g(t),\forall t>0$, where $g:[0,\infty)\to\mathbb{R}$ is a continuous function. The function f is said to be of class $C_\mu^{(n)}$ ($n\in\mathbb{N}$) if $f^{(n)}\in C_\mu$.

Definition 2. The Riemann – Liouville fractional integral operator of order $\alpha\geq 0$ of a function $y\in C_{-1}$ is defined as

$$J^{\alpha}y(t)=\begin{cases}\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^t(t-s)^{\alpha-1}y(s)ds, & \alpha>0,\\ y(t), & \alpha=0,\end{cases} \tag{1}$$

for $t\geq 0$. This operator has the following properties:

$$J^{\alpha}\left(\lambda_1 y_1(t)+\lambda_2 y_2(t)\right)=\lambda_1 J^{\alpha}y_1(t)+\lambda_2 J^{\alpha}y_2(t),\ \forall\lambda_1,\lambda_2 \text{ constants;} \tag{2}$$

$$J^{\alpha}J^{\beta}y(t)=J^{\beta}J^{\alpha}y(t)=J^{\alpha+\beta}y(t),\quad \forall\alpha,\beta\geq 0; \tag{3}$$

$$J^{\alpha}t^{\gamma}=\frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\alpha+1)}t^{\gamma+\alpha},\quad \forall\alpha\geq 0,\gamma>-1. \tag{4}$$

Definition 3. Let $n=\lceil\alpha\rceil$ (where $\lceil.\rceil$ denotes the ceiling function: $\lceil x\rceil=\min\{z\in\mathbb{Z}:z\geq x\}$). The Caputo fractional derivative of order $\alpha\geq 0$ of a function $y\in C_{-1}^{(n)}$ is defined as

$$D^{\alpha}y(t)=J^{n-\alpha}y^{(n)}(t). \tag{5}$$

The definition is equivalent to:

$$D^{\alpha}y(t)=\begin{cases}\dfrac{1}{\Gamma(n-\alpha)}\displaystyle\int_0^t\frac{y^{(n)}(s)}{(t-s)^{1+\alpha-n}}ds, & \alpha\notin\mathbb{N}\\ y^{(\alpha)}(t), & \alpha\in\mathbb{N}\end{cases}.$$

The Caputo fractional derivative has the following properties:

$$D^{\alpha}\left(\lambda_1 y_1(t)+\lambda_2 y_2(t)\right)=\lambda_1 D^{\alpha}y_1(t)+\lambda_2 D^{\alpha}y_2(t),\ \forall\lambda_1,\lambda_2 \text{ constants;}$$

$$D^{\alpha}J^{\alpha}y(t)=y(t),\quad \forall\alpha\geq 0;$$

$$J^{\alpha}D^{\alpha}y(t)=y(t)-\sum_{k=0}^{\lceil\alpha\rceil-1}y^{(k)}(0)\frac{t^k}{k!},\quad \forall\alpha>0; \tag{6}$$

$$D^{\alpha}t^{\gamma}=\frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)}t^{\gamma-\alpha},\quad \text{if } \gamma\geq\alpha; \tag{7}$$

$$D^{\alpha}t^{\gamma}=0,\ \text{if } \gamma\in\mathbb{N},\gamma<\alpha.$$

3. Fractional power series. Generalized Taylor's formula

Theorem 1. Let $\sum_{n\geq 0}a_n t^{n\alpha}$, $\alpha\in(0,1]$ be a fractional power series, let $R>0$ be the radius of convergence of the power series $\sum_{n\geq 0}a_n x^n$ and $r=\begin{cases}R^{\frac{1}{\alpha}}, \text{if } R<\infty\\ \infty, \text{if } R=\infty\end{cases}$. Then:

i) the series $\sum_{n\geq 0}a_n t^{n\alpha}$ is uniformly convergent on $[0,b]$, $\forall b\in(0,r)$;

ii) if $f:[0,r)\to\mathbb{R}$ is the sum of the fractional power series,

$$f(t)=\sum_{n\geq 0}a_n t^{n\alpha},\ \forall t\in[0,r),$$

then *f* is continuous and there exists the Caputo derivative, $D^{\alpha}f:[0,r)\to\mathbb{R}$. Moreover, the series of the Caputo derivatives, $\sum_{n\geq 0}a_n D^{\alpha}\left(t^{n\alpha}\right)=\sum_{n\geq 1}a_n\frac{\Gamma(n\alpha+1)}{\Gamma\left((n-1)\alpha+1\right)}t^{(n-1)\alpha}$ is uniformly convergent on $[0,b]$, $\forall b\in(0,r)$ and:

$$D^{\alpha}f(t)=\sum_{n\geq 0}a_n D^{\alpha}\left(t^{n\alpha}\right)=\sum_{n\geq 1}a_n\frac{\Gamma(n\alpha+1)}{\Gamma\left((n-1)\alpha+1\right)}t^{(n-1)\alpha},\ \forall t\in[0,r).$$

The next theorem is a generalization of Taylor's Formula.

Theorem 2. [8] If $\alpha \in (0,1]$ and there exist $D^{k\alpha} f = \underbrace{D^{\alpha} D^{\alpha} \ldots D^{\alpha}}_{k-times} f \in C([0,b])$ for $k = 0,1,\ldots,n+1$, then, for any $t \in [0,b]$, there exists $\tau \in [0,t]$ such that

$$f(t) = \sum_{k=0}^{n} \frac{D^{k\alpha} f(0)}{\Gamma(k\alpha+1)} t^{k\alpha} + \frac{D^{(n+1)\alpha} f(\tau)}{\Gamma((n+1)\alpha+1)} t^{(n+1)\alpha}. \quad (8)$$

Definition 4. A real function f is said to be representable by an $\alpha-$fractional Taylor series on $[0,b]$, if it can be written as the sum of a fractional power series:

$$f(t) = \sum_{n\geq 0} a_n t^{n\alpha}, \forall t \in [0,b], \quad (9)$$

Theorem 3. If a function f is representable by an $\alpha-$fractional Taylor series on $[0,b]$ ($\alpha \in (0,1]$), then the coefficients of the series (9) are given by the formula:

$$a_n = \frac{D^{n\alpha} f(0)}{\Gamma(n\alpha+1)}.$$

In this case, the function can be approximated by the fractional Taylor polynomial:

$$f(t) \simeq T_n^{\alpha}(t) = \sum_{k=0}^{n} \frac{D^{k\alpha} f(0)}{\Gamma(k\alpha+1)} t^{k\alpha}. \quad (10)$$

4. Bagley-Torvik equation

The Bagley – Torvik equation is a linear fractional differential equation containing the derivative of order 3/2:

$$AD^2 y(t) + BD^{3/2} y(t) + Cy(t) = f(t), \quad t \in [0,1]. \quad (11)$$

We consider the equation (11) with constant coefficients ($A, B, C \in \mathbb{R}$) subject to the initial conditions

$$y(0) = y_0, \quad y'(0) = y_1. \quad (12)$$

Suppose that f is representable by an $\alpha-$fractional Taylor series on $[0,1]$ with $\alpha = \frac{1}{2}$, so it can be approximated by the fractional Taylor polynomial:

$$f(t) \simeq \sum_{k=0}^{2n} \frac{b_k}{\Gamma\left(\frac{k}{2}+1\right)} t^{k/2}. \quad (13)$$

The solution of the problem (11) - (12) is also approximated by a fractional Taylor polynomial:

$$y(t) = \sum_{k=0}^{2m} \frac{c_k}{\Gamma\left(\frac{k}{2}+1\right)} t^{k/2}, \ m \geq n. \quad (14)$$

By the condition $y \in C^2([0,1])$ it follows that $c_1 = c_3 = 0$ and from the initial conditions (12) we obtain that $y(0) = c_0 = y_0$ and $y'(0) = c_2 = y_1$. The derivatives of the polynomial (14) are calculated using (7):

$$D^2 y(t) = \sum_{k=4}^{2m} \frac{c_k}{\Gamma\left(\frac{k}{2}-1\right)} t^{k/2-2} = \sum_{k=0}^{2m-4} \frac{c_{k+4}}{\Gamma\left(\frac{k}{2}+1\right)} t^{k/2}$$

$$D^{3/2} y(t) = \sum_{k=3}^{2m} \frac{c_k}{\Gamma\left(\frac{k-1}{2}\right)} t^{(k-3)/2} = \sum_{k=0}^{2m-3} \frac{c_{k+3}}{\Gamma\left(\frac{k}{2}+1\right)} t^{k/2}$$

and replaced in the equation (11):

$$A\sum_{k=0}^{2m-4}\frac{c_{k+4}}{\Gamma\left(\frac{k}{2}+1\right)}t^{k/2}+B\sum_{k=0}^{2m-3}\frac{c_{k+3}}{\Gamma\left(\frac{k}{2}+1\right)}t^{k/2}+C\sum_{k=0}^{2m}\frac{c_k}{\Gamma\left(\frac{k}{2}+1\right)}t^{k/2}=\sum_{k=0}^{2m}\frac{b_k}{\Gamma\left(\frac{k}{2}+1\right)}t^{k/2}$$

Thus, the following recurrence relation is obtained:

$$Ac_{k+4}+Bc_{k+3}+Cc_k=b_k,\ k=0,1,\ldots,2m-4. \tag{15}$$

5. Numerical examples

Example 1. Let us consider the Bagley-Torvik equation:

$$D^2y(t)+D^{3/2}y(t)+y(t)=1+t,\quad t\in[0,1] \tag{16}$$

with the initial conditions:

$$y(0)=1,\quad y'(0)=1. \tag{17}$$

Here, we have $A=B=C=1$, $f(t)=1+t$ and $y_0=y_1=1$.

We take $m=3$. The recurrence relation (15) is:

$$c_{k+4}+c_{k+3}+c_k=b_k,\ k=0,1,2,$$

where

$$b_0=b_2=1,\ b_k=0,\ k\neq 0,2.$$

We find $c_0=c_2=1$ and $c_i=0$ for $i\neq 0,2$. Thus, the polynomial obtained is

$$y(t)=1+t,$$

which is the exact solution of the problem.

Example 2. Consider the equation:

$$D^2y(t)+D^{3/2}y(t)+y(t)=2+\frac{4}{\sqrt{\pi}}\sqrt{t}-t+t^2,\quad t\in[0,1] \tag{18}$$

with the initial conditions:

$$y(0)=0,\quad y'(0)=-1. \tag{19}$$

We approximate the solution by a fractional polynomial (14) with $m=4$.

The recurrence relation is:

$$c_{k+4}+c_{k+3}+c_k=b_k,$$

where

$$b_0=b_1=b_4=2,\ b_2=-1 \text{ and } b_k=0 \text{ for } k\notin\{0,1,2,4\}.$$

Since $y_0=0, y_1=-1$ we have $c_0=0$ and $c_2=-1$. By the condition $y\in C^2([0,1])$ it follows that $c_1=c_3=0$ and so we get $c_2=-1, c_4=2$, $c_i=0$ for $i\neq 2,4$. The polynomial obtained is

$$y(t)=-t+\frac{2}{\Gamma(3)}t^2=t^2-t,$$

which is the exact solution of the equation (18) with the initial conditions (19).

Example 3. Let us consider the equation:

$$D^2y(t)-\frac{1}{\sqrt{\pi}}D^{3/2}y(t)+3y(t)=3t^3+2\sqrt{t}\left(12t^2-\frac{4}{\pi}t+15\right),\quad t\in[0,1] \tag{20}$$

with the initial conditions:

$$y(0)=0,\quad y'(0)=3. \tag{21}$$

The recurrence relation is:

$$c_{k+4}-\frac{1}{\sqrt{\pi}}c_{k+3}+3c_k=b_k,$$

where

$b_1 = 30\Gamma\left(\frac{3}{2}\right)$, $b_3 = -\frac{8}{\pi}\Gamma\left(\frac{5}{2}\right)$, $b_5 = 24\Gamma\left(\frac{7}{2}\right)$, $b_6 = 3\Gamma(4)$ and $b_k = 0$ for $k \notin \{1,3,5,6\}$.

We take $m = 4$ and we obtain $c_2 = 3$, $c_5 = 15\sqrt{\pi}$, $c_6 = 6$ and $c_i = 0$ for $i \notin \{2,5,6\}$. Thus, the polynomial is

$$y(t) = t^3 + 8t^{5/2} + 3t,$$

the exact solution of the equation (20) with the initial conditions (21).

Example 4. Consider now a Bagley-Torvik equation with variable coefficients:

$$D^2 y - D^{3/2} y - (16t^2 - 4)y = \sum_{k=0}^{\infty} \frac{(-1)^k 2^{k+2}(2k+1)!}{k!} \cdot \frac{t^{\frac{4k+1}{2}}}{\Gamma\left(\frac{4k+1}{2}+1\right)} \tag{22}$$

with the initial conditions:

$$y(0) = 1, \quad y'(0) = 0. \tag{23}$$

The coefficients of the equation are: $A = 1, B = -1, C(t) = -16t^2 + 4$.

We approximate the solution by the fractional polynomial (14): $y(t) \simeq \sum_{k=0}^{2m} \frac{c_k}{\Gamma\left(\frac{k}{2}+1\right)} t^{k/2}$.

By replacing into the equation (22) we obtain:

$$\sum_{k=0}^{2m-4} \frac{c_{k+4}}{\Gamma\left(\frac{k}{2}+1\right)} t^{k/2} - \sum_{k=0}^{2m-3} \frac{c_{k+3}}{\Gamma\left(\frac{k}{2}+1\right)} t^{k/2} + (4-16t^2) \sum_{k=0}^{2m} \frac{c_k}{\Gamma\left(\frac{k}{2}+1\right)} t^{k/2} = \sum_{k=0}^{2m} \frac{b_k}{\Gamma\left(\frac{k}{2}+1\right)} t^{k/2},$$

or, equivalently:

$$\sum_{k=0}^{2m-4} \frac{c_{k+4}}{\Gamma\left(\frac{k}{2}+1\right)} t^{k/2} - \sum_{k=0}^{2m-3} \frac{c_{k+3}}{\Gamma\left(\frac{k}{2}+1\right)} t^{k/2} + 4\sum_{k=0}^{2m} \frac{c_k}{\Gamma\left(\frac{k}{2}+1\right)} t^{k/2} - 16\sum_{k=4}^{2m+4} \frac{c_{k-4}}{\Gamma\left(\frac{k}{2}-1\right)} t^{k/2} = \sum_{k=0}^{2m} \frac{b_k}{\Gamma\left(\frac{k}{2}+1\right)} t^{k/2}.$$

Since $\Gamma\left(\frac{k}{2}-1\right) = \frac{4}{k(k-2)}\Gamma\left(\frac{k}{2}+1\right)$ we obtain the following relations:

$$c_{k+4} - c_{k+3} + 4c_k - 4k(k-2)c_{k-4} = b_k, \text{ for } k \geq 4$$

and

$$c_{k+4} - c_{k+3} + 4c_k = b_k, \text{ for } k = 0,1,2,3,$$

where

$$b_k = \begin{cases} \frac{(-1)^j 2^{j+2}(2j+1)!}{j!}, k = 4j+1 \\ 0, k = 4j, 4j+2, 4j+3 \end{cases}.$$

The exact solution of the equation (22) with the initial conditions (23) is $y(t) = e^{-2t^2}$. Let $E(t)$ be the difference (in absolute value) between the exact and the approximated value of $y(t)$:

$$E(t) = \left| e^{-2t^2} - T_{2m}^{1/2}(t) \right| = \left| e^{-2t^2} - \sum_{k=0}^{2m} \frac{c_k}{\Gamma\left(\frac{k}{2}+1\right)} t^{k/2} \right|.$$

The table bellow shows the values of $E(t)$ for $t = 0, t = 0.2, \ldots, t = 1$ and for polynomials of order $m = 10$, $m = 20$ and $m = 30$.

Table 1

t	$E(t),\ m=10$	$E(t),\ m=20$	$E(t),\ m=30$
0	0	0	0
0.2	$3.599\cdot10^{-10}$	$1.1\cdot10^{-16}$	$1.1\cdot10^{-16}$
0.4	$1.425\cdot10^{-6}$	$8.782\cdot10^{-14}$	$1.1\cdot10^{-16}$
0.6	$1.752\cdot10^{-4}$	$6.369\cdot10^{-10}$	$3.3\cdot10^{-16}$
0.8	$5.147\cdot10^{-3}$	$3.418\cdot10^{-7}$	$2.3\cdot10^{-12}$
1	$6.866\cdot10^{-2}$	$4.391\cdot10^{-5}$	$2.8\cdot10^{-9}$

6. Conclusion

In the present paper we present a numerical method for solving fractional differential equations. The method approximates the solution by fractional Taylor polynomials and has been applied for solving the Bagley – Torvik equation with constant coefficients, and with variable coefficients, as well. The coefficients of the polynomial are easy to be determined by a recurrence relation. The method can be also applied for other types of fractional differential equations.

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A CR δ -INVARIANT FOR QUATERNIONIC CR-SUBMANIFOLDS IN QUATERNIONIC SPACE FORMS

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Abstract: By using the method of constrained extrema, we establish an inequality for an intrinsic invariant of Chen-type defined on quaternionic CR-submanifolds in quaternionic space forms, in terms of the squared mean curvature, an extrinsic invariant.
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Keywords: CR-submanifolds, invariants.

1. Introduction

The present article is based on the presentation at the 14-th Scientific Communications Session of the Department of Mathematics and Computer Science of Technical University of Civil Engineering Bucharest, dedicated to Professor Gavriil Paltineanu, at the 75-th anniversary, May 27, 2017.

Starting from the results presented in [1], we give a similar inequality in the case of an quaternionic CR-submanifold of a quaternionic space form.

In the geometry of submanifolds, the study of intrinsic and extrinsic invariants of a submanifold play an important role. Among intrinsic invariants, the δ-invariants are very important and finding relations between them and the extrinsic invariants (especially the squared mean curvature) is one of the most fundamental problems in the theory of submanifolds.

The δ-invariants are very different in nature from the classical Ricci and scalar curvature, due to the fact that the non-trivial δ-invariants are obtained from scalar curvature by substracting a certain amount of sectional curvatures.

In the second section of this paper definitions of quaternionic space forms, quaternionic CR-submanifolds are given.

The third section recalls definitions of the invariant δ(D), results from the paper of Al-Solamy, Chen and Deshmukh [1] and the result obtained by the authors of the present article, in the case of a quaternionic CR-submanifold of a quaternionic space form (see [5]).

2. Preliminaries

Let $(\widetilde{M},\widetilde{g})$ be a Riemannian manifold and $M \subset \widetilde{M}$ a Riemannian submanifold of $\widetilde{M}$ with the Riemannian metric induced by the metric of $\widetilde{M}$. We denote by TM and $T^{\perp}M$ the tangent bundle, respectively the normal bundle of M, with ∇ and $\widetilde{\nabla}$ the Levi-Civita connections of M, respectively $\widetilde{M}$.

The Gauss and Weingarten formulae are:

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X,Y),$$
$$\widetilde{\nabla}_X V = -A_V X + \nabla^{\perp}{}_X Y,$$

$\forall X,Y \in T_x M, V \in T_x{}^{\perp}M, x \in M$, where $\nabla^{\perp}$ is the normal connection on $TM^{\perp}$, h is the second fundamental form and A the shape operator.

We also have the relation:

$$g(h(X,Y),V) = g(A_V X,Y).$$

If $\widetilde{M}$ is a $4m$-dimensional manifold with the Riemannian metric $\widetilde{g}$, then $\widetilde{M}$ is called a *quaternionic Kaehler manifold* if there exists a 3-dimensional vector bundle σ of type $(1,1)$ with local basis of almost Hermitian structures $\{J_1, J_2, J_3\}$ such that

$$J_\alpha \circ J_{\alpha+1} = -J_{\alpha+1} \circ J_\alpha = J_{\alpha+2}, J_\alpha^2 = -Id,$$

where $\alpha, \alpha+1, \alpha+2$ are taken modulo 3.

In this case, σ is called the *almost quaternionic structures* on $\widetilde{M}$, $\{J_1, J_2, J_3\}$ is the *canonical local basis of* σ. So, $(\widetilde{M},\sigma)$ is called an *almost quaternionic manifold*, with $\dim \widetilde{M} = 4m, m \geq 1$.

A Riemannian metric $\widetilde{g}$ on $\widetilde{M}$ is said to be *adapted to the almost quaternionic structure* σ if it satisfies

$$\widetilde{g}(J_\alpha X, J_\alpha Y) = \widetilde{g}(X,Y), \forall \alpha = \overline{1,3}.$$

$(\widetilde{M},\sigma,\widetilde{g})$ is called an *almost quaternionic Hermitian manifold*.

If σ is parallel with respect to $\widetilde{\nabla}$ of $\widetilde{g}$, then $(\widetilde{M},\sigma,\widetilde{g})$ is called a *quaternionic Kaehler manifold*. Equivalently, locally defined 1-forms $\omega_1, \omega_2, \omega_3$ exist such that

$$\forall \alpha = \overline{1,3}, \widetilde{\nabla}_X J_\alpha = \omega_{\alpha+2}(X) J_{\alpha+1} - \omega_{\alpha+1}(X) J_{\alpha+2},$$

where $\alpha, \alpha+1, \alpha+2$ are taken modulo 3.

Let $(\widetilde{M},\sigma,\widetilde{g})$ be a quaternionic Kaehler manifold and X be a non-null vector on $\widetilde{M}$. Then the 4-plane spanned by $\{X, J_1X, J_2X, J_3X\}$, denoted by $Q(X)$, is called a *quaternionic* 4*-plane*. Any 2-plane in $Q(X)$ is called a *quaternionic plane*. The sectional curvature of a quaternionic plane is called a *quaternionic sectional curvature*.

A quaternionic Kaehler manifold is called a *quaternionic space form* if its quaternionic sectional curvature is constant, say c.

$(\widetilde{M},\sigma,\widetilde{g})$ is a quaternionic space form if and only if

$$\widetilde{R}(X,Y)Z = \frac{c}{4}[\widetilde{g}(Y,Z)X - \widetilde{g}(X,Z)Y] +$$

$$+ \sum_{\alpha=1}^{3}[\widetilde{g}(Z,J_\alpha Y)J_\alpha X - \widetilde{g}(Z,J_\alpha X)J_\alpha Y + 2\widetilde{g}(X,J_\alpha Y)J_\alpha Z], \forall X,Y,Z \in T_x\widetilde{M}.$$

For a submanifold M of $\widetilde{M}$, if $\{e_1, e_2, ..., e_n\}$ is an orthonormal basis of T_xM, $\{e_{n+1}, e_{n+2}, ..., e_{4m}\}$ an orthonormal basis of $T_xM, x \in M$,

$$H(x) = \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i)$$

is called *the mean curvature vector*.

One denotes by

$$h_{ij}^r = g(h(e_i, e_j), e_r), i, j = \overline{1,n}, r = \overline{n+1, 4m},$$

$$\|h\|^2 = \sum_{i,j=1}^{n} g(h(e_i, e_j), h(e_i, e_j)).$$

Let $\widetilde{M}$ be a quaternionic Kaehler manifold and M be a real submanifold of $\widetilde{M}$. A distribution $D: x \to D_x \subset T_xM$ is called a *quaternionic distribution* if we have $J_\alpha(D) \subset D$, $\forall \alpha = 1,2,3$, so D is carried into itself by the quaternionic structure.

M is called a *quaternionic CR-submanifold* if it admits a differential quaternionic distribution D such that its orthogonal complementary distribution $D^\perp$ is totally real, i.e., $J_\alpha(D_x^\perp) \subset T_x^\perp M, \alpha = 1,2,3, \forall x \in M$.

A submanifold M in a quaternionic manifold $\widetilde{M}$ is called a *quaternionic submanifold* (respectively, a *totally real* submanifold) if $dim D_x^\perp = 0$ (respectively $dim D_x = 0$). A quaternionic CR-submanifold is called *proper* if it is neither totally real nor quaternionic.

Let $N_{\alpha x} = J_\alpha(D_x^\perp), v_x^\perp = N_{1x} \oplus N_{2x} \oplus N_{3x}$ a $3q$-dimensional distribution $v^\perp: x \to v_x^\perp$ globally defined on M, where $q = dim D_x^\perp$ and v the orthogonal complementary distribution of $v^\perp$.

M is called *mixed totally geodesic* if $h(X,Y) = 0, \forall X \in \Gamma(D), Y \in \Gamma(D^\perp)$.

Let $\pi = Sp\{X,Y\}$ be a tangent plane to $\widetilde{M}$ at a point $x \in \widetilde{M}$. The sectional curvature of π is

$$K(\pi) = \frac{\widetilde{R}(X,Y,X,Y)}{\widetilde{g}(X,X)\widetilde{g}(Y,Y) - \widetilde{g}^2(X,Y)}.$$

By the Gauss equation and the expression of the Riemannian curvature tensor $\widetilde{R}$, we have

$$K(X \wedge Y) = \frac{c}{4}\left[1 + 3\sum_{\alpha=1}^{3} \widetilde{g}^2(J_\alpha X, Y)\right] + \widetilde{g}(h(X,X), h(Y,Y)) - \widetilde{g}(h(X,Y), h(X,Y)).$$

3. An inequality for the CR δ-invariant in the case of a quaternionic CR-submanifold of a quaternionic space form

In this section we recall the definition of the invariant δ(D) and a result from the paper of Al-Solamy, Chen and Deshmukh [1] and we present the corresponding result obtained in the case of a quaternionic CR-submanifold (with minimal codimension) of a quaternionic space form.

If M is a quaternionic CR-submanifold of minimal codimension, i.e., $dim v_x = 0$, for all $x \in M$, we have the following orthonormal basis:

$$\{e_1, e_2, ..., e_n\} \subset D_x,$$

$$\{e_{n+1}, e_{n+2}, ..., e_{n+q}\} \subset D_x^{\perp};$$

so

$$\{e_1, e_2, ..., e_n; e_{n+1}, e_{n+2}, ..., e_{n+q}\} \subset T_x M$$

and

$$\{J_1 e_{n+1}, ..., J_1 e_{n+q}; J_2 e_{n+1}, ..., J_2 e_{n+q}; J_3 e_{n+1}, ..., J_3 e_{n+q}\} \subset T_x^{\perp} M.$$

We define the following *CR δ-invariant* $\delta(D)$ by

$$\delta(D)(x) = \tau(x) - \tau(D_x), x \in \tilde{M},$$

where τ and $\tau(D)$ denotes the scalar curvature of M, respectively the scalar curvature of the quaternionic distribution $D \subset TM$.

We define the two partial mean curvature vectors $\overrightarrow{H_D}$ and $\overrightarrow{H_{D^{\perp}}}$ of M by

$$\overrightarrow{H_D} = \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i), \ \overrightarrow{H_{D^{\perp}}} = \frac{1}{q}\sum_{r=n+1}^{n+q} h(e_r, e_r).$$

In [1], the authors prove an inequality for $\delta(D)$ in case of an anti-holomorphic submanifold of a complex space form:

Theorem 1. [1] *Let N be an anti-holomorphic submanifold of a complex space form $\tilde{M}^{h+p}(c)$ with $h = dim_C D \geq 1$ and $p = dim D^{\perp} \geq 2$. Then we have*

$$\delta(D) \leq \frac{p}{2}(4h + p - 1)\frac{c}{4} + \frac{(2h+p)^2}{2} \cdot \frac{p-1}{p+2}\|H\|^2. \quad (1)$$

The equality case holds identically if and only if the following three conditions are satisfied:

(a) N is D-minimal, i.e., $\overrightarrow{H_D} = 0$,

(b) N is mixed totally geodesic, and

(c) there exists an orthonormal frame $\{e_{2h+1}, ..., e_n\}$ of $D^{\perp}$ such that the second fundamental form σ of N satisfies

(i) $\sigma_{rr}^r = 3\sigma_{ss}^r$, for $2h+1 \leq r \neq s \leq 2h+p$,

(ii) $\sigma_{st}^r = 0$, for distinct $r, s, t \in \{2h+1, ..., 2h+p\}$.

We prove the following inequality involving $\delta(D)$ in case of a quaternionic CR-submanifold of a quaternionic space form:

Theorem 2. [5] *If M is a quaternionic CR-submanifold of a quaternionic space form $\tilde{M}(c)$, of minimal codimension, i.e., $dim v_x = 0$, for $x \in M$, $dim D_x = n$, $dim D_x^{\perp} = q$ and $dim v_x^{\perp} = 3q = dim T_x^{\perp} M$, then*

$$\delta(D) \leq \frac{q}{2}(2n + q - 1)\frac{c}{4} + \frac{(n+q)^2}{2} \cdot \frac{q+2}{q+5}\|H\|^2. \quad (2)$$

The equality sign holds at a point $x \in M$ if and only if the following conditions are satisfied:

(a) M is mixed totally geodesic,

(b) there is an orthonormal basis $\{e_1, e_2, ..., e_{n+q}\}$ *at* x *such that with respect to this basis the second fundamental form h takes the following form:*

$$\sum_{i=1}^{n} \widetilde{g}(h(e_i,e_i),J_\alpha e_r) = \widetilde{g}(h(e_r,e_r),J_\alpha e_r) = 3\widetilde{g}(h(e_s,e_s),J_\alpha e_r), n+1 \le r \ne s \le n+q,$$

$$\widetilde{g}(h(e_r,e_s),J_\alpha e_t) = 0, r,s,t = \overline{n+1,n+q}, \alpha = \overline{1,3}, r \ne s \ne t \ne r.$$

Sketch of the proof (the complete proof is given in [5]):

The CR-δ invariant $\delta(D)$ is given by

$$\delta(D)(x) = \sum_{r<s=n+1}^{n+q} K(e_r \wedge e_s) + \sum_{i=1}^{n} \sum_{r=n+1}^{n+q} K(e_i \wedge e_r).$$

For $A,B = \overline{1,n+q}, t = \overline{n+1,n+q}$, we consider the following notations:

$$h_{AB}^t = \widetilde{g}(h(e_A,e_B),J_1 e_t);$$
$$\widetilde{h}_{AB}^t = \widetilde{g}(h(e_A,e_B),J_2 e_t);$$
$$\widetilde{\widetilde{h}}_{AB}^t = \widetilde{g}(h(e_A,e_B),J_3 e_t).$$

By using the Gauss equation, we obtain for $\delta(D)$ the following inequality:

$$\delta(D)(x) \le \frac{q}{2}(2n+q-1)\frac{c}{4} + \sum_{r<s=n+1}^{n+q} \sum_{t=n+1}^{n+q} (h_{rr}^t h_{ss}^t + \widetilde{h}_{rr}^t \widetilde{h}_{ss}^t + \widetilde{\widetilde{h}}_{rr}^t \widetilde{\widetilde{h}}_{ss}^t) +$$

$$+ \sum_{i=1}^{n} \sum_{r,t=n+1}^{n+q} (h_{ii}^t h_{rr}^t + \widetilde{h}_{ii}^t \widetilde{h}_{rr}^t + \widetilde{\widetilde{h}}_{ii}^t \widetilde{\widetilde{h}}_{rr}^t) - \sum_{r<s=n+1}^{n+q} \sum_{t=n+1}^{n+q} [(h_{rs}^t)^2 + (\widetilde{h}_{rs}^t)^2 + (\widetilde{\widetilde{h}}_{rs}^t)^2).$$

Next, we will consider the sum

$$S = \sum_{r<s=n+1}^{n+q} \sum_{t=n+1}^{n+q} h_{rr}^t h_{ss}^t + \sum_{i=1}^{n} \sum_{r,t=n+1}^{n+q} h_{ii}^t h_{rr}^t - \sum_{r<s=n+1}^{n+q} \sum_{t=n+1}^{n+q} (h_{rs}^t)^2$$

and the analogous formulae for $\widetilde{S}$ and $\widetilde{\widetilde{S}}$.

For each of these sums we consider some quadratic functions

$$f_v(h_{11}^v, h_{22}^v, ..., h_{n+q;n+q}^v) = \sum_{r<s=n+1}^{n+q} h_{rr}^v h_{ss}^v + \sum_{i=1}^{n} \sum_{r=n+1}^{n+q} h_{ii}^v h_{rr}^v - \sum_{n+1 \le r \le n+q}^{r \ne v} (h_{rr}^v)^2, v = \overline{n+1,n+q}.$$

For each of these functions we must find an upper bound, searching for a critical point, under some conditions and using the method of constrained maximum.

We get the relation:

$$f_v \le \frac{q+2}{2(q+5)} \cdot (n+q)^2 \|H^v\|^2, v = \overline{n+1,n+q},$$

where $H^v = \frac{1}{n+q} \sum_{A=1}^{n+q} h_{AA}^v$.

Summing the relations of this type, we get the inequality (2).

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MATHEMATICAL MODELING FOR DETERMINING THE PREDICTIVE PRODUCTION LEVEL RELATED TO A SAFETY STOCK LEVEL

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Abstract: The study presents a model to determine stock levels in a window sills manufacturing company. The conventional inventory methods are not sufficient to determine the safety stock quantity. We use mathematical models such as the genetic algorithm in order to solve the production capacity constraints. The out of stock situations determine significant gaps in the supply chain. In addition, the high quantity of raw materials is another problem that companies want to avoid. In this paper we explore the idea of overcoming issues such as out of stock or excess supply using mathematical models.
Mathematical Subject Classification (2010): 91B38, 91B32, 91B68
Key words: safety stock level, production capacity, mathematical modeling, genetic algorithm.

1. Introduction

The purpose of the study is to propose a model for determining the safety stock in a window sills' production unit. This model is necessary because one of the major tasks for a company is to match the production capacity to the forecasted demand. There are many factors which contribute to the organization of the production process such as: sales levels, optimum resource usage production capacity and delivery time. Considering all the factors, each company tries to deal with the problem of matching the production capacity to the forecasted demand. By determining a safety stock, companies avoid the out of stock situation or the excess of inventory one. The most common situation companies have to deal with is the out of stock situation. The out of stock case creates two possible issues: the cancelation of the order or the increased waiting time for the customer until the order is ready to be delivered. The last situation mentioned is also known as backorder. In real life cases, clients are usually prone to cancel the order if backorder happens. In order to avoid unwanted cancelations, researchers started developing models which not only optimize the production process, but also reduces or avoids the incidence of backorder.

Helopal Romania is a production company in the building industry which produces window sills. Because window sill types cause a large product variety, there is an increased backorder probability. In a company that has product diversity, it is especially necessary to develop a production plan which would minimize the backorder. There are several models that are taken into consideration: mixed linear programming, genetic algorithm and greedy algorithm. The difficulties in the optimizing process are dependent on the great number of products and the nonlinear demand. Besides the mathematical model we also develop the analysis of the greedy algorithm and genetic algorithm in order to reach the goal of creating a safety stock.

2. Industrial context

Helopal Romania is a leading company in the window sills' production and has developed products from an entire range of materials such as marble, wood, aluminum, steel. The process consists in a few steps:

1. input of raw materials,
2. molding process,
3. optimizing the cutting process,
4. cutting process.

The company produces more than 150 window sills of different colors and dimensions from different types of materials. The production capacity is of 70 types monthly. The demand is nonlinear and is also depending on the season and the type of building. To avoid the backorder situation the company decides to maintain on stock not only a large quantity of the final product, but also raw materials. The benefits are canceled out by the high costs of this policy. A model which provides a better stock control is aimed using the heuristic algorithms proposed: the genetic algorithm and the greedy algorithm.

3. Mathematical programming model

In this section, we propose a mixt integer linear programming model (MILP) which deals with the safety stock problem in the window sills manufacturing industry. Due to the production of a wide variety of window sills, the company Helopal Romania encountered backorder and/or high inventory level situations. While backorder causes loss of prestige and hence, a possible profit loss, high inventory requires the increase in the size of the storage area which may cause higher holding costs. The company searches out an optimum solution to avoid this potential situation. In this problem, all constraints are determined with the production planning department of the company.

The company Helopal Romania has to plan de production of 150 type window sills denoted by n for a 12- month period denoted by m. Due to the machine and mold capacities and the company policy, at most 70 types of product can be produced each month. Let w denote the window mold code and T_i denote the daily mold capacity. We also let L_m and N_i to denote the manufacturing machine type and associated capacity, respectively. These variables are summarized in Table 1. Moreover, the available number of mold codes is limited. We use M_w to present the number of available molds which are presented in Table 2. For each window sill, holding cost CL_i and backorder cost CS_i are defined and presented in Table 3. The available number of window sill manufacturing machines MA_s is defined as in Table 4.

On one hand, if Helopal Romania overproduces window sills to prevent backorder, higher inventory may occur. On the other hand, producing window sills to minimize holding costs may cause backorder. Our goal is to obtain a production policy that provides the optimal trade-off between these costs.

Before proceeding to model formulation, we state our assumptions as follows: the lead time and the deteriorated products are ignored, the transaction cost is not related to the quantity

and product variation, the production demand is known through the planning period. Next, we present in detail the proposed mathematical formulation. Due to the company policy, we ignore the production cost in this problem.

Parameters:

$i = 1, \dots, n$: number of window sills

$j = 1, \dots, m$: month

$w = 1, \dots, s$: number of cutting machines

$CL_i, \forall i = 1, \dots, n$: holding cost for one unit window sill code i

$CS_i, \forall i = 1, \dots, n$: backorder cost for one unit window sill code i

$D_j, \forall j = 1, \dots, m$: working day in month j

$S_{ij}, \forall i = 1, \dots, n, \forall j = 1, \dots, m$: demand of window sill code i at month j

$T_i, \forall i = 1, \dots, n$: daily mold capacity of window sill code i

$N_i, \forall i = 1, \dots, n$: daily manufacturing (cutting machines) capacity of window sill code i

M_w: number of cutting machines w

A: number of machines type A

B: number of machines type B

$Maxstock$: maximum inventory level

$Maxcode$; number of window sill code at the same month

$$a_{ik} = \begin{cases} & if\ window\ sill\ i\ uses\ machine\ type\ k, \quad k \in \{A, B\} \\ 0, & otherwise \end{cases}$$

Variables:

$P_{ij}, \forall i = 1, \dots, n, \forall j = 1, \dots, m$: the number of production for window sill code i at month j

$E_{ij}, \forall i = 1, \dots, n, \forall j = 1, \dots, m$: amount of inventory of window sill code i at the beginning of month j

$B_{ij}, \forall i = 1, \dots, n, \forall j = 1, \dots, m$: backorder of window sill code i at month ji

$$y_{ij} = \begin{cases} & if\ window\ sill\ i\ is\ produced\ at\ month\ j \\ 0, & otherwise \end{cases}$$

Table 1

Window sill code	Mold code (w)	Daily mold capacity (Ti)	Cutting machine (Lm)	Window sills cutting capacity (Ni)
alu 15	A150	50	B	280
alu 20	A200	50	B	280
alu 25	A250	50	B	356
alu 30	A300	35	A	356

Table 2 – Resource capacity table

Mold code(w)	Mold number (Mw)
A150	2
A200	1
A250	1
A300	1

Table 3 – Costs

Window sill code	Holding cost	Backorder cost of one unit
A15	CL1	CS1
A20	CL2	CS2
A25	CL3	CS3
A30	CL4	CS4

Table 4 – Window sill production machine

Window sill manufacturing machine	Machine number (Mas)
A	2
B	1

Mathematical model:

$$minimize \sum_{j=1}^{m}\sum_{i=1}^{n}(CL_i\, E_{ij} + CS_i B_{ij})$$

subject to:

$$E_{ij} = E_{i(j-1)} + P_{i(j-1)} - S_{i(j-1)} + B_{i(j-1)}\ \forall i, j$$

$\sum_{i=1}^{n}[(P_{ij} \,/\, D_j)/T_i\,] \le M_w\ \forall j, w,$

$$\sum_{i=1}^{n} a_{iA}[(P_{ij} \,/\, D_j)/N_i\,] \le A\ \forall j,$$

$$\sum_{i=1}^{n} a_{iB}\,[(P_{ij} \,/\, D_j)/N_i\,] \le B\ \forall j,$$

$$\sum_{i=1}^{n} E_{ij} \le Maxstock\ \forall j,$$

$$\sum_{i=1}^{n} y_{ij} \le Maxcode\ \forall j,$$

$$P_{ij}\ \le M y_{ij}\ \forall i,j$$

$$E_{ij}\,, P_{ij}\,, B_{ij}\ \in \mathbb{Z}_+\,, y_{ij} \in \{0,1\}.$$

Available mold numbers were defined for 150 different mold codes that were essential for window sill production. Constraint (1) ensures that the daily total mold usage $2 \times n \times m + m \times s + 4m$ cannot exceed the number of available mold. Window sills are produced by two machines denoted as A and B codes respectively. Constraint (2) and (3) guarantee that the daily total manufacturing machine usage in respect to the manufacturing machine code cannot exceed the number of available machines. These constraints derive from the limited mold and machine capacities of the window sill production process. Constraint (4) ensures that the total amount of inventory cannot exceed the capacity of storage area. Constraint (5) presents that the total types of window sills produced at each month cannot exceed the limit defined by the company. Finally, constraint (6) guarantees that window sill *i* can be produced if and only if the window sill type is selected for the production for month *j*. This model involves $4 \times n \times m$ variables and constraints. The model aims to minimize the total stock quantity and the backorder incidence.

Because of production constraints we have to decide the 70 window sills' types in order to avoid backorders and excessive inventory. The mathematical model presented is compared with heuristic models like the greedy and the genetic algorithms.

The criteria used in the mathematical model were the backorder number and inventory levels. The most accurate results are generated by the mathematical model even though the computation speed of the greedy algorithm was superior. The most inadequate model is provided by the genetic algorithm with a great number of backorders and high level of inventory.

4. Conclusion

The topic presented is current and it influences the economic yield of companies. The most affected parameters are the cash flow and the client satisfaction.

The mathematical model, the greedy algorithm and the genetic algorithm provide different results. The greedy algorithm is better when it comes to the computation time and the inventory level, but the mathematical model provides the lowest backorder level, which is desired. In this way the mathematical model generates the highest level of client satisfaction. The inventory level computed by the mathematical model is not the lowest possible, but it is not so far from the one computed by the greedy algorithm. The performance of the genetic

algorithm is poor compared to the ones of the greedy algorithm and the mathematical model. The contribution of this paper is to provide practical solutions for companies which try to improve their performance.

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TEACHING AND LEARNING MATHEMATICS EFFECTIVELY IN A TECHNOLOGY ERA

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Abstract: Mathematics is one of the key subjects for any career in engineering, science or business. Yet, over the years, the process of teaching mathematics has faced more and more challenges and poorer performances. One reason behind these results is the significant number of changes taking place in the last two decades. The rapid development of technology is one of the changes that have profoundly influenced the teaching of mathematics at the university level. The considerable differences among the mathematical knowledge and skills of students, as well as the differences in their ability to learn mathematics, are also important factors. The approach of FutureMath project is to take into account all of these factors in order to create a framework suitable for individual and self-paced learning. Some of the materials developed within this project are already used in the partner universities and their influence in the process of teaching and learning mathematics is a positive one.
Mathematics Subject Classification (2010): 97D40, 97U50
Key words: Mathematics education, technology, online exercises, short videos

1. Introduction

Mathematics is one of the key subjects for any career in engineering, science or business and a good mathematical knowledge is required for understanding and mastering different engineering disciplines. Unfortunately, in the recent decades, the mathematical competence of students have weakened and all those involved in teaching mathematics are facing more and more challenges. There are several factors that had a substantial impact on the process of teaching mathematics.

The rapid development of technology and computer programs is one of them. There is a wide variety of mathematical computer software (Maple, Mathematica, Mathcad, Matlab etc.) that can offer quick and easy solutions to different kind of mathematical or engineering problems. Spreadsheet programs, such as Excel, also support complex calculations through their complex library of functions. On one hand, these tools are important in solving different kind of problems, but on the other hand, they may be one cause of low students' motivation on acquiring substantial mathematical knowledge.

Another factor that is causing difficulties in the process of teaching mathematics at university level is that the study groups are large and becoming more heterogeneous in recent years. As the students' background studies and level of competence in mathematics varies greatly, the organization of the teaching process becomes difficult. In these circumstances, taking into account the individual needs of students or organizing dynamic and insightful activities becomes almost impossible during conventional classes.

A meaningful and efficient process of teaching mathematics has to take into account all the factors who have a substantial impact on the process. The new ways of teaching must rely on

technology to create flexible and friendly learning environments in which students can easily acquire and understand new information. Also, the existing technology must be used to its fullest potential to solve a wide variety of problems, make connection between math problems and real life situations and enhance the quality of learning. The teachers must encourage deeper knowledge through geometric exploration programs where learners can visualize functions and data, and experiment with mathematical formulas.

2. FutureMath project. Goal and innovation

FutureMath project started from the need of high quality mathematics' education in an era of technology and computers. The goal of the project is to make mathematics learning more motivational, interesting, effective and accessible. The project aims to develop technology-based innovative pedagogical methods, techniques, materials and resources, for teaching and learning mathematics, and also for assessing mathematics' learning [1]. The learning resources developed within the project had to pay attention to different learner types, effective feedback and assessment. They offer alternative modern methods for mathematics learning, individual learning solutions and flexibility.

All the learning resources developed within the project will be made available for free under the idea of Open Source or Open Educational Resource (OER) at the end of the project [1].

The FutureMath project develops pedagogical methods and resources to teach and learn mathematics more effectively by providing personalized learning possibilities with the help of ubiquitous technology. The underlying notion is to support digitalization of European engineering mathematics education in a large scale. By these means, it is supposed to improve the efficiency, accessibility and quality of mathematics teaching and learning on European level which, in fact, is one of the four common objectives of EU's Strategic Framework of Education and Training 2020. Additionally, as an impact of the project, improving of transversal and basic skills (ET2020), such as digital skills and mathematical skills, will be a central focus. With these actions, it is expected not only to develop innovative learning approaches but also to enrich the teaching, support personalized learning and increase the flexibility and attractiveness.

In addition to the mathematics learning platform (MLP), the proposed project aims to develop innovative pedagogical methods, techniques, materials and resources not only to teach and learn mathematics but also to assess mathematics' learning. The key approaches while planning the resources are i.e. collective thinking, collaboration and shared problem solving skills - the skills that are necessary for success in working life. Furthermore project resources will respect individual learning solutions. Therefore, different learning types will be taken into account in the project’s material production. In this way, it is also possible to decrease the inequality among different kinds of learners.

Overall, the one main objective of this project is to increase the global large-scale awareness about the possibilities ubiquitous technology offers for mathematics learning throughout MLP. Our aim is to make mathematics learning more motivational, interesting and increase accessibility and the alternative modern methods for mathematics learning [1].

The project is funded by the EU under the ERASMUS+ Program, with reference 2015-1-FI01-KA203-009044.

3. Online resources

The **FutureMath** project develops pedagogical methods and resources to teach and learn mathematics more effectively by providing personalized learning possibilities with the help of technology.

The materials developed within the project aim to:

Offer a generous database of resources for learning and teaching mathematics. The materials are short videos, interactive materials, stack exercises and quizzes. They offer theoretical explanations, worked examples, visualization of abstract mathematical problems or concepts, and the possibility to test your own knowledge.
Encourage self-paced learning. Besides being flexible and convenient, online materials are available anytime, so students can access their course information any time of the day and as many times as they feel the need to do so. Another advantage is that students who work at a fast pace have the opportunity to gain competences quickly, while the others, who learn at a slower pace, have the opportunity to review materials.
Develop mathematical visualization skills. The use of computers enables learners to manipulate diagrams dynamically. This not only supports learning by visualizing actual diagrams, graphs or methods of solving problems, it also encourages learners to make connections between abstract mathematical notions and real life problems.
Encourage the search of connections. The computer enables formulae, tables of numbers and graphs to be linked. Making changes in one representation and seeing changes in the others helps students to better understand the connections between them. Working through a medium which enables learners to switch effortlessly between these representations enhances their conceptual development.
Observing patterns. When exploring certain mathematical problems, the speed of computers and computer programs enables students to produce many similar examples. This supports their observation of patterns. As a consequence, it encourages them to make their own conjectures and to test them out (eventually modifying their ideas if it is the case) Also this supports and justify the generalizations in different mathematical problems.
Learning from feedback. The computer quizzes provide a fast and reliable feedback which is impartial. This can encourage students to test their knowledge and improve it, if necessary.
Most of the materials developed in our university in the first stage of the project consists in educational screencast videos. These are actually recordings of a screen while the teacher is narrating as it can be seen from the figures 1 and 2.

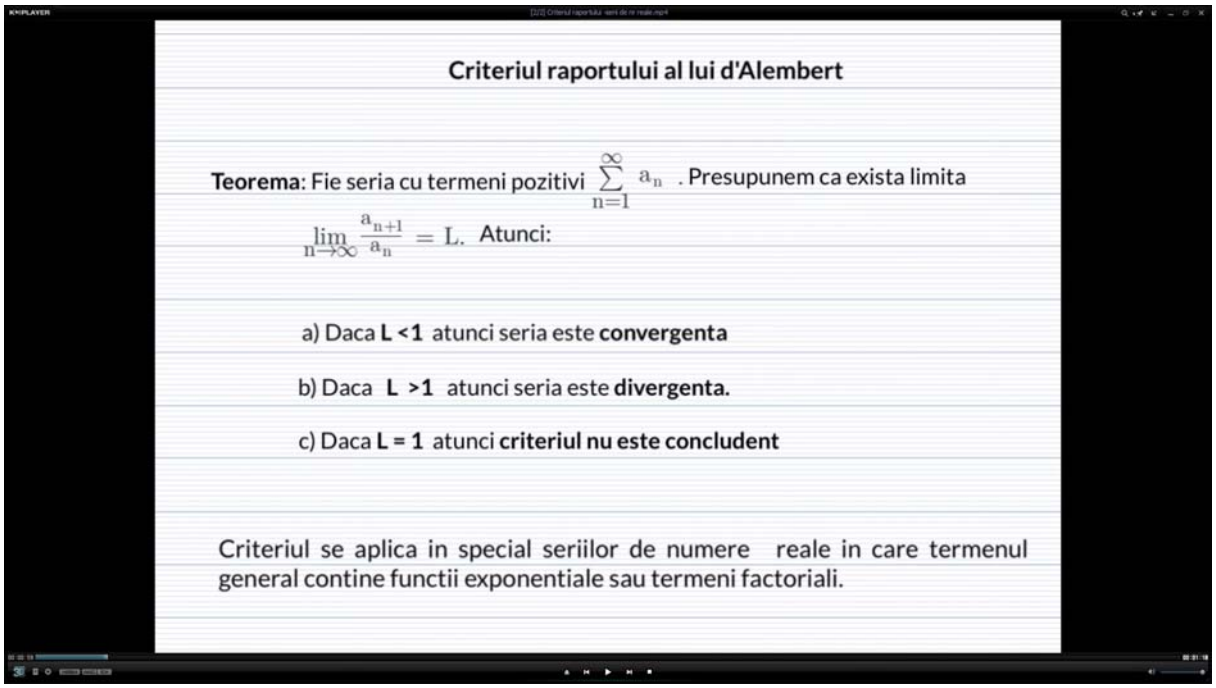

Figure 1.

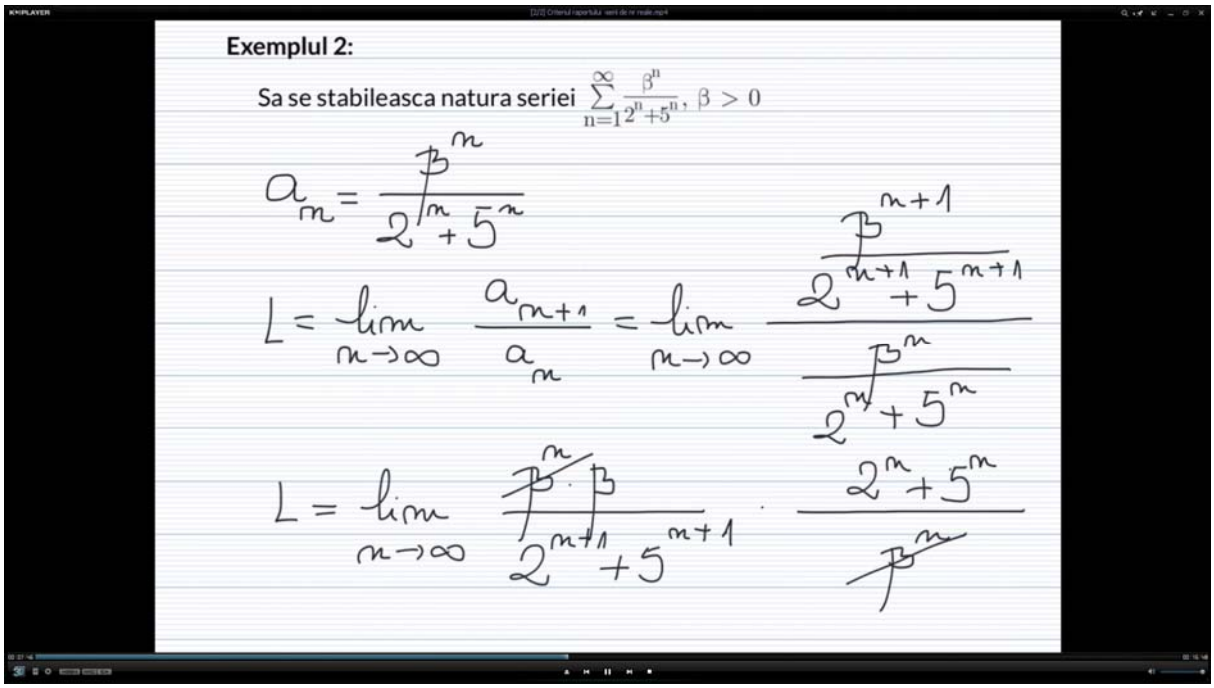

Figure 2.

They comprise a theoretical subject and some explained examples with visualization when necessary. All these materials are now available and complement the classical methods of teaching and learning mathematics. In order to support the process of efficiently learning mathematics, the materials have to be correlated with the curriculum of all mathematical courses and some additional quizzes and stack exercises have to be added. Also, their influence and efficiency in the teaching process will have to be assessed in the future, but students have already shown interest in working with this type of resources.

4. Conclusions

Current research reveals the positive effects of technology assisted teaching and learning, when this is done at its fullest potential. Therefore it is important for teachers to find a way to integrate technology into their classrooms [3]. Still, more important than if technology is used, is the way that computers are used in teaching and learning mathematics [2]. Therefore, teachers' ability to select appropriate software and materials, or to create their own materials, plays an essential role in the process of effectively and successfully integrate technologies into classroom teaching.

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EFFICIENCY AND DUALITY FOR MULTITIME VECTOR FRACTIONAL CONTROL PROBLEMS VIA (ρ, b)-QUASIINVEXITY

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Abstract: In this paper we establish necessary efficiency conditions and develop a duality of Mond-Weir type for a multitime vector fractional control problem using the (ρ, b)-quasiinvexity notion. Particularly, there are obtained similar results for multitime vector control problems.
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Key words : optimal control, efficiency, multitime vector programming, (ρ, b)-quasiinvexity.

1. Introduction and preliminaries

In 1961 Berkowicz [2] approached the optimal control problem as a particular case of variational problem, establishing necessary conditions of optimum. After the apparition of the invexity notion in 1989 [Hanson], the optimal control problem knows an intensive development(vectorial problems with mixed constraints , sufficient conditions of efficiency, duality with generalized invexities). In this direction we quote Preda [9,1991], Bhatia and Kumar [3,1995], Bector [1,1996], Gramatovici [2005]. In 2009 Mititelu [6] established optimality conconditions and a Wolfe duality type for invex multitime control problems (using multiple integrals, according to Udrişte [10,2006]) with mixed constraints. In this paper we present the efficiency conditions and develop a duality of Mond-Weir type for a multitime vector fractional problem using the (ρ, b)-quasiinvexity notion.

Consider a measurable set Ω in R^m and the continuously differentiable functions

$$X_\alpha^i : \Omega \times R^n \times R^{nm} \to R,\ Y_\beta : \Omega \times R^n \times R^{nm} \to R,\ h : \Omega \times R^n \times R^{nm} \to R,$$

$$f = (f_r) : \Omega \times R^n \times R^{nm} \to R^p,\ g = (g_r) : \Omega \times R^n \times R^{nm} \to R^p,$$

where $n, m, p, q \in N^*$, $r = \overline{1,n}$, $\alpha = \overline{1,m}$ and $\beta = \overline{1,q}$.

Denote be $t = (t^1, ..., t^m) = (t^\alpha)$ the current element of Ω, be $x = (x^1, ..., x^n) = (x^i)$ the current element of R^n and be $u = (u^1, ..., u^k) = (u^j)$ the current element of R^k. The arguments of X_α^i, Y_β, f, g, h are $(t, x, u) = (t, x(t), u(t))$, where

$$x : \Omega \to R^n : \quad x = x(t) = (x^1(t), ..., x^n(t)) = (x^i(t)) \in R^n,\ t \in \Omega \text{ and}$$

$$u : \Omega \to R^k : \quad u = u(t) = ((u^1(t), ..., u^k(t)) = (u^j(t)) \in R^k,\ t \in \Omega.$$

We suppose that the functions $x(t)$ and $u(t)$ there are piecewise smooth functions on Ω and we define the set $X = \{x : \Omega \to R^n \mid x(t) \in R^n\}$, endowed with the distance $d(x, x^0) = d(x(.), x^0(.))$

$= \sup_{t\in\Omega}(x(t), x^0(t))$, where d_e is the Euclidean distance in R^n. Let and $U = \left\{u : \Omega \to R^k \,\middle|\, u(t) \in R^k\right\}$.

Throughout in the paper, for two vectors $v = (v_1,..., v_n)$ and $w = (w_1,..., w_n)$ the relations of the form $v = w, v < w, v \leqq w, v \leq w$ are defined as follows

$$v = w \Leftrightarrow v_i = w_i,\ i = \overline{1,n}\ ;\ v < w \Leftrightarrow v_i < w_i,\ i = \overline{1,n}\ ;$$

$$v \leqq w \Leftrightarrow v_i < w_i \text{ or } v_i = w_i,\ i = \overline{1,n}\ ;\ v \leq w \Leftrightarrow v \leqq w \text{ and } v \neq w.$$

Consider the relations:

$$\frac{\partial x^i}{\partial t^\alpha} = X^i_\alpha(t, x(t), u(t)),\ i = \overline{1,n},\ \alpha = \overline{1,m},\ t \in \Omega, \tag{1.1}$$

$$Y_\beta(t, x(t), u(t)) \leq 0,\ \beta = \overline{1,m},\ t \in \Omega, \tag{1.2}$$

$$x(t) \in X, u(t) \in U \subset R^k, x(t)\big|_{\partial\Omega} = \varphi(t) \text{ (given)}, \forall t \in \Omega, \tag{1.3}$$

and we define the set

$$D = \left\{(x,u) \,\middle|\, (x,u) \in X \times U \text{ satisfying } (1.1), (1.2), (1.3)\right\}.$$

Consider now the following multitime vector fractional control problem:

$$\text{(VFCP)}\quad \begin{cases} \text{Maximize Pareto } J(x,u) = \left(\dfrac{\int_\Omega f_1(t,x(t),u(t))dv}{\int_\Omega g_1(t,x(t),u(t))dv}, ..., \dfrac{\int_\Omega f_p(t,x(t),u(t))dv}{\int_\Omega g_p(t,x(t),u(t))dv}\right) \\ \text{subject to} \\ \quad (x,u) \in D, \\ \quad\quad \text{where} \quad G_r(x,u) = \int_\Omega g_r(t,x(t),u(t)) \neq 0,\ r = \overline{1,p} \end{cases}$$

and $dv = dt_1 dt_2 ... dt_n$; $x = x(t)$ is the state function, and $u = u(t)$ is the control function in this problem.

The set D is the domain of the problem (VFCP).

Particularly, we obtain the following multitime vector nonfractional control problem

$$\text{(VCP)}\begin{cases} \text{Minimize Pareto } F(x,u) = \int_\Omega f(t,x(t),u(t))dv \\ \text{subject to} \\ \quad (x,u) \in D\,. \end{cases}$$

The domain of (VCP) is also D, and $F(x,u) = (F_1(x,u),..., F_p(x,u))$, where

$$F_r(x,u) = \int_\Omega f_r(t,x(t),u(t))\,dv\,,\ r = \overline{1,p}\,.$$

First, în this paper we establish necessary efficiency conditions to problem (VCP). Then, we use these resultats to obtain similarly results to the multitime vector fractional control problem (VFCP).

2. Necessary efficiency conditions for (VCP) and (VFCP)

In this section we establish necessary conditions of efficiency for solutions in the control problems (VCP) and (VFCP).

I. Efficiency to (VCP).

Definition 2.1. A feasible solution $(x^0, u^0) \in D$ of (VCP) is called *efficient solution* in this problem if there exist no other feasible solution $(x,u) \in D$ such that $F(x,u) \leq F(x^0, u^0)$.

Theorem 2.1 (Necessary efficiency conditions for (VCP))[9]. *Let* $(x^0,u^0)\in D$ *be an efficient solution to problem* (VCP). *Then there exist a vector* $\theta=(\theta^r)\in R^p$ *and the piecewise smooth functions* $\lambda(t)=(\lambda_i^\alpha(t))\in R^{nm}$ *and* $\mu(t)=(\mu^\beta(t))\in R^q$ *fulfilling the next conditions:*

$$(\text{VFJ})\begin{cases}\theta^r\dfrac{\partial f_r}{\partial x^i}+\lambda_i^\alpha(t)\dfrac{\partial X_\alpha^i}{\partial x^i}+\mu^\beta(t)\dfrac{\partial Y_\beta}{\partial x^i}+\dfrac{\partial\lambda_i^\alpha}{\partial t^\alpha}=0,\ i=\overline{1,n}\\ \theta^r\dfrac{\partial f_r}{\partial u^j}+\lambda_i^\alpha(t)\dfrac{\partial X_\alpha^i}{\partial u^j}+\mu^\beta(t)\dfrac{\partial Y_\beta}{\partial u^j}=0,\ j=\overline{1,k}\\ \mu^\beta(t)Y_\beta(t,x^0,u^0)=0\ \ (\textit{no summation}),\ \beta=\overline{1,q}\\ \theta\geqq 0,\ \lambda_i^\alpha(t)\in R,\ \mu^\beta(t)\geq 0,\ \forall t\in\Omega,\end{cases}$$

where the derivatives are calculated at (t,x^0,u^0): $\dfrac{\partial f_r}{\partial x^i}=\dfrac{\partial f_r}{\partial x^i}(t,x^0,u^0)$ *etc.*

Proof. See the proof of Theorem 3.1 from [7].

Definition 2.2. A feasible solution $(x^0,u^0)\in D$ is called a *normal efficient solution* in (VCP) if the conditions (VFJ) hold with $\theta\geq 0$ and $e'\theta=1,\ e'=(1,...,1)\in R^p$.

II. Efficiency to (VFCP).

Now we discuss the efficiency of the vector fractional problem (VFCP). This problem has the same domain *D*.

Definition 2.3. A feasible solution $(x^0,u^0)\in D$ of (VFCP) is called *efficient solution* of this problem if there is no other $(x,u)\in D$ such that $J(x,u)\leq J(x^0,u^0)$.

Theorem 2.2 (Necessary efficiency conditions for (VFCP)). *Let* $(x^0,u^0)\in D$ *be an efficient solution of the problem* (VFCP). *Then there exist the vector* $\theta=(\theta^r)\in R^p$ *and the piecewise smooth functions* $\lambda(t)=(\lambda_i^\alpha(t))\in R^{nm}$ *and* $\mu(t)=(\mu^\beta(t))\in R^q$ *satisfying the following conditions:*

$$(\text{VFFJ})\begin{cases}\theta^r\left[G_r(x^0,u^0)\dfrac{\partial f_r}{\partial x^i}-F_r(x^0,u^0)\dfrac{\partial g_r}{\partial x^i}\right]+\lambda_i^\alpha(t)\dfrac{\partial X_\alpha^i}{\partial x^i}+\mu^\beta(t)\dfrac{\partial Y_\beta}{\partial x^i}+\dfrac{\partial\lambda_i^\alpha}{\partial t^\alpha}=0,\ i=\overline{1,n}\\ \theta^r\left[G_r(x^0,u^0)\dfrac{\partial f_r}{\partial u^j}-F_r(x^0,u^0)\dfrac{\partial g_r}{\partial u^j}\right]+\lambda_i^\alpha(t)\dfrac{\partial X_\alpha^i}{\partial u^j}+\mu^\beta(t)\dfrac{\partial Y_\beta}{\partial u^j}=0,\ j=\overline{1,k}\\ \mu^\beta(t)Y_\beta(t,x^0,u^0)=0\ \ (\textit{no summation}),\ \beta=\overline{1,q}\\ \theta\geqq 0,\ \lambda_i^\alpha(t)\in R,\ \mu^\beta(t)\geq 0,\ \forall t\in\Omega,\end{cases}$$

where the derivatives are calculated at (t,x^0,u^0).

Proof. See the proofs of Theorems 4.3 and 4.4 from [7].

Definition 2.4. A feasible solution $(x^0,u^0)\in D$ is called a *normal efficient solution* in (VFCP) if the conditions (VFFJ) in Theorem 2.2 hold with $\theta\geq 0$ and $e'\theta=1,\ e'=(1,...,1)\in R^p$.

3. (ρ,b) – quasiinvexity functionals

The concept of invexity has been introduced by Hanson and Craven in 1981 [3]. Then there have been introduced the notions of ρ – invexity in 1991 by Preda[9], of ρ – strictly

quasiinvexity in 1988 by Bhatia and Kumar[3], of $(\rho,b)-$ invexity by Bector [1] in 1996 in the optimization theory of the variational problems and of the control problems.

According to Berkowics [2,1961], the control problems are variational problems which don't contains the derivative $\dot{x}(t)$. Then, let $\rho\in\mathbf{R}$, $x\in X, u\in U$ be and consider the functional

$$H: X\times U\to\mathbf{R}, \text{ defined by } H(x,u)=\int_\Omega h(t,x(t),u(t))dv.$$

Now we present the $(\rho,b)-$*quasiinvexity* notion for multitime functionals for to develop a duality of Mond-Weir type to the vector fractional control problem (VFCP).

Definition 3.2.[9] The functional H is said to be [*strictly*] (ρ,b)-*quasiinvex* at (x^0,u^0) if there exist vector functions $\eta(t)=\eta(t,x,x^0,\dot{x},\dot{x}^0,u,u^0)=(\eta^1(t),...,\eta^n(t))\in\mathbf{R}^n$ of C^1-class with $\eta(t)\big|_{\partial\Omega}=0$, $\xi(t)=\xi(t,x,x^0,\dot{x},\dot{x}^0,u,u^0)=(\xi^1(t),...,\xi^k(t))\in\mathbf{R}^k$ of C^0-class for any $(x,u)\in X\times Y$ $[(x,u)\neq(x^0,u^0)]$ such that $H(x,u)-H(x^0,u^0)\leqq 0\ \Rightarrow$

$$\Rightarrow b(x,u,x^0,u^0)\int_\Omega\left\{\eta(t)'h_x(t,x^0(t),u^0(t))+\xi(t)'h_u(t,x^0(t),\dot{u}(t))\right\}dv\ [<]\leqq$$

$$[<]\leqq\ -\rho b(x,u,x^0,u^0)d^2((x,u),(x^0,u^0)).$$

Example. The following functional

$$H(x,u)=\int_0^1 h(t,x(t),u(t))\,dt$$

is $(\rho,1)$-quasiinvex, for $\rho\leq 0$ and a distance function *d,* at (x^0,u^0) with respect to

$$\eta(t)=\xi(t)=\left(H(x,u)-H(x^0,u^0)\right)\left(\frac{\partial h}{\partial x^1}(t,x^0(t),u^0(t)),\ \frac{\partial h}{\partial x^2}(t,x^0(t),u^0(t))\right)\in R^2,\ t\in[0,1].$$

4. A duality of Mond-Weir type for (VFCP)

We consider the set $Q=\{1,...,q\}$ and let $\{Q_1,Q_2,...,Q_s\}$ $(s<q)$ be one his partition, that is $Q_\tau\subseteq Q$, $Q_\tau\cap Q_\upsilon=\Phi$ *if* $\tau\neq\upsilon$, $\bigcup_{\upsilon=1}^{s}Q_\upsilon=Q$ and we denote $N=\sum_{\upsilon=1}^{s}\upsilon$.

Consider functions $(y,v)\in X\times U$ and we associate to (VFCP) the next multitime vector fractional variational dual problem of Mond-Weir[8] type:

$$\text{(WFCD)}\ \begin{cases}\text{Maximize Pareto}\left(\dfrac{\int_\Omega f_1(t,y(t),v(t))dv}{\int_\Omega g_1(t,y(t),v(t))dv},...,\dfrac{\int_\Omega f_p(t,y(t),v(t))dv}{\int_\Omega g_p(t,y(t),v(t))dv}\right)\\ \text{subject to}\\ \theta^r\left[G_r(y,v)\dfrac{\partial f_r}{\partial y^i}-F_r(y,v)\dfrac{\partial g_r}{\partial y^i}\right]+\lambda_i^\alpha(t)\dfrac{\partial X_\alpha^i}{\partial y^i}+\mu^\beta(t)\dfrac{\partial Y_\beta}{\partial y^i}+\dfrac{\partial\lambda_i^\alpha}{\partial t^\alpha}=0,\ i=\overline{1,n}\\ \theta^r\left[G_r(y,v)\dfrac{\partial f_r}{\partial v^j}-F_r(y,v)\dfrac{\partial g_r}{\partial v^j}\right]+\lambda_i^\alpha(t)\dfrac{\partial X_\alpha^i}{\partial v^j}+\mu^\beta(t)\dfrac{\partial Y_\beta}{\partial v^j}=0,\ j=\overline{1,m}\\ \lambda_i^\alpha(t)\left[X_\alpha^i(t,y(t),v(t))-\dfrac{\partial y^i}{\partial t^\alpha}\right]\dfrac{\upsilon}{N}+\mu^{Q_\upsilon}(t)'Y_{Q_\upsilon}(t,y(t),v(t))\geqq 0,\ \upsilon=\overline{1,s},\\ \theta=(\theta^r)\geqq 0,\quad e'\theta=1,\ \lambda_i^\alpha\in R,\mu(t)\geqq 0,\ y(t)\big|_{\partial\Omega}=\varphi(t),\ t\in\Omega,\end{cases}$$

(summations by all *r, i, j,* α,β), where the derivative there are calculated at $(t,y(t),v(t))$.

Remark. The dual (WFCD) is of Mond-Weir type because it have the same objective as the primal (VFCP) and because of his the inequality constraint**.**

The term $\mu^{Q_\upsilon}(t)'Y_{Q_\upsilon}(t,\, y(t),\, v(t))$ signifies $\sum_{\beta\in Q_\upsilon}\mu^{\beta}(t)'Y_{\beta}(t,y(t),v(t))$**.**

Denote by $\pi(x,u)$ the value of problem (VFCP) at $(x,u)\in\mathbf{D}$ and be $\delta(y,v,\theta,\lambda,\mu)$ the value of the dual (WFCD) at $(y,v,\theta,\lambda,\mu)\in\Delta$, where Δ is the domain of the dual (WFCD).

In what follows we develop a duality theory between (VFCP) and (WFCD).

THEOREM 4.1 (Weak duality). *Let* $(x,u)\in D$ *and* $(y,v,\theta,\lambda,\mu)\in\Delta$ *be feasible solutions of the problems* (VFCP) *and* (WFCD). *Assume that the next conditions are satisfied* :

a) *for each* $r=\overline{1,p}$ *the integral*

$$\int_\Omega [G_r(x,u)f_r(t,x(t),u(t))-F_r(x,u)g_r(t,x(t),u(t))]\,dv$$

is $(\rho_r,b)-$*quasiinvex at* (y,v) *with respect to* η *and* ξ *(as in Definition* 3.2*).*

b) *for each* $\upsilon=\overline{1,s}$ *the integral*

$$\int_\Omega\left\{\lambda^i_\alpha(t)\left[X^\alpha_i(t,x(t),u(t))-\frac{\partial x^i}{\partial t^\alpha}\right]\frac{\upsilon}{N}+\mu^{Q_\upsilon}(t)Y_{Q_\upsilon}(t,x(t),u(t))\right\}dv$$

(summation by i,α) *is* (ρ_υ,b)*-quasiinvex at* (y,v) *with respect to* η *and* ξ *(as in Definition* 3.2*).*

c) *one of the functions of* a) and b) *is strictly* (ρ,b)*-quasiinvex at* (y,v) *with respect to* η *and* ξ.

d) $\theta^r\rho_r+\sum_{\upsilon=1}^{s}\rho_\upsilon\geqq 0$**.** *Then* $\pi(x,u)\leq\delta(y,v,\theta,\lambda,\mu)$ *is false.*

Proof. By reduction ad absurdum, suppose that $\pi(x,u)\leq\delta(y,v,\theta,\lambda,\mu)$**,** or from components

$$\frac{\int_\Omega f_r(t,x(t),u(t))dv}{\int_\Omega g_r(t,x(t),u(t))dv}\leqq\frac{\int_\Omega f_r(t,y(t),v(t))dv}{\int_\Omega g_r(t,y(t),v(t))dv},\ r=\overline{1,p}\text{, etc.}$$

Writting these relations under the forms $\int_\Omega[G_r(y,v)f_r(t,x(t),u(t))-F_r(y,v)g_r(t,x(t),u(t))]dv\leqq 0$ and using a) , we obtain

$$(4.1)\quad b(x,u,y,v)\int_\Omega\left\{\eta^i(t)\left[G_r(y,v\frac{\partial f_r}{\partial y^i}-F_r(y,v)\frac{\partial g_r}{\partial y^i}\right]+\xi^j(t)\left[G_r(y,v\frac{\partial f_r}{\partial v^j}-F_r(y,v)\frac{\partial g_r}{\partial v^j}\right]\right\}dv$$
$$\leqq-\rho_r b(x,u,y,v)\,d^2((x,u),(y,v))\,.$$

For each υ, from the domain D and Δ we have the inequalities

$$\int_\Omega\left\{\lambda^\alpha_i(t)\left[X^i_\alpha(t,x(t),u(t))-\frac{\partial x^i}{\partial t^\alpha}\right]\frac{\upsilon}{N}+\mu^{Q_\upsilon}(t)Y_{Q_\upsilon}(t,x(t),u(t))\right\}dv\ \leqq$$
$$\leqq\int_\Omega\left\{\lambda^\alpha_i(t)\left[X^i_\alpha(t,y(t),v(t))-\frac{\partial x^i}{\partial t^\alpha}\right]\frac{\upsilon}{N}+\mu^{Q_\upsilon}(t)Y_{Q_\upsilon}(t,y(t),v(t))\right\}dv$$

that, according to b), there implies

$$(4.2)\quad b(x,u,y,v)\int_\Omega\left\{\eta^i(t)\left[\lambda^\alpha_i(t)\frac{\partial X^i_\alpha}{\partial y^i}\frac{\upsilon}{N}+\mu^{Q_\upsilon}(t)\frac{\partial Y_{Q_\upsilon}}{\partial y^i}\right]+\xi^j(t)\left[\lambda^\alpha_i(t)\frac{\partial X^i_\alpha}{\partial v^j}\frac{\upsilon}{N}+\mu^{Q_\upsilon}(t)\frac{\partial Y_{Q_\upsilon}}{\partial v^j}\right]\right\}dv\ \leqq$$
$$\leqq-\rho b(x,u,y,v)\,d^2((x,u),(y,v))\qquad(\textit{summation from } i,\alpha,\, j).$$

Multiplying (4.1) by θ^r and summing over *r,* and summing over υ in (4.2), then summing now the two obtained relations and taking into account the hypothesis c) , we have

$$(4.3)\qquad b(x,u,y,v)\int_\Omega \Big\{\eta^i(t)\Big\{\theta^r\left[G_r(y,v)\frac{\partial f_r}{\partial y^i}-F_r(y,v)\frac{\partial g_r}{\partial y^i}\right]+\lambda_i^\alpha(t)\frac{\partial X_\alpha^i}{\partial y^i}+\mu^\beta(t)\frac{\partial Y_\beta}{\partial y^i}\Big\}+$$

$$+\xi^j(t)\Big\{\theta^r\left[G_r(y,v)\frac{\partial f_r}{\partial v^j}-F_r(y,v)\frac{\partial g_r}{\partial v^j}\right]+\lambda_i^\alpha(t)\frac{\partial X_\alpha^i}{\partial v^j}+\mu^\beta(t)\frac{\partial Y_\beta}{\partial v^j}\Big\}\Big\}dv <$$

$$< \quad -b(x,u,y,v)(\theta^r\rho_r+\sum_{\upsilon=1}^{s}\rho_\upsilon)\,d^2((x,u),(y,v)).$$

Using the hypotheses c) and d) from (4.3) we obtain $\int_\Omega \eta^i(t)\frac{\partial\lambda_i^\alpha}{\partial t^\alpha}dv>0$, that becomes $0<0$, that is false.

THEOREM 4.2 (Direct duality). *Let* (x^0,u^0) *be a normal efficient solution of the primal control problem* (VFCD) *and suppose satisfied the hypotheses of Theorem* 4.1. *Then there are a vector* $\theta^0\in\mathbf{R}^p$ *and the piecewise smooth vector functions* $\lambda^0:\Omega\to\mathbf{R}^{nm}$ *and* $\mu^0:\Omega\to\mathbf{R}^q$ *such that* $(x^0,u^0,\theta^0,\lambda^0,\mu^0)\in\Delta$ *and moreover,* $\pi(x^0,u^0)=\delta(x^0,u^0,\theta^0,\lambda^0,\mu^0)$.

Then $(x^0,u^0,\theta^0,\lambda^0,\mu^0)$ *is an efficient solution to the dual control problem* (WFCD).

Proof. Using Theorem 2.2 for $(x^0,u^0)\in D$ we find $(x^0,u^0,\theta^0,\lambda^0,\mu^0)\in\Delta$. Moreover, $\pi(x^0)= =\delta(x^0,u^0,\theta^0,\lambda^0,\mu^0)$.

THEOREM 4.3 (Converse duality). *Let* $(x^0,u^0,\theta^0,\lambda^0,\mu^0)$ *be an efficient solution of the dual* (WFCD) *and suppose satisfied the following conditions*:

i) $(\bar{x},\bar{u})$ *is a normal efficient solution of the primal problem* (VFCP).

ii) *The hypotheses of Theorem* 4.1 *are satisfied for* $(y,v,\theta,\lambda,\mu)=(x^0,u^0,\theta^0,\lambda^0,\mu^0)$.

Then $\bar{x}=x^0$ *and moreover,* $\pi(x^0,u^0)=\delta(x^0,u^0,\theta^0,\lambda^0,\mu^0)$.

Proof. On the contrary, suppose that $\bar{x}\neq x^0$ and we will obtain a contradiction. Finally, we have $\pi(\bar{x},\bar{u})=\delta(x^0,u^0,\theta^0,\lambda^0,\mu^0)$.

5. Particular case: a Mond-Weir duality for (VCP)

For $G_r(x)=1,\ r=\overline{1,p}$ the primal control problem (VFCP) becomes(VCP) and the dual control problem (WFCD) becomes the dual control problem of (VCP), denoted (WCD). It results a duality of Mond-Weir type between the multitime control problems (VCP) and (WCD).

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PROBLEMS OF ALGEBRA AND GEOMETRY SOLVED WITH MAXIMA. THE NATIONAL "TRAIAN LALESCU" MATHEMATICS COMPETITION

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Abstract: Maxima is a free symbolic-based mathematical software used at the Technical University of Civil Engineering, Faculty of Building Services for a quick computation of many mathematical expressions and plotting functions and data in two or three dimensions. We use this software to solve two Traian Lalescu Competition problems.
Mathematics Subject Classification (2010): 97H60, 97G70,68T20, 65D18, 97-04, 68-04.
Key words: Maxima, symbolic computations, linear algebra, conics, quadrics

1. Introduction

Maxima is a symbolic-based mathematical software providing a number of functions for algebraic manipulation, calculus operations, matrix and linear algebra and other mathematical calculations [1]. Maxima is complete at this stage, with abilities such as symbolic integration, 2D and 3D plotting, as well as an ODE solver. It can also be used as a calculator with arbitrary precision. Maxima is implemented in Lisp and it is derived from the Macsyma system, developed at MIT in the years 1968 through 1982 as part of Project MAC. Macsyma was the first of a new breed of computer algebra systems, leading the way for programs such as Maple, Matlab or Mathematica.

2. A Geometry Problem – National Competition 2013

We consider the following matrix $A=\begin{pmatrix} 2 & 1 & 1 \\ -1 & 4 & 2 \\ -1 & -2 & 1 \end{pmatrix}$ and $X=\begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

Determine the coordinates of the center of symmetry of the curve obtained from the intersection of the surface $X^t\left(A+A^t\right)X-32=0$ and the plane $x+y+z-2=0$, where X^t and A^t represent the transpose of the matrices [2].

```
A:matrix([2,1,1],[-1,4,2],[-1,-2,1]);
```

$$\begin{bmatrix} 2 & 1 & 1 \\ -1 & 4 & 2 \\ -1 & -2 & 1 \end{bmatrix}$$

```
X:matrix([x],[y],[z]);
```

The surface is determined by

```
transpose(X).(A+transpose(A)).X-32
```

$$2z^2+8y^2+4x^2-32$$

We determine the curve of intersection

```
eliminate([4*x^2+8*y^2+2*z^2-32=0,x+y+z-2=0],[z]);
```

$$[2\left(5y^2+(2x-4)y+3x^2-4x-12\right)]$$

Simpler, it becomes

```
ratsimp([2*(3*x^2+5*y^2+(2*x-4)*y-4*x-12)]);
```

$$10y^2+4xy-8y+6x^2-8x-24$$

The associated quadratic form

$$Q(x,y)=6x^2+4xy+10y^2$$

Matrix of the quadratic form

```
matrix([6,2],[2,10]);
```

$$\begin{bmatrix} 6 & 2 \\ 2 & 10 \end{bmatrix}$$

The determinant of the matrix of the quadratic form

```
determinant(matrix([6,2],[2,10]));
```

$$56$$

Thus, the intersection is an elliptical conical and there is a center of symmetry. Its matrix is

```
matrix([6,2,-4],[2,10,-4],[-4,-4,-24]);
```

$$\begin{bmatrix} 6 & 2 & -4 \\ 2 & 10 & -4 \\ -4 & -4 & -24 \end{bmatrix}$$

And the determinant

```
determinant(matrix([6,2,-4],[2,10,-4],[-4,-4,-24]));
```

$$-1536$$

It is different from zero, so we have a non-degenerate conical, an ellipse.
The coordinates of the center of symmetry

```
f(x,y):=10*y^2+4*x*y-8*y+6*x^2-8*x-24
```

The derivative with respect to x

```
diff(f(x,y),x);
```

$$4y+12x-8$$

The derivative with respect to y

```
diff(f(x,y),y);
```
$20y + 4x - 8$

The coordinates of the center of symmetry x_0 and y_0

```
linsolve([diff(f(x,y),x)=0, diff(f(x,y),y)=0],[x,y]);
```

We find z_0

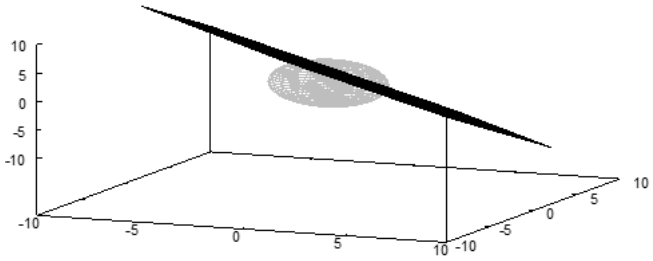
```
z0:2-x0-y0
```

The graph

```
load(draw)
```

```
C:/PROGRA~2/MAXIMA~1.3/share/maxima/5.16.3/share/draw/draw.lisp
```

```
draw(
  gr3d(
  color        = grey,
  parametric_surface(sin(v)*cos(u)*(2*sqrt(2)),
               sin(v)*sin(u)*2,
               4*cos(v),
               u,0,2*%pi,v,0,%pi),
  color        = black,
  implicit(x+y+z-2=0,x,-10,10,y,-10,10,z,-10,10)))$
```



3. An Algebra problem – National Competition Sibiu 2016

Let $V = M_2(R)$ be the vector space of the real 2×2 matrices and

$B = \left\{E_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, E_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, E_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, E_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\}$ its canonical basis. Let

$T : V \to V$ be a liniar operator defined by $T(A) = 2A + 3A^T$, where A^T denotes the transpose of the matrix A.

a) Show that the matrix $m_B(T)$ of the operator T in basis B is

$$m_B(T) = \begin{pmatrix} 5 & 0 & 0 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 3 & 2 & 0 \\ 0 & 0 & 0 & 5 \end{pmatrix}$$

```
E11:matrix([1,0],[0,0]);E12:matrix([0,1],[0,0]);
E21:matrix([0,0],[1,0]);E22:matrix([0,0],[0,1]);
```

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

```
T(X):=2*X+3*transpose(X);
```

By definition, $m_B(T)$ is a 4×4 matrix whose columns are coordinates of the matrices $T(E_{11})$, $T(E_{12})$, $T(E_{21})$, $T(E_{22})$ with respect to the basis B.

```
E1:T(E11); E2:T(E12); E3:T(E21); E4:T(E22);
```

$$\begin{bmatrix}5&0\\0&0\end{bmatrix}\quad\begin{bmatrix}0&2\\3&0\end{bmatrix}\quad\begin{bmatrix}0&3\\2&0\end{bmatrix}\quad\begin{bmatrix}0&0\\0&5\end{bmatrix}$$

Each matrix $E_1,...,E_4$ must be written as a sum of multiples of basis elements.

```
[m,n]:matrix_size(E11)
 for i:1 thru m do for j:1 thru n do e1[j+(i-1)*n,1]:E1[i,j];
 for i:1 thru m do for j:1 thru n do e2[j+(i-1)*n,1]:E2[i,j];
 for i:1 thru m do for j:1 thru n do e3[j+(i-1)*n,1]:E3[i,j];
 for i:1 thru m do for j:1 thru n do e4[j+(i-1)*n,1]:E4[i,j];
```

$$\begin{bmatrix}5\\0\\0\\0\end{bmatrix}\quad\begin{bmatrix}0\\2\\3\\0\end{bmatrix}\quad\begin{bmatrix}0\\3\\2\\0\end{bmatrix}\quad\begin{bmatrix}0\\0\\0\\5\end{bmatrix}$$

The matrix of the operator T in basis B is

```
A:addcol(e1,e2,e3,e4)
```

$$\begin{bmatrix}5&0&0&0\\0&2&3&0\\0&3&2&0\\0&0&0&5\end{bmatrix}$$

b) Write down the matrix of T in basis

$$B'=\left\{F_1=\begin{pmatrix}1&1\\0&0\end{pmatrix},F_2=\begin{pmatrix}0&1\\1&0\end{pmatrix},F_3=\begin{pmatrix}0&0\\1&1\end{pmatrix},F_4=\begin{pmatrix}0&0\\0&1\end{pmatrix}\right\}$$

```
F1:matrix([1,1],[0,0]); F2:matrix([0,1],[1,0]);
F3:matrix([0,0],[1,1]); F4:matrix([0,0],[0,1]);
```

The transition matrix C from the basis B to the basis B' is a 4×4 matrix whose columns are coordinates of the matrices F_1, F_2, F_3, F_4 with respect to the basis B [3].

```
for i:1 thru m do for j:1 thru n do f1[j+(i-1)*n,1]:F1[i,j];
for i:1 thru m do for j:1 thru n do f2[j+(i-1)*n,1]:F2[i,j];
for i:1 thru m do for j:1 thru n do f3[j+(i-1)*n,1]:F3[i,j];
for i:1 thru m do for j:1 thru n do f4[j+(i-1)*n,1]:F4[i,j];
```

$$\begin{bmatrix}1\\1\\0\\0\end{bmatrix}\quad\begin{bmatrix}0\\1\\1\\0\end{bmatrix}\quad\begin{bmatrix}0\\0\\1\\1\end{bmatrix}\quad\begin{bmatrix}0\\0\\0\\1\end{bmatrix}$$

```
C:addcol(f1,f2,f3,f4)
```

Suppose that A and A' are the matrices for T with respect to the basis B and with respect to the basis B' respectively, then $A' = C^{-1}AC$.

```
A':invert(C).A.C;
```

$$\begin{bmatrix} 5 & 0 & 0 & 0 \\ -3 & 5 & 3 & 0 \\ 6 & 0 & -1 & 0 \\ -6 & 0 & 6 & 5 \end{bmatrix}$$

c) Find the kernel and the image of the operator and check that Rank-Nullity Theorem holds.

We must find all solutions to $A\mathbf{x} = \mathbf{0}$. Reducing the augmented matrix of the system

```
A:matrix([5,0,0,0],[0,2,3,0],[0,3,2,0],[0,0,0,5]);
b:matrix([0],[0],[0],[0]);
A1:addcol(A,b)
```

Yields

```
echelon(A1)
```

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & \frac{3}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

The nonzero number of A1 is 4. Since the only solution of $A\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = \mathbf{0}$, the nullspace of A consists of the zero vector alone [4]. Then the kernel of T, is $\{\mathbf{0}\}$ and its dimension is 0.

or

```
nullspace(A);nullity(A)
span( ) 0
columnspace(A);rank(A)
```

d) Find the eigenvalues and the eigenvectors (in the basis B) of the operator T.

The characteristic polynomial

```
P(x):=charpoly(A, x), expand;
```

$$x^4 - 14x^3 + 60x^2 - 50x - 125$$

Eigenvalues are the roots of the characteristic polynomial

```
allroots(P(x))
```

$$[\, x = -1.0,\ x = 6.2306172958669816\,10^{-5}\,\%i + 4.999937843515951,\ x = 4.999937843515951 - 6.2306172958669816\,10^{-5}\,\%i,\ x = 5.000124312968097\,]$$

or

```
eigenvalues(A)
[[5,-1],[3,1]]
v:eigenvectors(A)
```

```
[[[5,-1],[3,1]],[1,0,0,0],[0,1,1,0],[0,0,0
,1],[0,1,-1,0]]
```

e) Calculate $T^{2016}\left(\begin{pmatrix} 1 & 2 \\ -2 & 0 \end{pmatrix}\right)$.

The eigenvalues of *A* are 5 and -1 and the corresponding eigenvectors are

```
v1:columnvector(v[2]);v2: columnvector(v[3]);
v3:columnvector(v[4]);v4: columnvector(v[5])
```

A is nondefective, hence diagonalizable. Let $S = [v_1 \quad v_2 \quad v_3 \quad v_4]$.

```
S:addcol(v1,v2,v3,v4)
```

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Then will have $D = S^{-1}AS$ where *D* is a matrix of which the diagonal entries are the eigenvalues of *A*.

```
D:invert(S).A.S
```

$$\begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Since $D = S^{-1}AS$, $A = SDS^{-1}$. Then, as the matrix product is associative, $A^2 = (SDS^{-1}) \cdot (SDS^{-1}) = SD^2S^{-1}$, thus $A^{2016} = SD^{2016}S^{-1}$ [5]. We can compute D^{2016} by raising each entry to the 2016-th power and then

```
A2016:S.(D^^2016).invert(S)
(S.(D^^2016).invert(S)).matrix([1],[-2],[2],[0]);
```

$$\begin{bmatrix} 5^{2016} & -2 \\ 2 & 0 \end{bmatrix}$$

4. Conclusions

A lot of engineering computations [6], can be easily done using computer software, thus it is important for our students to use computers in their projects. We chose Maxima because is a free complete software.

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THE DEPENDENCE OF THE HUTCHINSON VECTOR MEASURE WITH RESPECT TO A PARAMETER

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Abstract: In this paper we obtain a generalization of the results given in [4], regarding the dependence of the Hutchinson measure (also called : the fractal measure) with respect to a parameter. In [4] is considered the classical case (in the frame of the probability measures) and we work here with vector measures, taking values in a Hilbert space. The main result is that, if any contraction used for the construction of the Markov-type operator is a continuous function of a parameter, then the Hutchinson measure is, also, a continuous function of that parameter.
Mathematics Subject Classification (2010): 28C20, 4GG12
Key words: iterated function system, Hutchinson measure, Markov- type operator

1. Preliminary Definitions

We recall here some definitions that we will use. Let (T,d) be a compact metric space and $\mathcal{B}$ the Borel subsets of *T*. Let, also, $(X,\langle\ \rangle)$ be a Hilbert space. For any *T*-additive measure $\mu:\mathcal{B}\to X$ and any $A\in\mathcal{B}$, we define *the variation of* μ *on A* by the formula: $|\mu|(A)=\sup\{\sum_i \|\mu(A_i)\|,(A_i)_i$ being a finite partition of *A* with Borel subsets$\}$. If $|\mu|(T)<\infty$ we say that μ has *bounded variation* (for more details, one can consult [1]). We denote $cabv(X)=\{\ \mu:\mathcal{B}\to X\ /|\mu|(T)<\infty\}$. It can be proved that $\|\mu\|:=|\mu|(T)$ is a norm and $(cabv(T),\|\ \|)$ is a Banach space. Now we consider the sets of functions: $S(X)=\{f=\sum_{i=1}^{n}\varphi_{A_i}x_i\ /\ A_i\in\mathcal{B},x_i\in X\}$; (here, φ_{A_i} is the characteristic function of the set *A*)

$TM(X)=\{f:T\to X\ /\ \exists(f_n)_n\subset S(X) f_n\xrightarrow{u}f\}$;
$C(X)=\{f:T\to X\ /\ f$ is continuous on $T\}$;
$BL(X)=\{f:T\to X\ /\ f$ is a Lipschitz function$\}$;

Remark. For $f\in BL(X)$, denoting by $\|f\|_\infty=\sup\{\|f(t)\|/t\in T\}$, $\|f\|_L$ – the Lipschitz constant of *f*, and $\|f\|_{BL}=\|f\|_\infty+\|f\|_L$, it is easy to prove that $\|\ \|_{BL}$ is a norm on the vector space $BL(X)$.

We denote $BL_1(x)=\{f\in BL(X)/\|f\|_{BL}\le 1\}$. For $a\ge 0$, $B_a(X)=\{\mu\in cabv(X)/\|\mu\|\le a\}$. We introduce now the concept of integral that we will use: for $\mu\in cabv(X)$ and

$f \in S(X), f = \sum_{i=1}^{n} \varphi_{A_i} x_i$, we define $\int f d\mu = \sum_{i=1}^{n} \langle x_i, \mu(A_i) \rangle$. Then, for $f \in TM(X)$, let $(f_n)_n \subset S(X), f_n \xrightarrow{u} f$. We define $\int f d\mu = \lim_{n\to\infty} \int f_n d\mu$ (for details, see [2]). Using this integral, we can define the *Monge-Kantorovich-type norm* on $cabv(X)$:

$$\|\mu\|_{MK} = \sup\{\left|\int f d\mu\right| / f \in BL_1(X)\}$$

2. The Hutchinson measure

We denote by $\mathcal{L}(X) = \{F : X \to X / F \text{ is linear and continuous}\}$. Let $(R_i)_{1\le i\le N} \subset \mathcal{L}(X), (\omega_i)_{1\le i\le N}, \omega_i : T \to T$ contractions of ratios $r_i \le 1$. We define the *Markov-type operator*: $H(\mu) = \sum_{i=1}^{N} R_i \circ \mu \circ \omega_i^{-1}$, that is $H(\mu)(A) = \sum_{i=1}^{N} R_i(\mu(\omega_i^{-1}(A))), \forall A \in \mathcal{B}$. It is easy to prove that $H(\mu) \in cabv(X)$. For the following theorem, we supose that $X = \boldsymbol{K}^n, n \in \boldsymbol{N}^*, \boldsymbol{K}$ being $\boldsymbol{R}$ or $\boldsymbol{C}$.

Theorem 2.1 *With the previous notations, let us suppose that* $\sum_{i=1}^{N} \|R_i\|(1+r_i) < 1$ ($\|R_i\|$ *being the operatorial norm of* R_i*). Let* $a > 0, \mu^0 \in cabv(X)$. *We define* $P : cabv(X) \to cabv(X)$, $P(\mu) = H(\mu) + \mu^0$. *Let* $A \subset B_a(X)$, *weak-*$*$ *close, such that* $P(A) \subset A$ *and denote also by P its restriction of A. Then,* $\exists! \mu^* \in A$ *such that* $P(\mu^*) = \mu^*$. *(*μ^* *is called the Hutchinson measure or fractal measure)*

For the proof of this theorem, one can consult [3].

3. The (continuous)dependence of the fractal measure with respect to a parameter

Now, we will give a generalization of a result given in [4], regarding the fractal measure.

Let (Y, d_Y) be a metric space, $\omega_i : Y \times T \to T$ contractions of ratios $r_i : Y \to [0,1)$. For any $t \in Y$ we define: $H_t(\mu)(A) = \sum_{i=1}^{N} R_i(\mu(\omega_i^{-1}(t, A))), \forall A \in \mathcal{B}$, and, for any $\mu^0 \in cabv(\boldsymbol{K}^n)$, $P_t(\mu) = H_t(\mu) + \mu^0$. For any $t \in Y$ we denote by μ_t^* the Hutchinson measure associated with P_t.

Theorem 3.1. *We suppose that for any* $i \in \{1,...,N\}$ *the function* $t \mapsto \omega_i(t,\cdot)$ *is continuous. Then, the function* $t \mapsto \mu_t^*$ *is continuous.*

Proof We will need two lemmas:

Lemma 3.2. $\forall f \in C(X), \forall \mu \in cabv(X)$, $\int f dH(\mu) = \int \sum_{i=1}^{N} R_i^* \circ f \circ \omega_i d\mu$ (see[3]).

Lemma 3.3. *Considering again the contractions* $\omega_i : Y \times T \to T$, *the function* $t \mapsto \omega_i(t,\cdot)$ *beingcontinuous on Y,* $\forall i \in \{1,...,N\}$, *if* $t_k \to t_0$ *(as* $k \to \infty$*), then* $\sup_{t\in T} d(\omega_i(t_k, x), \omega_i(t_0, x)) \to 0, \forall i \in \overline{1, N}$.

For the proof of Lemma 3.3., see [4].

We can give now the proof of Theorem 3.1. In the proof of Theorem 2.1. it is proved that the operator P is a contraction on the set A. We will denote by r_t the contraction of P_t. We have, successively:

$$\left\|\mu^*_{t_k}-\mu^*_{t_0}\right\|_{MK}=\left\|P_{t_k}(\mu^*_{t_k})-P_{t_0}(\mu^*_{t_0})\right\|\le\left\|P_{t_k}(\mu^*_{t_k})-P_{t_k}(\mu^*_{t_0})\right\|+\left\|P_{t_k}(\mu^*_{t_0})-P_{t_0}(\mu^*_{t_0})\right\|\le$$

$$\le r_{t_k}\left\|\mu^*_{t_k}-\mu^*_{t_0}\right\|+\left\|H_{t_k}(\mu^*_{t_0})-H_{t_0}(\mu^*_{t_0})\right\|\Rightarrow\left\|\mu^*_{t_k}-\mu^*_{t_0}\right\|\le\frac{1}{1-r_{t_k}}\left\|H_{t_k}(\mu^*_{t_0})-H_{t_0}(\mu^*_{t_0})\right\|\ (1).$$

If we prove that, when $t_k\to t_0$ (as $k\to\infty$), then $\left\|H_{t_k}(\mu^*_{t_0})-H_{t_0}(\mu^*_{t_0})\right\|\to 0$, from (1), we get $\mu^*_{t_k}\to\mu^*_{t_0}$, and the proof will be complete. We can write:

$$\left|\sum_{i=1}^{N}(\int R_i^*\circ(f\circ\omega_i(t_k,\cdot))d\mu^*_{t_0}-\int R_i^*\circ(f\circ\omega_i(t_0,\cdot))d\mu^*_{t_0}\right|=\left|\sum_{i=1}^{N}(\int R_i^*\circ(f\circ\omega_i(t_k,\cdot)-f\circ\omega_i(t_0,\cdot))d\mu^*_{t_0}\right|\le$$

$$\le\sum_{i=1}^{N}\left\|R_i^*\right\|\cdot\int\max_i\sup_{t\in T}d(\omega_i(t_k,x),\omega_i(t_0,x))d\left|\mu^*_{t_0}\right|\to 0$$, using Lemma 3.3.

Now, using Lemma 3.2. we obtain: $\sup_{f\in BL_1(X)}\left|\int f d(H_{t_k}(\mu^*_{t_0})-H_{t_0}(\mu^*_{t_0}))\right|\to 0$, hence,

$\left\|H_{t_k}(\mu^*_{t_0})-H_{t_0}(\mu^*_{t_0})\right\|\to 0$

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LOCAL FORMULA OF THE POINCARE-CARTAN FORM

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Abstract: We give a formula for the Poincaré-Cartan form of a variational problem using local coordinates and give some properties
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1. Introduction

In [2] the authors present a geometric theory for the calculus of variations on J^1Y. Here we present an analogous theory for the calculus of variations on J^kY. More on geometric structures on jet spaces one finds in [1]. The extensions of the concepts of [2] from J^1Y to J^kY one finds in [3] and in short form in [4]. The results of this note appeared in the little known book [3].

For $\pi_{XY} : Y \to X$ a fiber bundle with local coordinates (x_i, y^A) a jet of order k at $x \in X$ is an equivalence class of sections $s : X \to Y$ where two sections S_1, S_2 which look in local coordinates as $(x_0, x_1, x_2, ...x_n) \to \left(s^A(x_0, x_1, x_2, ...x_n)\right)_{A=1,2,..N}$ are equivalent iff

$$\frac{\partial^{|I|} s_1^A}{\partial x_{i_1} \partial x_{i_2} ... \partial x_{i_p}} = \frac{\partial^{|I|} s_2^A}{\partial x_{i_1} \partial x_{i_2} ... \partial x_{i_p}},\ I = \{i_1, i_2, ...i_p\},\ p \le k .$$

The equivalence class of s at $x \in X$ is $j^k s(x)$. The set of jets of order k has a natural structure of a differentiable manifold. If $(x_0, x_1, x_2, ...x_n)$ are local coordinates in X and $(y^1, y^2, ...y^N)$ are local coordinates on the fibre then (x_i, y^j, y_J^A) is a local coordinate chart for J^kY. Here $J = (j_1, j_2, ..j_p),\ p \le k$ and $y_J^A = \frac{\partial^{|J|} s^A}{\partial x_{i_1} \partial x_{i_2} .. \partial x_{i_p}}$ where s is the section which represents the jet.

Any local diffeomorphism $\eta_Y : Y \to Y$ which covers local diffeomorphism $\eta_X : X \to X$ is extendable to a local diffeomorphism $\eta_{J^kY} : J^kY \to J^kY$. A local vector field on Y which covers a vector field on X, $V = V^i(x)\frac{\partial}{\partial x^i} + V^A(x,y)\frac{\partial}{\partial y^A}$ generates a local 1 parameter group of diffeomorphisms of Y which covers a local 1 parameter of diffeomorphisms of X. The generator of this group is by definition the extension j^kV of the vector field V to J^kY. We have

$$j^kV = V^i(x)\frac{\partial}{\partial x^i} + V^A(x,y)\frac{\partial}{\partial y^A} + \sum_{1 \le |I| \le k} V_I^A \frac{\partial}{\partial y_I^A}$$

where $V_I^A = D_I\left(V^v\right)^A + V^j y_{I,j}^A$, $\left(V^v\right)^A := V^A - y_j^A V^j$, $D_i = \frac{\partial}{\partial x^i} + \sum_{|I|\le k-1} y_{I,i}^A \frac{\partial}{\partial y_I^A}$.

The forms $\theta_J^A = dy_J^A - y_{J,i}^A dx^i$, $|J| \le k-1$ are called horizontal forms and $\theta_J^A(D_i) = 0$.

2. The variational problem

As in the bibliography we set the variational problem. Let U a C^∞ differentiable manifold of dimension n+1, with smooth boundary $\pi_{X,Y} : Y \to X$ a fiber bundle with $\dim(X) = n+1$ and fiber dimension N. Let

$$C_U^\infty = \{\varphi | \} \varphi : U \to Y, \varphi \text{ of class } C^\infty, \pi_{X,Y} \circ \varphi : U \to X \text{ embeeding}\}$$

And let $\varphi_X : U \to X$, $\varphi_X = \pi_{X,Y} \circ \varphi$ and $U_X = \varphi_X(U) = \pi_{X,Y}(\varphi(U))$. Let C_U the closure of C_U^∞ in some Banach space topology. The tangent space at $\varphi \in C_U^\infty$ contains

$$T_\varphi C_U^\infty = \left\{\widetilde{V} \in C^\infty(U, TY),\ \pi_{Y,TY} \circ \widetilde{V} = \varphi\right\}$$

For each $\widetilde{V}$ it exists a projectable vector field on a neighborhood of $\varphi(U)$ in Y, $V = V^i(x)\frac{\partial}{\partial x^i} + V^A(x,y)\frac{\partial}{\partial y^A}$, such that $\widetilde{V} = V \circ \varphi$.

The one parameter group of V projects to a one parameter group defined in some neighborhood of $\varphi_X(U)$ in X such that

$$\begin{array}{ccc} Y & \overset{\eta_{Y,\lambda}}{\to} & Y \\ \downarrow \pi_{X,Y} & & \downarrow \pi_{X,Y} \\ X & \overset{\eta_{X,\lambda}}{\to} & X \end{array}$$

This family of embeddings of U gives a variation of φ, $\varphi^\lambda = \phi(\eta_{Y,\lambda}, \varphi) = \eta_{Y,\lambda} \circ \varphi$.

A k order lagrangian is n+1 horizontal differential form on $J^k Y$, that is

$$\begin{aligned} \Lambda &= L\left(x^i, y^A, y_i^A, y_{i_1 i_2}^A, y_{i_1 i_2 i_3}^A, \ldots y_{i_1 i_2 \ldots i_k}^A\right) dx^0 \wedge dx^1 \wedge \ldots \wedge dx^n \\ &= L\left(x^i, y^A, y_i^A, y_{i_1 i_2}^A, y_{i_1 i_2 i_3}^A, \ldots y_{i_1 i_2 \ldots i_k}^A\right) \omega \end{aligned} \tag{1}$$

We use the notations

$$\omega = dx^0 \wedge dx^1 \wedge dx^2 \wedge \ldots \wedge dx^n,\ \omega_i = i_{\frac{\partial}{\partial x^i}} \omega, \quad \omega_{i,j} = i_{\frac{\partial}{\partial x^j}} i_{\frac{\partial}{\partial x^i}} \omega \tag{2}$$

The action of Λ is defined as

$$S(\varphi) = \int_{U_X} j^k\left(\varphi \circ \varphi_X^{-1}\right)^* \Lambda \tag{3}$$

We see that $\varphi \circ \varphi_X^{-1}$ is a section of Y over $\varphi_X(U)$.

Let

$$\varphi^\lambda = \phi(\eta_{Y,\lambda}, \varphi),\ \varphi_X^\lambda = \pi_{X,Y} \circ \varphi^\lambda,\ U_X^\lambda = \varphi_X^\lambda(U) = \eta_{X,\lambda}(U_X) \tag{4}$$

The section of φ^λ over U_X^λ is $\varphi^\lambda \circ \left(\varphi_X^\lambda\right)^{-1} = \eta_Y^\lambda \circ \left(\varphi \circ \varphi_X^{-1}\right) \circ \left(\eta_X^\lambda\right)^{-1}$.

Let $V = V^i(x)\frac{\partial}{\partial x^i} + V^A(x,y)\frac{\partial}{\partial y^A}$ the infinitesimal generator of $\eta_{Y,\lambda}$ and let $j^k(V)$ its prolongation on $J^k Y$. Then

$$S\left(\varphi^\lambda\right) = \int_{U_X^\lambda} j^k\left(\varphi^\lambda \circ \left(\varphi_X^\lambda\right)^{-1}\right)^* \Lambda \tag{5}$$

Then we have the following

Proposition 1. *If* $\left(\varphi^{\lambda}\circ\left(\varphi_X^{\lambda}\right)^{-1}\right)(U_X)$ *is embedded in a coordinate chart then*

$$\begin{aligned}&\frac{\partial}{\partial\lambda}\Big|_{\lambda=0}S\left(\varphi^{\lambda}\right)=\int_{U_X}\sum_{A=1}^{N}\sum_{0\le|J|\le k}\left(j^{2k}\left(\varphi\circ\varphi_X^{-1}\right)\right)^{*}\left((-1)^{|J|}D_J\frac{\partial L}{\partial y_J^A}\cdot\left(V^{v}\right)^{A}\cdot\omega\right)\\&+\int_{\partial U_X}\left(j^{2k-1}\left(\varphi\circ\varphi_X^{-1}\right)\right)^{*}\theta_{L,V}\end{aligned}\tag{6}$$

where

$$\theta_{L,V}=\sum_{A=1}^{N}\sum_{1\le|J|\le k}\sum_{J=\left(J',j^i,J''\right)}\left(-1\right)^{|J''|}D_{J''}\frac{\partial L}{\partial y_J^A}\cdot D_{J'}\left(V^{v}\right)^{A}\cdot\omega_{j^i}+L\sum_{i=0}^{n}V^i\omega_i\tag{7}$$

In (7) the multiindexes J,J',J'' *are not decreasing sequences and if* $J=\left(j^1j^2\ldots j^l\right)$, *then* $J'=\left(j^1..j^{i-1}\right)$, $J''=\left(j^{i+1}\ldots j^l\right)$, *for* $i=1,2,\ldots l$.

Proof. (sketch)

$$\frac{\partial}{\partial\lambda}\Big|_{\lambda=0}S\left(\varphi^{\lambda}\right)=\int_{U_X}\left(j^k\left(\varphi\circ\left(\varphi_X\right)^{-1}\right)\right)^{*}\left(L_{j^k(V)}\left(L\omega\right)\right)$$

where $L_{j^k(V)}$ is the Lie derivative along $j^k(V)$ and $j^k(V)=j^k(V)^{v}+j^k(V)^{h}$.

After some computations we find

$$\int_{U_X}\left(j^k\left(\varphi\circ\left(\varphi_X\right)^{-1}\right)\right)^{*}\left(L_{j^k(V)}\left(L\omega\right)\right)=\int_{U_X}\left(j^k\left(\varphi\circ\left(\varphi_X\right)^{-1}\right)\right)^{*}\left(L_{V^iD_i}\left(L\omega\right)\right)=\int_{\partial U_X}L\left(j^k\left(\varphi\circ\left(\varphi_X\right)^{-1}\right)\right)V^i\omega_i$$

For the vertical derivative we get

$$\begin{aligned}&\int_{U_X}\left(j^k\left(\varphi\circ\left(\varphi_X\right)^{-1}\right)\right)^{*}\left(L_{D_J\left(V^{v}\right)^{A}\frac{\partial}{\partial y_J^A}L\omega}\right)=\int_{U_X}L_{D_J\left(V^{v}\right)^{A}\frac{\partial}{\partial y_J^A}}\left(L\omega\right)\left(j^k\left(\varphi\circ\left(\varphi_X\right)^{-1}\right)\right)\\&=\int_{U_X}\sum_{A,J}\frac{\partial L}{\partial y_J^A}\left(\left(j^k\left(\varphi\circ\left(\varphi_X\right)^{-1}\right)\right)\right)\cdot D_J\left(V^{v}\right)^{A}\cdot\omega\end{aligned}\tag{8}$$

Using the Stokes formula and $\left(D_jL\right)\left(j^{k+1}\left(\varphi\circ\left(\varphi_X\right)^{-1}\right)\right)=\frac{\partial}{\partial x^j}\left(L\left(j^k\left(\varphi\circ\left(\varphi_X\right)^{-1}\right)\right)\right)$ we find

$$\begin{aligned}&\int_{U_X}\frac{\partial L}{\partial y^A_{j^1j^2..j^l}}\cdot D_{j^1j^2..j^l}\left(V^{v}\right)^{A}\cdot\omega=\int_{U_X}\left(D_{j^l}\left(\frac{\partial L}{\partial y^A_{j^1j^2..j^l}}D_{j^1j^2..j^{l-1}}\left(V^{v}\right)^{A}\right)-D_{j^l}\frac{\partial L}{\partial y^A_{j^1j^2..j^l}}\cdot D_{j^1j^2..j^{l-1}}\left(V^{v}\right)^{A}\right)\omega\\&=-\int_{U_X}D_{j^l}\frac{\partial L}{\partial y^A_{j^1j^2..j^l}}\cdot D_{j^1j^2..j^{l-1}}\left(V^{v}\right)^{A}\cdot\omega+\int_{U_X}d\left(\frac{\partial L}{\partial y^A_{j^1j^2..j^l}}D_{j^1j^2..j^{l-1}}\left(V^{v}\right)^{A}\omega_{j^l}\right)\\&=-\int_{U_X}D_{j^l}\frac{\partial L}{\partial y^A_{j^1j^2..j^l}}\cdot D_{j^1j^2..j^{l-1}}\left(V^{v}\right)^{A}\cdot\omega+\int_{\partial U_X}\frac{\partial L}{\partial y^A_{j^1j^2..j^l}}D_{j^1j^2..j^{l-1}}\left(V^{v}\right)^{A}\omega_{j^l}\end{aligned}\tag{9}$$

Using repeatedly (9) one finds

$$\int_{U_X}\frac{\partial L}{\partial y^A_{j^1j^2..j^l}}\cdot D_{j^1j^2..j^l}\left(V^{v}\right)^{A}\cdot\omega=\left(-1\right)^{|J|}\int_{U_X}D_J\frac{\partial L}{\partial y_J^A}\cdot\left(V^{v}\right)^{A}\cdot\omega+\sum_{J=\left(J',j^i,J''\right)}\int_{\partial U_X}\left(-1\right)^{|J''|}D_{J''}\frac{\partial L}{\partial y^A_{j^1j^2..j^l}}\cdot D_{J'}\left(V^{v}\right)^{A}\cdot\omega_{j^i}$$

that is (6)+(7).

QED.

3. The Poincare-Cartan form

Using the forms $\theta_J^A = dy_J^A - y_{J,i}^A dx^i$ we define on $J^{2k-1}Y$ the *Poincare-Cartan* form

$$\theta_\Lambda = \sum_{A=1}^{N} \sum_{1\le|J|\le k} \sum_{J=(J',j^i,J'')} (-1)^{|J''|} D_{J''} \frac{\partial L}{\partial y_J^A} \cdot \theta_{J'}^A \wedge \omega_{j^i} + L\omega \tag{10}$$

and we get from the proposition 1:

Proposition 2. *If* $\left(\varphi^\lambda \circ \left(\varphi_X^\lambda\right)^{-1}\right)(U_X)$ *is embedded in a coordinate chart then*

$$\begin{aligned} \frac{\partial}{\partial\lambda}\Big|_{\lambda=0} S(\varphi^\lambda) &= \int_{U_X} j^{2k}\left(\varphi\circ\varphi_X^{-1}\right)^* \left(\int_{U_X} \sum_{A=1}^{N} \sum_{0\le|J|\le k} (-1)^{|J|} D_J \frac{\partial L}{\partial y_J^A} \cdot \left(V^\nu\right)^A \cdot \omega \right) \\ &+ \int_{\partial U_X} j^{2k-1}\left(\varphi\circ\varphi_X^{-1}\right)^* \left(i_{j^{2k-1}(V)}\ \theta_\Lambda\right) \end{aligned} \tag{11}$$

Now the integral over U_X can be further transformed .

Proposition 3. *If* $\left(\varphi^\lambda \circ \left(\varphi_X^\lambda\right)^{-1}\right)(U_X)$ *is embedded in a coordinate chart then*

$$\int_{U_X} j^{2k}\left(\varphi\circ\varphi_X^{-1}\right)^* \left(\sum_{A=1}^{N} \sum_{0\le|J|\le k} (-1)^{|J|} \int_{U_X} D_J \frac{\partial L}{\partial y_J^A} \cdot \left(V^\nu\right)^A \cdot \omega \right) = \int_{U_X} j^{2k-1}\left(\varphi\circ\varphi_X^{-1}\right)^* \left(i_{j^{2k-1}(V)}\, d\theta_\Lambda\right) \tag{12}$$

Proof. Locally θ_L may be considered a form on $J^{2k}Y$ **.** We have

$$d\theta_\Lambda = \sum_{A=1}^{N} \sum_{1\le|J|\le k} \sum_{J=(J',j^i,J'')} (-1)^{|J''|} d\left(D_{J''} \frac{\partial L}{\partial y_J^A} \right) \wedge \theta_{J'}^A \wedge \omega_{j^i} + \sum_{A=1}^{N} \sum_{1\le|J|\le k} \sum_{J=(J',j^i,J'')} (-1)^{|J''|} D_{J''} \frac{\partial L}{\partial y_J^A} \cdot \left(-dy_{J',j^i}^A\right) \wedge \omega + dL \wedge \omega \tag{13}$$

For the horizontal component of $j^{2k}(V)$:

$$\int_{U_X} j^{2k}\left(\varphi\circ\varphi_X^{-1}\right)^* \left(i_{j^{2k}(V)^h}\, d\theta_\Lambda\right) = \int_{U_X} j^{2k}\left(\varphi\circ\varphi_X^{-1}\right)^* \left(j^{2k}\left(\varphi\circ\varphi_X^{-1}\right)_* i_{V_i \frac{\partial}{\partial x^i}}\, d\theta_\Lambda \right) = \int_{U_X} i_{V_i \frac{\partial}{\partial x^i}} \left(j^{2k}\left(\varphi\circ\varphi_X^{-1}\right)^* d\theta_\Lambda \right) = 0$$

because $j^{2k}\left(\varphi\circ\varphi_X^{-1}\right)^* d\theta_L$ is a n+2 form. For the vertical part of the field:

$$\begin{aligned} i_{j^{2k}(V)^v} d\theta_\Lambda &= \sum_{A=1}^{N} \sum_{1\le|J|\le k} \sum_{J=(J',j^i,J'')} (-1)^{|J''|} d\left(D_{J''} \frac{\partial L}{\partial y_J^A} \right) \left(j^{2k}(V)^v\right) \cdot \theta_{J'}^A \wedge \omega_{j^i} + \sum_{A=1}^{N} \sum_{1\le|J|\le k} \sum_{J=(J',j^i,J'')} (-1)^{|J''|+1} d\left(D_{J''} \frac{\partial L}{\partial y_J^A} \right) \wedge \left(V^\nu\right)_{J'}^A \omega_{j^i} \\ &+ \sum_{A=1}^{N} \sum_{1\le|J|\le k} \sum_{J=(J',j^i,J'')} (-1)^{|J''|+1} D_{J''} \frac{\partial L}{\partial y_J^A} \cdot (V)_{J',j^i}^A \omega + \sum_{A=1}^{N} \sum_{0\le|J|\le k} \frac{\partial L}{\partial y_J^A} \cdot \left(V^\nu\right)_J^A \cdot \omega \end{aligned} \tag{14}$$

Now we apply $j^{2k}\left(\varphi\circ\varphi_X^{-1}\right)^*$ to (14) and we get

$$j^{2k-1}\left(\varphi\circ\varphi_X^{-1}\right)^* \left(i_{j^{2k-1}(V)^v} d\theta_\Lambda\right) = S_1 + S_2 + S_3$$

where

$$S_1 = \sum_{A=1}^{N} \sum_{1\le|J|\le k} \sum_{J=(J',j^i,J'')} (-1)^{|J''|+1} \left(\frac{\partial^{|J''|+1}}{\partial x^{j^i,J''}} \left(\frac{\partial L}{\partial y_J^A} \left(\varphi\circ\varphi_X^{-1}\right) \right) \right) \cdot \left(V^\nu \left(\varphi\circ\varphi_X^{-1}\right) \right)_{J'}^A \omega$$

$$S_2 = \sum_{A=1}^{N} \sum_{1\le|J|\le k} \sum_{J=(J',j^i,J'')} (-1)^{|J''|+1} \frac{\partial^{|J''|}}{\partial x^{J''}} \left(\frac{\partial L}{\partial y_J^A} \left(\varphi\circ\varphi_X^{-1}\right) \right) \cdot \left(V \left(\varphi\circ\varphi_X^{-1}\right) \right)_{J',j^i}^A \omega$$

$$S_3 = \sum_{A=1}^{N} \sum_{0\le|J|\le k} \frac{\partial L}{\partial y_J^A} \left(\varphi\circ\varphi_X^{-1}\right) \cdot \left(V^\nu \left(\varphi\circ\varphi_X^{-1}\right) \right)_J^A \cdot \omega$$

Cancellations of terms between S_1, S_2 and S_3 give

$$j^{2k}\left(\varphi\circ\varphi_X^{-1}\right)^*\left(i_{j^{2k}(V)^v}d\theta_\Lambda\right)=\sum_{A=1}^N\sum_{0\le|J|\le k}(-1)^{|J|}\left(\frac{\partial^{|J|}}{\partial x^J}\left(\frac{\partial L}{\partial y_J^A}\left(\varphi\circ\varphi_X^{-1}\right)\right)\right)\cdot\left(V^v\left(\varphi\circ\varphi_X^{-1}\right)\right)$$

Because θ_L is a form on $J^{2k-1}Y$ then $j^{2k}\left(\varphi\circ\varphi_X^{-1}\right)^*\left(i_{j^{2k}(V)^v}d\theta_\Lambda\right)=j^{2k-1}\left(\varphi\circ\varphi_X^{-1}\right)^*\left(i_{j^{2k-1}(V)^v}d\theta_\Lambda\right)$.

QED.

Proposition 4. *There is only one n+1 form θ_Λ on $J^{2k-1}Y$ defined in some coordinate chart such that:*

i. $L\left(j^k\left(\varphi\circ\varphi_X^{-1}\right)\right)\omega=\left(j^{2k-1}\left(\varphi\circ\varphi_X^{-1}\right)\right)^*\theta_\Lambda$

ii. $\frac{\partial}{\partial\lambda}\Big|_{\lambda=0}S\left(\varphi^\lambda\right)=\int_{U_X}j^{2k-1}\left(\varphi\circ\varphi_X^{-1}\right)^*\left(i_{j^{2k-1}V}d\theta_\Lambda\right)+\int_{\partial U_X}j^{2k-1}\left(\varphi\circ\varphi_X^{-1}\right)^*\left(i_{j^{2k-1}(V)}\theta_\Lambda\right)$

iii. *θ_Λ depends at $j^{2k-1}(s)(x)$ only on dx^i, i=0..n and dy_J^A, with $|J|\le k-1$ and for R and S two $\pi_{X,J^{2k-1}Y}$ vertical fields one has $i_Ri_S\theta_L=0$.*

iv. *For W a vector field on $J^{2k-1}Y$, tangent to the fibers of $\pi_{Y,J^{2k-1}Y}$, then* $j^{2k-1}\left(\varphi\circ\varphi_X^{-1}\right)^*\left(i_Wd\theta_\Lambda\right)=0$

Proof. θ_Λ defined by (10) satisfies i-iv as a consequence of propositions 1, 2, 3. Now let θ another form. From iii) it follows

$$\theta=G\omega+\sum_{i=0}^n\sum_{A=1}^N\sum_{|J|\le k-1}G_J^{A;\,i}dy_J^A\wedge\omega_i=F\omega+\sum_{i=0}^n\sum_{A=1}^N\sum_{|J|\le k-1}G_J^{A;\,i}\theta_J^A\wedge\omega_i$$

From i) it follows $F=L$. Let $\Delta=\theta-\theta_L=\sum_{i=0}^n\sum_{A=1}^N\sum_{|J|\le k-1}F_J^{A;\,i}\theta_J^A\wedge\omega_i$ with $F_J^{A;i}$ some functions on $J^{2k-1}Y$. The condition ii) implies $\int_{U_X}j^{2k-1}\left(\varphi\circ\varphi_X^{-1}\right)^*\left(i_{j^{2k-1}(V)}d\Delta\right)=0$ for any field $V=V^A(x,y)\frac{\partial}{\partial y^A}$ with support compact inside $\pi_{X,Y}^{-1}\left(\overset{\circ}{U}_X\right)$. Combined with iv) it follows that $\int_{U_X}j^{2k-1}\left(\varphi\circ\varphi_X^{-1}\right)^*\left(i_W\,d\Delta\right)=0$ for any field $W=\sum_{A=1}^N\sum_{|J|\le 2k-1}W_J^A\frac{\partial}{\partial y_J^A}$ with compact support in $\pi_{X,Y}^{-1}\left(\overset{\circ}{U}_X\right)$. We have

$$i_Wd\Delta=dF_J^{A;\,i}(W)\cdot\theta_J^A\wedge\omega_i-\theta_J^A(W)\cdot\,dF_J^{A;\,i}\wedge\omega_i-F_J^{A;\,i}\cdot dy_{J,i}^A(W)\cdot\omega$$

whence

$$j^{2k-1}(\varphi\circ\varphi_X^{-1})^*\left(i_Wd\Delta\right)=-W_J^A\cdot\left(D_iF_J^{A;\,i}\right)\left(j^{2k-1}(\varphi\circ\varphi_X^{-1})\right)\cdot\omega-W_{J,i}^A\cdot F_J^{A;\,i}\left(j^{2k-1}(\varphi\circ\varphi_X^{-1})\right)\cdot\omega$$

$$=-W_{J,i}^A\cdot\left(F_J^{A;\,i}\left(j^{2k-1}(\varphi\circ\varphi_X^{-1})\right)+\left(D_kF_{J,i}^{A;\,k}\right)\left(j^{2k-1}(\varphi\circ\varphi_X^{-1})\right)\right)-W^A\cdot\left(D_kF^{A;\,k}\right)\left(j^{2k-1}(\varphi\circ\varphi_X^{-1})\right)\cdot\omega$$

Now $\int_{U_X}j^{2k-1}\left(\varphi\circ\varphi_X^{-1}\right)^*\left(i_Wd\Delta\right)=0$ for any section $s=\varphi\circ\varphi_X^{-1}$ implies

$$0=F_J^{A;\,i}\left(j^{2k-1}(\varphi\circ\varphi_X^{-1})\right)+\left(D_kF_{J,i}^{A;\,k}\right)\left(j^{2k-1}(\varphi\circ\varphi_X^{-1})\right)$$

Because $F_{J,i}^{A;\,k}=0$ for $|J,i|\ge k$ it follows $F_J^{A;\,i}=0$ for any J with $|J|\ge 0$ whence $\Delta=\theta-\theta_L=0$ and $\theta=\theta_L$

QED.

The form $\omega_\Lambda = -d\theta_\Lambda$ is called the multisymplectic form of Λ.
The variation according to previous formula is equivalent to the classical one.

Proposition 5. *Let the lagrangian* $\Lambda = L\omega$ *on* $J^k Y$ *and let* $S(\varphi) = \int_{U_X} j^k\left(\varphi \circ \varphi_X^{-1}\right)^* \Lambda$ *the associated action.* Then *at* $\varphi \in C_U^\infty$ *the Euler-Lagrange equations*

$$\sum_{0\le|J|\le k} (-1)^{|J|}\left(\frac{\partial^{|J|}}{\partial x^J}\left(\frac{\partial L}{\partial y_J^A}\left(\varphi \circ \varphi_X^{-1}\right)\right)\right) = 0,\ A = 1,2,..N$$

are equivalent to

$$j^{2k-1}\left(\varphi \circ \varphi_X^{-1}\right)^*\left(i_W\ d\theta_\Lambda\right) = 0$$

for any field W *on* $J^{2k-1}Y$.

Proof. In the proof of proposition 3 we got

$$j^{2k-1}\left(\varphi \circ \varphi_X^{-1}\right)^*\left(i_{j^{2k-1}(V)^v} d\theta_\Lambda\right) = \sum_{A=1}^N \sum_{0\le|J|\le k} (-1)^{|J|}\left(\frac{\partial^{|J|}}{\partial x^J}\left(\frac{\partial L}{\partial y_J^A}\left(\varphi \circ \varphi_X^{-1}\right)\right)\right)\cdot\left(V^v\left(\varphi \circ \varphi_X^{-1}\right)\right) \qquad (15)$$

For a vector field $W = j^{2k-1}V + W_1$ on $J^{2k-1}Y$ with W_1 tangent to the fibers of $\pi_{Y,J^{2k-1}Y}$ we have according to (15) and proposition 4, iv.

$$j^{2k-1}\left(\varphi \circ \varphi_X^{-1}\right)^*\left(i_W\ d\theta_L\right) = j^{2k-1}\left(\varphi \circ \varphi_X^{-1}\right)^*\left(i_{j^{2k-1}V+W_1}\ d\theta_L\right) = j^{2k-1}\left(\varphi \circ \varphi_X^{-1}\right)^*\left(i_{j^{2k-1}V}\ d\theta_L\right)$$
$$= \sum_{A=1}^N \sum_{0\le|J|\le k} (-1)^{|J|}\left(\frac{\partial^{|J|}}{\partial x^J}\left(\frac{\partial L}{\partial y_J^A}\left(\varphi \circ \varphi_X^{-1}\right)\right)\right)\cdot\left(V^v\left(\varphi \circ \varphi_X^{-1}\right)\right)$$

Thanks to arbitrariness of $V^v\left(\varphi \circ \varphi_X^{-1}\right)$ we have the proposition.
QED.

Example. For *k=1, n,N arbitrary*

$$\Lambda = L\left(x^0, x^1,..x^n, y^1, y^2,..y^N, y_0^1, y_1^1,..y_n^N\right)dx^0 \wedge dx^1 \wedge .. \wedge dx^n$$

$$\theta_\Lambda = \sum_{A,j} \frac{\partial L}{\partial y_j^A}\theta_e^A \wedge \omega_j + L\omega = \sum_{A,j} \frac{\partial L}{\partial y_j^A} dy^A \wedge \omega_j + \left(L - \sum_{A,j} y_j^A \frac{\partial L}{\partial y_j^A}\right)\omega$$

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DYNAMICS IN BUCKINGHAM-TYPE PROBLEMS

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Abstract: The force field described by the Buckingham potential models various phenomena from physics, chemistry, astrophysics etc. We present the motion equations and the first integrals of energy and angular momentum of the problem, using McGehee transformations to blow up the collision singularity and to paste the resulting manifold on the phase space. The global flow in the phase plane is depicted for all possible values of the parameters of the potential.
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Keywords: two-body problem, Buckingham potential, global flow, phase portrait.

1. Basic equations

The Buckingham potential is a function proposed by Richard Buckingham in a theoretical study of the equation of state for gaseous helium, neon and argon ([1]). This potential describes the Pauli repulsion energy and van der Waals energy for the interaction of two atoms that are not directly bonded, as a function of the interatomic distance r,

$$U(r) = A\exp(-Br)-(M/(r^6)),$$

where A, B and M are positive constants. The Buckingham potential being central, the associated two-body problem can be reduced to a central-force problem. The motion is confined to a plane. We fix one particle as centre at the origin of this plane and study the relative motion of the other particle.

Denoting the position (or configuration) vector of this particle by $\mathbf{q} = (q_1, q_2)$ and the momentum vector by $\mathbf{p} = \dot{\mathbf{q}}$, $\mathbf{p} = (p_1, p_2)$, the Buckingham potential reads

$$U(\mathbf{q}) = A\exp(-B|\mathbf{q}|)-(M/(|\mathbf{q}|^6)), \quad (1)$$

where A, B and M are positive constants and the kinetic energy of the unit-mass particle is

$$T(\mathbf{p}) = ((|\mathbf{p}|^2)/2).$$

The motion is described by the equations

$$\dot{\mathbf{q}} = ((\partial H)/(\partial \mathbf{p})), \quad \dot{\mathbf{p}} = -((\partial H)/(\partial \mathbf{q})) \quad (2)$$

for the Hamiltonian

$$H(\mathbf{q},\mathbf{p}) := T(\mathbf{p})-U(\mathbf{q}) = ((|\mathbf{p}|^2)/2)-A\exp(-B|\mathbf{q}|)+(M/(|\mathbf{q}|^6)).$$

The phase space is Q×P, where $Q=\mathbb{R}^2-\{(0,0)\}$ is the configuration space and $P=\mathbb{R}^2$ is the momentum space. For given initial conditions, the existence and uniqueness of a real

analytic solution (q_1, q_2, p_1, p_2) of the system are ensured by standard results of the theory of differential equations.

The Hamiltonian function is a first integral of the system of equations, the integral of energy:

$$H(\mathbf{q},\mathbf{p}) = h/2 = \text{const.}, \tag{3}$$

where h is the energy constant.

The field $U(\mathbf{q})$ being central, the angular momentum is conserved, hence we obtain another first integral

$$L(\mathbf{q},\mathbf{p}) = q_1 p_2 - q_2 p_1 = C = \text{const.}, \tag{4}$$

where C stands for the constant of angular momentum.

We observe that the potential $U(\mathbf{q})$, the motion equations and the energy integral have an isolated singularity at the origin $\mathbf{q} = (0,0)$. This singularity corresponds to a collision particle-centre. We shall apply a sequence of McGehee-type transformations of the second kind ([4], [5]) to remove this singularity and to regularize the above differential equations,

$$r = |q|$$

$$\theta = arctg\left(\frac{q_2}{q_1}\right)$$

$$\xi = \dot{r} = \frac{q_1 p_1 + q_2 p_2}{|q|} \quad .$$

$$\eta = r\dot{\theta} = \frac{q_1 p_2 - q_2 p_1}{|q|}$$

We obtain (see [6]):

$$\begin{aligned} \dot{r} &= \xi \\ \dot{\theta} &= \eta / r \\ \dot{\xi} &= ((\eta^2)/r) - ABexp(-Br) + 6M(1/(r^7)) \\ \dot{\eta} &= -((\xi\eta)/r). \end{aligned} \tag{5}$$

We denote

$$x = r^3 \xi, \qquad y = r^3 \eta \,.$$

Rescaling the time through

$$ds = r^{-4} dt$$

and writing $'=d/ds$, the equations of motion become:

$$\begin{aligned} r' &= rx \\ \theta' &= y \\ x' &= 3x^2 + y^2 - ABr^7 \, exp(-Br) + 6M \\ y' &= 2xy. \end{aligned} \tag{6}$$

The first integrals now read respectively

$$x^2 + y^2 = hr^6 + 2Ar^6 \, exp(-Br) - 2M \tag{7}$$

$$y = Cr^2. \tag{8}$$

2. The global flow

This section intends to report some results concerning the global flow for the two-body problem associated to the Buckingham potential (see for details [8], [7], [6]). We shall investigate the global flow for all positive parameters A, B and M of the field, and for all values of the energy constant h. To obtain phase portraits, we reduce the dimension of the phase space from 4 to 2 as follows. The reduction from dimension 4 to 3 is performed by factorizing the flow by S^1 and for the reduction from dimension 3 to 2, we use the fact that the motion equations have two first integrals. Thus, the phase-space structure can be analyzed in (r, x) coordinates. The equation for our analysis is:

$$x^2 = hr^2 + 2Ar^2 \exp(-Br) - C^2 r^2 - 2Mr := f(r). \tag{9}$$

We consider separately the situations $h < 0$, $h = 0$ and $h > 0$. The phase portraits in terms of physical orbits are interpreted.

For the negative-energy case ($h < 0$) we have two situations: if $C \neq 0$ the motion is curvilinear, while $C = 0$ the motion is radial. If $C \neq 0$, there exists a critical energy level $h_c^- = -2A < 0$ that creates two situations: if $h \leq h_c^-$, the motion of the particle in the phase-space is not possible, or if $h > h_c^-$, the phase portrait in the (r, x)-plane is drawn in Fig. 1, where SE represents a stable circular orbit, while the curves that surround it mean periodic or quasiperiodic orbits.

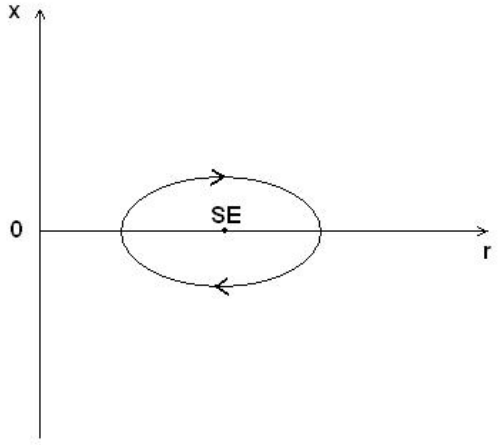

Fig. 1

If $C = 0$, the phase portrait is similar to that plotted in Fig. 1, but SE represents a stable rest, while the curves that surround it mean periodic or quasiperiodic radial oscillatory motion.

For the zero-energy case ($h = 0$), if $C \neq 0$, the phase portrait is identical to that given in Fig 1, with the same physical interpretation as in the negative-energy case. If $C=0$, the phase portrait is identical to the one plotted in Fig 1, and has the physical interpretation for radial motion.

For the positive-energy case ($h > 0$), if $C \neq 0$, we have the following motions which do not encounter collisions (see Fig. 2), where UE is an unstable circular motion and SE is a stable circular motion:

- spiral orbits that come asymptotically from infinity reach a minimum distance from centre, then tend back asymptotically to infinity, denoted by 1 ;

- capture orbits that spiral coming asymptotically from infinity and tending asymptotically to the unstable circular orbit UE, denoted by 2;
- spiral escape orbits that imitate inversely the trajectories 2, denoted by 2';
- bounded trajectory that starts asymptotically from the unstable circular orbit UE, spiral towards the centre, reaches a minimum distance, then spirals back asymptotically towards UE, denoted by 3;
- periodic orbits 4, that close after a finite number of rotations around the centre or quasiperiodic orbits, that never close, filling densely an annulus;
- spiral orbits 5 analogous to the curves 1, but they approach more the centre encompassing the distance of UE and SE, whereas trajectories 1 keep farther than UE and SE.

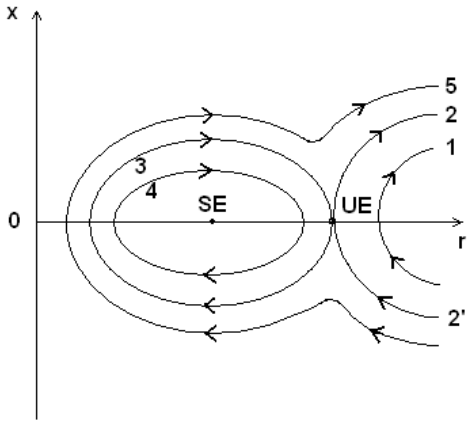

Fig. 2

If $C = 0$ the phase portrait is similar to that plotted in Fig 2, where UE is an unstable rest and SE is a stable rest:

- radial capture/ejection orbits that come asymptotically from infinity, reach a minimum distance from centre, then tend back asymptotically to infinity, on a radial escape path, denoted by 1;
- radial capture orbits 2 that come asymptotically from infinity and tending asymptotically to the unstable rest UE;
- radial escape orbits that imitate inversely the trajectories 2, denoted by 2';
- bounded trajectory 3 that starts asymptotically from the unstable rest UE, radial move towards the centre, reaches a minimum distance, then radial move back asymptotically towards UE;
- radial librations periodic or quasiperiodic (filling densely an segment), denoted by 4;
- radial orbits 5, analogous to the orbits 1, but they approach more the centre encompassing the distance of UE and SE, whereas trajectories 1 keep farther than UE and SE.

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A DEDEKIND STRUCTURE IN AN INFINITE ALGEBRAIC NUMBER FIELD OF DEGREE 2^{∞}

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Abstract. We construct a tower of finite algebraic number fields

$$Q = K_0 \subset K_1 \subset ... \subset K_n \subset K_{n+1} \subset ... \subset K = \bigcup_{n=0}^{\infty} K_n,$$

where $\left[K_{n+1} : K_n\right] = 2, n = 0, 1, ...$ and the integral closure O_K of Z in K is a Dedekind domain with quotient field K

Mathematics Subject Classification (2000): 11R04, 12F04, 13F05.
Key words: algebraic numbers, Dedekind rings, ramification theory, extensions of Dedekind structures.

1. Basic general results on Dedekind structures

For this section we use some facts from [1], [2] and [4].

Let A be a nontrivial unitary commutative ring such that $xy = 0$ implies $x = 0$ or $y = 0$ for any $x, y \in A$. We say in this last situation that A is a *domain*. Any domain A has a quotient field $Q(A) = \left\{ \frac{a}{b} : a, b \in A, b \neq 0 \right\}$. An element $\alpha \in Q(A)$ is said to be *integral over A* if it is a root of a monic polynomial with coefficients in *A*, and the polynomial $f_{\alpha}(X)$ of this type and of the least degree is called the *minimal polynomial of* α. A domain *A* is called *algebraically closed* if there are no other integral elements in $Q(A)$ over *A*, except those of *A*. A subset *M* of A is said to be an *ideal* of *A* if *M* is an additive subgroup of *A* and if *bM* is a subset of *M* for any element *b* of *A*. An ideal is said to be a *maximal ideal* in *A* if it is not equal to *A* and if it is not contained in another proper ideal of *A* distinct of *A*. An ideal *P* of *A* is a *prime ideal* if it is not *A* itself ($1 \notin P$) and if $x, y \in A$ such that $xy \in P$, then $x \in P$ or $y \in P$. Since our ring *A* is a domain, the zero ideal of *A* is a prime ideal. Any maximal ideal is a prime ideal. An ideal *P* is a prime ideal if and only if the residue ring A/*P* is a domain. It is a maximal ideal if and only if A/*P* is a field. We say that an ideal *M* is finitely generated if there exist $a_1, a_2, ..., a_n$ in *M* such that $M = Aa_1 + ... + Aa_n$. If any ideal of *A* is finitely generated, we say that *A* is a Noetherian ring.

Definition 1 Let *K* be a nontrivial (commutative) field and *D* a subring of it. We say that the pair (*D*, *K*) is a *Dedekind structure on K* if: 1) $K = Q(D)$, the quotient field of *D*; 2) *D* is integrally closed; 3) *D* is a Noetherian ring; 4) Any prime ideal of *D* is a maximal ideal.

Remark 1 On any finite algebraic number field we have an infinite number of Dedekind structures. On a finite field or on an algebraically closed field we have no Dedekind structure. This is true because on such last fields we have no discrete valuation and any Dedekind ring is

an intersection of discrete valuation rings [4]. The first statement is true because the integral closure O_K of Z in a finite algebraic number field K ($Q \subseteq K, [K:Q] < \infty$) is a Dedekind ring and any fraction ring of it is again a Dedekind ring [2].

Theorem 1 ([2]) (*D, K*) is a Dedekind structure on *K* if and only if any ideal *M* of *D* can be uniquely written as a product of powers of distinct maximal ideals $P_1, ..., P_k$:

(1) $M = P_1^{e_1} P_2^{e_2} ... P_k^{e_k}$

Theorem 2 ([2]) Let (*D, K*) be a Dedekind structure on a finite field of algebraic numbers and let *L/K* be a finite extension of *K*. Let O_L be the integral closure of *D* in *L*. Then (O_L, L) is a Dedekind structure on *L*. Moreover, if *P* is a maximal ideal in *D* and if :

(2) $PO_L = Q_1^{e_1} Q_2^{e_2} ... Q_t^{e_t}$

is the decomposition of the extended ideal PO_L of P in O_L, then one has:

(3) $n = [L:K] = \sum_{j=1}^{t} e_j f_j$, where $f_j = [O_L / Q_j : D / P]$, $j = 1, ..., t$ (the fundamental equality).

Theorem 3 (Chinese Reminder Theorem (CRT), [1]) Let *A* be a unitary nontrivial commutative ring and $I_1, I_2, ..., I_n$ be a set of comaximal ideals, i.e. $I_i + I_j = A, i \neq j$. Let $x_1, x_2, ..., x_n$ be elements of *A*. Then there exists $x \in A$ such that $x - x_j \in I_j$, $j = 1, ..., n$. We also say $x \equiv x_j \pmod{I_j}$, $j = 1, ..., n$.

2. Some results on finite fields

Let *F* be a nontrivial finite commutative field. Then, the least natural number $p \neq 0$ such that $p \cdot 1 = 0$ is a prime number called the characteristic of *F*. In this case the subset $\{0, 1, ..., p-1\}$ of F is a subfield of *F*, isomorphic to Z / pZ. So $|F| = p^m$, where $m = [F : Z / pZ]$.

Theorem 4 ([1]) Let *F* be a finite field and let $F^* = F \setminus \{0\}$ be the multiplicative subgroup of *F*. Then F^* is a cyclic group, i.e. there exists an element $\xi \in F^*$ such that $F^* = \{1, \xi, ..., \xi^{p^m - 2}\}$, where $|F| = p^m$.

Lemma 5 Let *F* be a finite field with characteristic $p \neq 2$. Then there exists at least one element $\gamma \in F^*$ which is not a perfect square in F^*, i.e. $\gamma \notin F^{*2}$.

Proof It is sufficient to take γ to be the generator of F^* (see Theorem 4). Indeed, if $\gamma = \delta^2, \delta \in F^*$, then $F^* = \{1, \delta, \delta^2, ..., \delta^{2(p^m - 2)}\}$ and $|F^*| = p^m$, a contradiction.

Theorem 3 and Lemma 5 are useful in proving the following basic result.

Theorem 6 Let *A* be a unitary commutative ring and $M_1, M_2, ..., M_k$ be *k* distinct maximal ideals of *A* such that $|A / M_j| < \infty$, and the characteristic of A / M_j is not *2* for any $j = 1, 2, ..., k$. Then there exists at least one element $a \in A$ such that its residue class $\bar{a}$ in any finite field A / M_j, $j = 1, 2, ..., k$ is not a square, i.e. *a* is not a square modulo M_j, $j = 1, 2, ..., k$.

Proof For any $j = 1, 2, ..., k$ we can take $a_j \in A$ such that $\bar{a}_j$, the residue class of a_j modulo M_j is not a square in A/M_j (see Lemma 5). From the CRT (Theorem 3) we can find $a \in A$ such that $a \equiv a_j (\mathrm{mod}\, M_j),\ j = 1, ..., k$. Thus, a cannot be a square modulo $M_j,\ j = 1, ..., k$.

3. Some basic arithmetic

Let us come back to some basic remarks in number theory.

Theorem 7 (The division theorem in Z, [1]) Let $a, b \in Z, b \neq 0$. Then there exists $q, r \in Z, 0 \leq r < |b|$ such that $a = bq + r$.

To prove this basic result, it is sufficient to consider $a > 0, b > 0, a > b$. Then q is the greatest natural number h such that $a - hb \geq 0$ and then take $r = a - qb$. The existence of q is guaranteed by the fact that any nonempty subset of N has a least element (r in our case) or by the axiom of Archimedes for the rational numbers: $Q = \bigcup_{n=0}^{\infty} [Q \cap [n, n+1)]$.

Corollary 8 Any ideal M of Z is generated by the least natural number n_0 of M, i.e. $M = n_0 Z$. In particular, $M = pZ$ is a maximal ideal if and only if p is a prime number. So, $(p, n) = 1$ if and only if $pZ + nZ = Z$, i.e. if and only if $1 = pu + nv, u, v \in Z$.

Theorem 9 (The fundamental theorem of arithmetic-[1]) (Z, Q) is the least Dedekind structure on Q. Thus, any integer n can be uniquely written as: $n = \pm p_1^{e_1} \cdot p_2^{e_2} \cdot ... \cdot p_k^{e_k}$, where $p_1, p_2, ..., p_k$ are distinct prime numbers.

Remark 2 Let (D, K) be a Dedekind structure on a field K and let T be a proper nonempty subset of *Spec(D)*, the set of all the maximal ideals of D. Let $S_T = D \setminus \bigcup_{M \in T} M$ and let $D_{S_T} = \{a/b : b \in S_T, a \in D\}$ be the quotient ring relative to the multiplicative system S_T. Then the pair (D_{S_T}, K) is a "greater" than (D, K) Dedekind structure. The maximal Dedekind structures in K are the pairs (O, M) where O is a discrete valuation ring [1] and M is its maximal ideal. For a given Dedekind structure (D, K), to construct Dedekind structures less than it, is a more difficult task. It is sufficient to find a discrete valuation ring O in K, distinct of any localization of D, relative to a maximal ideal of it, and to take the pair $(D \cap O, K)$. The difficulty consists in proving that this pair is a Dedekind pair. I do not know that this is true in general. In [3] we find a specific treatment of a similar problem. See also a discussion in [4].

4. An infinite extension of a Dedekind structure

Let (D, K) be a Dedekind structure on a field K and let L/K be an algebraic extension of K. We say that a Dedekind structure (E, L) on L *is an extension of* (D, K), or that $(E, L)/(D, K)$ is a *Dedekind extension*, if $D \subseteq E$ and this one is an integral extension of rings, i.e. E is integral over D. If $[L : K] < \infty$ Theorem 2 says that if O_L is the integral closure of D in L, and if (D, K) is a Dedekind structure, then $(O_L, L)/(D, K)$ is the unique Dedekind

extension of (D, K). If $[L:K]=\infty$, this last result is not always true. For instance, the ring of all algebraic integers is not Noetherian, so it is not a Dedekind domain. In the following we construct an infinite algebraic number field *K* of degree 2^{∞} over Q, the rational number field, such that the integral closure O_K of the Dedekind domain $D = Z_{(2)} = \left\{ \frac{a}{2^s} : a \in Z, s \in N \right\}$ in *K*, is a Dedekind domain. This last field *K* will be realized as a union of algebraic number fields $K_n, n = 1, 2, ...$ such that:

(4) $$K_0 = Q \subset K_1 \subset K_2 \subset ... \subset K_n \subset K_{n+1} \subset ... \subset K = \bigcup_{n=0}^{\infty} K_n \subset \overline{Q},$$

where $\overline{Q}$ is the algebraic closure of *Q* in the complex number field *C*, and $[K_{n+1}:K_n]=2$, thus $[K:Q]=2^{\infty}$.

Let $p_2 = 3D, p_3 = 5D, ..., p_nD, ...$ be the sequence of all the prime ideals of *D*. K_1 has a special construction. Namely, let $f_1(X) = X^2 + 1$ and let α_1 be a root of this irreducible (over *Q*) polynomial. Since $f_1(X)$ is irreducible modulo 3, by using the fundamental equality (3) and Proposition 8.2, Ch. II of [2], we see that in O_{K_1}, the integral closure of *D* in $K_1 = Q(\alpha_1)$, the ideal $3O_{K_1}$ is a prime ideal i.e. $t = 1, e_1 = 1, f_1 = 2, n = 2.$ Thus, the ideal p_2 is not ramified and does not split in O_{K_1}. Assume that we have constructed K_n such that $[K_n:Q]=2^n$ and $p_2, ..., p_n$ are not ramified and indecomposable in O_{K_n}. Let M_n be the set of all maximal ideals $P_1, P_2, ..., P_h$ of O_{K_n} which appear in the decompositions of the prime ideals $p_2, ..., p_n, p_{n+1}$ in the Dedekind domain O_{K_n}. Let us choose $a_n \in O_{K_n}$ which is not a square modulo all P_j, $j = 1, 2, ..., h$ (apply Theorem 6 and the fact that the residue ring O_{K_n}/P_j is a finite field as being a finite extension of a field of the form *Z/pZ*, with *p* a prime number in the set $\{p_2, ..., p_{n+1}\}$). If we take $\alpha_{n+1} = \sqrt{a_n}$ and write $K_{n+1} = K_n(\alpha_{n+1})$, we see that the polynomial $X^2 - a_n$ is irreducible modulo P_j, $j = 1, 2, ..., h$, so all ideals $P_1, P_2, ..., P_h$ do not split and do not ramify in $O_{K_{n+1}}$ the integral closure of O_{K_n} in K_{n+1} (see the same Proposition 8.2 cited above).

Thus, $p_2 = 3, ..., p_n, p_{n+1}$ are indecomposable and unramified in $O_{K_{n+1}}$. So, up to now, we know that any prime ideal *M* of O_K, the integral closure of *Z* in $K = \bigcup_{n=0}^{\infty} K_n$ is a maximal ideal because it is "locally" maximal, i.e. its intersection $M \cap O_{K_n}$ with any Dedekind ring O_{K_n} is a maximal ideal in O_{K_n}. Moreover, *M* is of the form $M = P_{n_0} O_K$, where P_{n_0} is a maximal ideal of $O_{K_{n_0}}$. Since P_{n_0} is finitely generated, we see that *M* is finitely generated in O_K. Thus, any prime ideal of O_K is finitely generated. Let us use Lemma 10 bellow and conclude that O_K is a Noetherian ring. So, this last ring is a Dedekind ring.

Lemma 10 Let *A* be a commutative unitary ring with the property that any prime ideal of it is finitely generated. Then *A* is a Noetherian ring, i.e. any other ideal of A is finitely generated.

Proof (some ideas were randomly taken from internet) Let *H* be the set of all non finitely generated ideals of *A*. Let us assume that *H* is not empty. It is easy to see that the union of all

ideals in a totally ordered by inclusion set *S* of ideals in *H* is a maximal element in *S*. Thus, we can apply Zorn Lemma and find a maximal element *M* of *H*. We shall prove that *M* is a prime ideal of *A*. Since $M \in H$, it is not finitely generated, so it is not equal to $A = 1 \cdot A$ itself. Let us assume that *M* is not a prime ideal, i.e. there exist $x, y \in A, x \notin M, y \notin M, xy \in M$. Thus, $M \subset M + Ax$ and $M \subset (M : x)$ because $y \in (M : x)$ and $y \notin M$. Since the last two inclusions cannot be equality and since *M* is maximal in *H*, the ideals $M + Ax$ and $(M : x)$ are finitely generated. Thus, let $x_1, x_2, ..., x_n \in A, y_1, y_2, ..., y_m \in A$ such that $M + Ax = Ax_1 + ... + Ax_n + Ax, (M : x) = Ay_1 + ... + Ay_m$. It is easy to see that we can take $x_1, x_2, ..., x_n \in M$. Let us prove that $M = \sum_{i=1}^{n} Ax_i + \sum_{j=1}^{n} Axy_j$. Since $x_1, x_2, ..., x_n \in M, xy_1, xy_2, ..., xy_m \in M$, it is sufficient to prove that $M \subseteq \sum_{i=1}^{n} Ax_i + \sum_{j=1}^{n} Axy_j$. Let $c \in M, c = \sum_{i=1}^{n} c_i x_i + c_0 x, c_0, ..., c_n \in A$. Since $c_0 x \in M$, $c_0 \in (M : x)$ and so, $c_0 = d_1 y_1 + ... + d_m y_m, d_j \in A, j = 1, 2, ..., m$. Finally, $c = \sum_{i=1}^{n} c_i x_i + d_1 xy_1 + ... + d_m xy_m$, and $M = \sum_{i=1}^{n} Ax_i + \sum_{j=1}^{n} Axy_j$, a contradiction. Thus, *H* is empty.

Remark 3 If all the rings O_{K_n} are principal ideal domains (for instance if they have only a finite numbers of prime ideals – see [4], in the case when instead of *Z* we take a greater Dedekind domain with a finite prime numbers), then O_K is also a principal ideal domain. Indeed, any prime ideal *M* of O_K is an extension of a prime ideal $p_m Z$, where p_m is the *m*-th prime number. Since, starting from O_{K_m} on all the prime ideal *S* of $O_{K_l}, l \geq m$ do not ramify and do not decompose, we can assume that $M = S_m O_K$, where S_m is a maximal ideal in the Dedekind domain O_{K_m}. Since $S_m = \xi_m O_{K_m}$, one sees that $M = \xi_m O_K$, i.e. *M* is a principal ideal. Thus, from the Lemma 11, we see that O_K is a principal ideal domain.

Lemma 11 Let *A* be a commutative unitary ring such that each prime ideal is principal. Then *A* is a principal ideal ring.

Proof (we took some ideas from internet) Let *H* be the set of all non principal ideals. If *H* is non empty, let *M* be a maximal element of *H* (see Zorn Lemma). Since $M \neq A \cdot 1$, we shall prove that *M* is a prime ideal. For this, if *M* is not a prime ideal, let $x \notin M, y \notin M, xy \in M$. Since $M + xA = cA, M : x = Ad$, with $c \in M, d \in A$. We shall prove that *M* is a principal ideal. Let $z \in M, z = ct_1 + xt_2, t_1, t_2 \in A$. Since $xt_2 \in M$, we see that $t_2 \in M : x$, so $t_2 = t_3 d$. Coming back to *z*, we find $z = ct_1 + xt_3 d = ct_1 + ct_4 t_3 d = c(t_1 + t_4 t_3 d)$, i.e. $M = cA$, a contradiction.

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SOME APPLICATIONS OF VECTOR CALCULUS WITH MATLAB

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Abstract. Applications of vector calculus are unlimited, in any engineering problem it will be used the vector calculus. So it is very important for every engineering student to understand very well the vector calculus and the many applications of it. For a profound study, it is a good way to mix the computer science with vector calculus. As a result, some applications of vector calculus made with MatLAB program it will be described in the following study. Some problems and applications of vector calculus will be explained and detailed, like how to find the collinearity or coplanarity between some vectors, how to calculate volumes and areas using the vectorial calculus.
Mathematics Subject Classification (2010): 97R20
Key words: vector calcul, matlab

1. Definition of vectors in MatLAB

In $\Re^2$: $\vec{a} = a_1 \vec{i} + a_2 \vec{j}$ for an orthonormat base $\{ \vec{i}, \vec{j} \}$.

In $\Re^3$: $\vec{a} = a_1 \vec{i} + a_2 \vec{j} + a_3 \vec{k}$ for an orthonormat base $\{ \vec{i}, \vec{j}, \vec{k} \}$.

In MatLAB for a vector it will be associated a line matrix witch will contain the components of the vector for an orthonormat base: in $\Re^2$: $a = [\, a_1 \; a_2 \,]$, in $\Re^3$: $a = [\, a_1 \; a_2 \; a_3 \,]$.

2. Operations with vectors in MatLAB

The sum of vectors: it will be used the sum operation of matrix, because they are define as line matrix.

The multiplication with scalar: it will be used the multiplication with a scalar operation of a matrix.

Example 1: Find the sum of the following vectors: $\vec{a} = 2\vec{i} - 3\vec{j} + 5\vec{k}$; $\vec{b} = \vec{i} - \vec{j} - \vec{k}$, and calculate the vector $2\vec{a}$.

The program sequence in MatLAB is:

```
>> syms a b s
>> a=[2 -3 5]
     a =
   2   -3    5
>> b=[1 -1 -1]
     b =
   1   -1   -1
```

```
>> s=a+b
    s =
3    -4     4
  >> 2*a
    ans =
4    -6    10
```

It can be observed the syntax of the line commands:

- For the variables previously defined it will be used equal sign;
- For calculus it will be no longer use the equal sign, the MatLAB program makes the operations directly.

3. The scalar product in MatLAB

The scalar product of 2 vectors can be made in MatLAB in 2 ways:

- using the command: **dot(a,b)**;
- by multiplying one vector with the other transpose**: a*b'.**

The program sequence in MatLAB for calculus of the scalar product is:

```
>> syms a b
>> a=[2 -3 5]
    a =
    2    -3     5
>> b=[1 -1 -1]
    b =
    1    -1    -1
 >> dot(a,b)
    ans =
         0
  >> a*b'
    ans =
         0
```

4. The norm of a vector in MatLAB

To calculate the norm of a vector in MatLAB it will be used the square root of the scalar product or the command **norm.**

The program sequence in MatLAB for calculate the norm of a vector:

```
>> syms a  normaa
>> a=[2 -3 5]
a =
     2    -3     5
>> normaa=norm(a)
normaa =
   6.1644
>> normaa=sqrt(dot(a,a))
normaa =
   6.1644
```

5. The vector product in MatLAB

The vector product in MatLAB can be calculate by using the command **cross(a,b)** only if the vectors have the dimension minimum 3.

The program sequence in MatLAB for calculate the vector product of 2 vectors is:

```
>> syms a b
>> a=[2 -3 5]
     a =
   2    -3     5
>> b=[1 -1 -1]
     b =
   1    -1    -1
>> cross(a,b)
     ans =
   8     7     1
```

6. The mixed product of vectors in MatLAB

For 3 given vectors, it will be formatted the square matrix with 3 lines and 3 columns, by aligning this 3 given vectors. Then it will be calculated the determinant of this matrix. All this it will made in MatLAB by the command det([a;b;c]).

The program sequence in MatLAB for calculate the mixed product of 3 vectors, is:

```
>> syms a b c
>> a=[2 -3 5]
a =
    2    -3     5
>> b=[1 -1 -1]
b =
    1    -1    -1
>> c=[1 2 3]
c =
    1     2     3
>> det([a;b;c])
ans =
   25
```

7. Collinearity, coplanarity and orthogonality with MatLAB

It can be determinate the collinearity of a two vectors by making the vectorial product of them. The coplanarity of a three vectors can be found from the mixed product. From the calculus of the scalar product of two vectors can determinate the orthogonality of those vectors.

The program sequence in MatLAB is:

```
>> syms a b c
>> a=[2 -3 5]
a =
    2    -3     5
>> b=[1 -1 -1]
b =
    1    -1    -1
>> c=[1 2 3]
c =
    1     2     3
>> if cross(a,b)==0 disp('vectors a and b are collinear')
```

```
else disp('vectors a and b are not collinear');
end
vectors a and b are not collinear
>> if dot(a,b)==0 disp('vectors a and b are orthogonal')
else disp('vectors a and b are not orthogonal');
end
vectors a and b are orthogonal
>> if det([a;b;c])==0 disp('vectors a,b and c are coplanar')
else disp('vectors a,b and c are  not coplanar');
end
vectors a,b and c are  not coplanar
```

8. Applications. Problems for exam of Algebra made in MatLAB.

Problem 1. Let be the points A(3,-1,-1); B(1,1,-1); C(2,0,1); D(-1,1,0).

a) Write the vectors AB, AC, BC, AD;
b) Calculate the perimeter and area of the triangle ABC;
c) Calculate the volume of the tetrahedral ABCD;
d) Calculate the distance from D to the plane (ABC).

The program sequence in MatLAB for solving this problem is:

```
>> syms a b c d AB AC AD BC Area Volume Perimeter Distance
>> a=[3 -1 1]
a =
    3    -1     1
>> b=[1 1 -1]
b =
    1     1    -1
>> c=[2 0 1]
c =
    2     0     1
>> d=[-1 1 0]
d =
   -1     1     0
>> AB=b-a
AB =
   -2     2    -2
>> AC=c-a
AC =
   -1     1     0
>> BC=c-b
BC =
    1    -1     2
>> AD=d-a
AD =
   -4     2    -1
>> Perimeter=sqrt(AB*AB')+sqrt(AC*AC')+sqrt(BC*BC')
```

```
Perimeter =
   7.3278
>> Area=0.5*sqrt(dot(cross(AB,AC),cross(AB,AC)))
Area =
   1.4142
>> Volume=(1/6)*abs(det([AB;AC;AD]))
Volume =
   0.6667
>> Distance=3*Volume/Area
Distance =
   1.4142
```

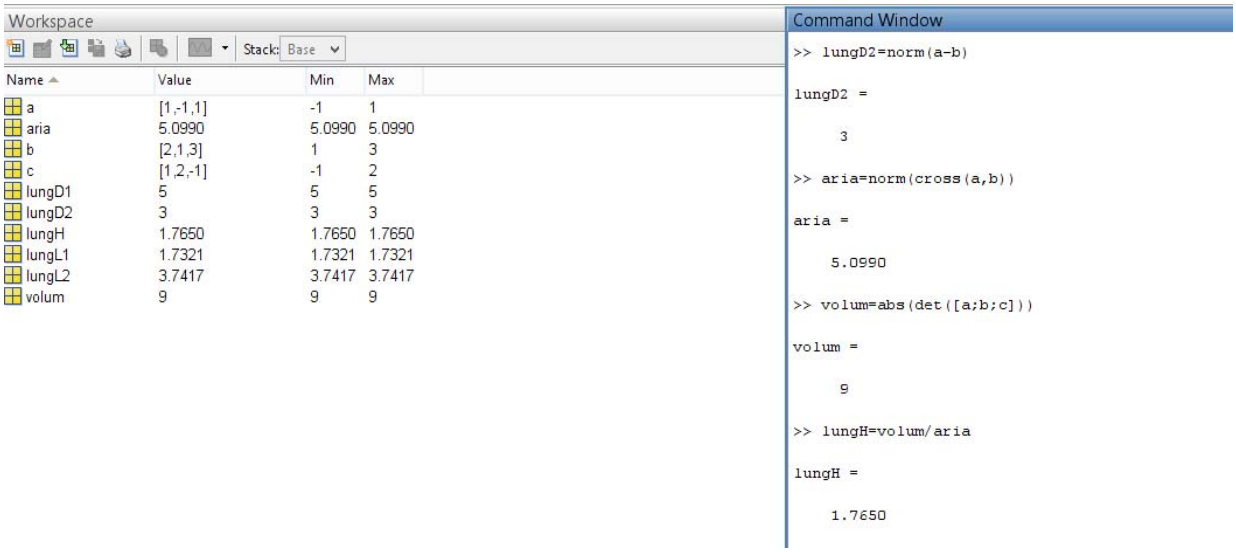
Problem 2. Let be the vectors in $\Re^3$: $\vec{a} = \vec{i} - \vec{j} + \vec{k}$; $\vec{b} = 2\vec{i} + \vec{j} - 3\vec{k}$; $\vec{c} = \vec{i} + 2\vec{j} - \vec{k}$.

a) Set whether the vectors a and b are collinear or orthogonal;
b) Calculate the length of the sides, diagonals and area of the parallelogram built on the vectors a and b;
c) Set whether the vectors a,b and c are coplanar;
d) Calculate the volume of the parallelepiped build on vectors a,b and c, and the length of height corresponding to the base built on vectors a and b.

The program sequence in MatLAB for solving this problem is:

```
>> syms a b c lungL1 lungL2 lungD1 lungD2 aria volum lungH
>> a=[1 -1 1]
a =
    1    -1     1
>> b=[2 1 3]
b =
    2     1     3
>> c=[1 2 -1]
c =
    1     2    -1
>> if cross(a,b)==0 disp('vectors a and b are collinear')
else disp('vectors a and b are not collinear')
end
vectors a and b are not collinear
>> if dot(a,b)==0 disp('vectors a and b are orthogonal')
else disp('vectors a and b are not orthogonal')
end
vectors a and b are not orthogonal
>> lungL1=norm(a)
lungL1 =
   1.7321
>> lungL2=norm(b)
lungL2 =
   3.7417
>> lungD1=norm(a+b)
lungD1 =
    5
```

```
>> lungD2=norm(a-b)
lungD2 =
     3
>> aria=norm(cross(a,b))
aria =
   5.0990
>> if det([a;b;c])==0 disp('vectors a,b and c are coplanar')
else disp('vectors a,b and c are not coplanar')
end
vectors a,b and c are not coplanar
>> volum=abs(det([a;b;c]))
volum =
     9
>> lungH=volum/aria
lungH =
      1.7650
```



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LOW-DIMENSIONAL PARAMETERIZATION OF CHANNELIZED RESERVOIRS USING HIGH ORDER SINGULAR VALUE DECOMPOSITION (HOSVD)

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Abstract. Prior to any estimation process of channelized reservoirs, in the context of an Assisted History Matching method, the parameterization of facies fields is a necessary task. Usually, the parameterization of channelized reservoirs (two facies types) consists of defining a numerical field (parameter field) with the dimension equals to the dimension of the reservoir so that a projection function recovers the facies field from the parameter field. The issue of dimensionality is becoming relevant when the history matching method is applied, especially due to the tremendous number of parameters involved in the estimation process of the channelized reservoirs. In this study, we present a low-dimensional parameterization of channelized reservoirs, obtained after a Tucker decomposition of a tensor using the high order singular value decomposition (HOSVD). We show how the facies fields are fully characterized by a linear combination of a small number of coefficients with basis functions. This parameterization helps further in the estimation process of the facies fields.
Mathematics Subject Classification (2010): 47A75, 15A18, 15A69
Keywords: Parameterization, Tucker decomposition, HOSVD, Channelized reservoirs.

1.Introduction

In the reservoir engineering world, the knowledge of spatial distribution of geological structures is a task that consumes energy, time and especially financial resources for oil companies. When speaking about spatial structures we are thinking about the distribution of geological bodies that compose the geology of reservoir. These geo-bodies are called facies. From all types of facies, one is very hard to be estimated after the primary stage of reservoir geology investigation. This type is called the channel type. The channels are created from old rivers and can have various shapes (meandering, sinuous) and directions. The distribution of channels is modeled using geological simulation methodologies like multipoint geostatistics (MPS, Caers and Zhang 2004) or object-based simulation models (Deutsch and Wang 1996). Any of these methodologies are able to create a realistic distribution of the channels, but the estimation of channel position is the most important issue and consumes a lot of time, energy and money. The estimation process involves the use of mathematics, which means that the spatial distribution of the facies should be linked with numerical fields. This relationship is named parameterization and the numerical field defined to characterize the facies field is called the parameter field. The parameterization of facies fields is very useful in a process named history matching (HM), where information about reservoir data (production data, hard data, seismic data etc.) are used in an inversion technique in order to estimate certain reservoir parameters and obtaining a good data match and predictions. Here we emphasize that the facies field is a field of objects spatially distributed in the reservoir domain and consequently is not a numerical field. Thus, for the estimation of such field a parameterization of it necessary. Of course, one may try to estimate the permeability field or porosity field and, from these fields to further infer the distribution of the facies. However, this has been proven that not always gives good results (Sebacher et al 2015). From this category, we mention the

work of Zhang et al. (2015) where the permeability field is estimated from the production and is projected onto a facies field using a post-processing technique based on wavelet transformation. In Zhao et al. (2016) the permeability field is parameterized with the discrete cosine transform and afterward, the facies fields are estimated using an optimization based post-processing procedure. In Jafarpour and Khodabakhshi (2011) the authors proposed a methodology named probability conditioning method (PCM) where firstly a probability field of the channel is inferred from the estimated permeability field and, secondly the MPS method is conditioned to the estimated probability field for the generation of the updated facies fields. In Lorentzen et al. (2012) the authors propose a parameterization of the facies fields using the distance from the current cell to the border of the channel. If the cell is inside of the channel the distance function is positive and if the cell is outside, the distance is negative.

In this paper, we define a parametrization of the channels using an extension of the singular value decomposition (SVD) method applied to tensors. The SVD method was first used for parametrization of the permeability field in Tavakoli and Reynolds (2011) and further developed with its extension, the principal component analysis (PCA), by many authors. In Insuasty et al. 2017 is presented a parameterization of the permeability field using the high order singular value decomposition (HOSVD, Lathauwer et al. 2000) of the permeability field. The authors parameterise the permeability field with the coefficients of decomposition. In this paper we are using this idea, but not on permeability field but on a parameterization of the permeability field. The parametrization is defined in Sebacher et al. 2015 and is based on a truncation of marginally Gaussian fields with thresholds defined by the normal score transform.

2. High order singular value decomposition (HOSVD) of tensors

The tensors are the natural extension of the vectors and matrices. If $A=\left(a_{ij}\right)_{i,j}\in M_{m,n}\left(\mathbb{R}\right)$ is a matrix with m rows and n columns having real entries, then it defines a bi-linear application $\varphi:\mathbb{R}^m\times\mathbb{R}^n\to\mathbb{R},\varphi\left(x,y\right)=\sum_{i=1}^{m}\sum_{j=1}^{n}a_{ij}x_iy_j$, where $x=\left(x_1,\ldots x_m\right)\in\mathbb{R}^m$ and $y=\left(y_1,\ldots,y_n\right)\in\mathbb{R}^n$. If we calculate the values of application φ in the elements of canonical bases $\left\{e_1^{(1)},e_2^{(1)},\ldots,e_m^{(1)}\right\}\subseteq\mathbb{R}^m$ and $\left\{e_1^{(2)},e_2^{(2)},\ldots,e_n^{(2)}\right\}\subseteq\mathbb{R}^n$ we obtain the entries of matrix A, i.e. $\varphi\left(e_i^{(1)},e_j^{(2)}\right)=a_{ij}$. With this in mind, we extend this approach to multilinear applications. We define a rank k tensor T as a multi-dimensional array (structure) which induces a multi-linear application $\varphi:\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\times\ldots\mathbb{R}^{n_k}\to\mathbb{R}$. If we consider the canonical basis $\left\{e_1^{(i)},e_2^{(i)},\ldots,e_{n_i}^{(i)}\right\}\subseteq\mathbb{R}^{n_i}$ in each of the linear space $\mathbb{R}^{n_i}$, then the element $\varphi\left(e_{i_1}^{(1)},e_{i_2}^{(2)},\ldots,e_{i_k}^{(k)}\right)=T_{i_1i_2\ldots i_k}$ is the entry of tensor T at the position $\left(i_1,i_2,\ldots,i_k\right)$. In addition, we have the relation $\varphi\left(x^{(1)},x^{(2)},\ldots,x^{(k)}\right)=\sum_{i_1,i_2,\ldots,i_k}T_{i_1i_2\ldots i_k}x_{i_1}^{(1)}x_{i_2}^{(2)}\ldots x_{i_k}^{(k)}$.

With this formulation, the arrays become rank 1 tensors and the matrices rank 2 tensors. Similar to singular value decomposition of matrices, it comes to the idea to decompose a tensor based on some directions with weights attached. The Tucker decomposition of a tensor was the first one and consists of decomposition of a rank k tensor T as

$T = \sum_{i_1,i_2,\ldots,i_k} \sigma_{i_1 i_2 \ldots i_k} u_{i_1}^{(1)} \otimes u_{i_2}^{(2)} \otimes \ldots \otimes u_{i_k}^{(k)}$ where $\otimes$ is the outer product of the arrays. The rank k tensor σ is named core tensor and the sets $\left\{u_1^{(r)}, u_2^{(r)}, \ldots, u_{n_r}^{(r)}\right\} \subseteq \mathbb{R}^{n_r}, r \in \overline{1,k}$ are basis in the linear spaces $\left\{\mathbb{R}^{n_r}, r \in \overline{1,k}\right\}$. The HOSVD decomposition is a particular case of Tucker decomposition with the property that the basis $\left\{u_1^{(r)}, u_2^{(r)}, \ldots, u_{n_r}^{(r)}\right\} \subseteq \mathbb{R}^{n_r}, r \in \overline{1,k}$ are orthonormal. The HOSVD methodology (Lathauwer et al. 2000) is the extension of SVD to tensors and can be summarized as follows:

(a) For a rank k tensor T defined on $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \ldots \mathbb{R}^{n_k}$, for each $r \in \overline{1,k}$,calculate the mode r flattening T_r , which is the matrix of dimension $n_r \times \left(\prod_{i \neq r} n_i\right)$ that corresponds to T.

(b) For each matrix T_r, calculate its SVD decomposition keeping its left singular vectors $\left\{u_1^{(r)}, u_2^{(r)}, \ldots, u_{n_r}^{(r)}\right\} \subseteq \mathbb{R}^{n_r}$ .In particular, it defines an orthonormal basis in $\mathbb{R}^{n_r}$.

(c) Calculate the entries $\sigma_{i_1 i_2 \ldots i_k} = T(u_{i_1}^{(1)}, u_{i_2}^{(2)}, \ldots, u_{i_k}^{(k)})$ for each sets $\left\{i_1, i_2, \ldots, i_k\right\}$.

Then, we have the Tucker decomposition of tensor $T = \sum_{i_1,i_2,\ldots,i_k} \sigma_{i_1 i_2 \ldots i_k} u_{i_1}^{(1)} \otimes u_{i_2}^{(2)} \otimes \ldots \otimes u_{i_k}^{(k)}$

We mention that in this summation the index $i_r \in \overline{1,n_r}, \forall\, r \in \overline{1,k}$. If we truncate the sum taking only first $\overline{n_r} < n_r, \forall\, r \in \overline{1,k}$ indexes we obtain a low-rank approximation of T which is the rank k tensor $\overline{T} = \sum_{i_1=1}^{\overline{n_1}} \sum_{i_2=1}^{\overline{n_2}} \ldots \sum_{i_k=1}^{\overline{n_k}} \sigma_{i_1 i_2 \ldots i_k} u_{i_1}^{(1)} \otimes u_{i_2}^{(2)} \otimes \ldots \otimes u_{i_k}^{(k)}$ and $T - \overline{T} = \overline{\overline{T}}$ is its approximation error. The size of approximation error can be quantified based on Frobenius norm $\left\|\overline{\overline{T}}\right\|_F^2 = \sum_{i_1=\overline{n_1}+1}^{n_1} \ldots \sum_{i_k=\overline{n_k}+1}^{n_k} \sigma_{i_1 \ldots i_k}^2$ knowing that the bases $\left\{u_1^{(r)}, u_2^{(r)}, \ldots, u_{n_r}^{(r)}\right\}$ are orthonormal.

3. Low dimensional parameterization of channelized reservoirs using HOSVD

We present the methodology for a channelized reservoir with a square domain of 100 grid cells in each direction. The geological simulation model is a particular multipoint geostatistical model named single normal equation simulation (SNESIM) which uses a training image (Fig. 1) from where plausible channelized reservoirs are simulated. All simulations keep the multipoint geostatistical properties of the training image. In Figure 1 (right) are presented four of them. We start by generating an ensemble of 120 channelized reservoirs from the training image (the prior in our case). At this point, each member of the ensemble is an image (facies field) of two objects: a channel and a background (non-channel). We want to define a numerical field on the same domain (10000 cells) and a rule so the original facies field is reconstructed from the associated parameter field.

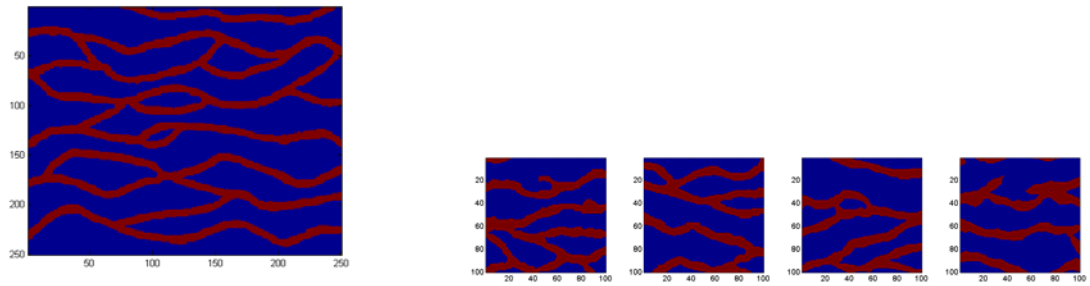

Figure 1: The Training Image (left) and four channelized reservoirs (right)

For each cell cell *j* of the domain we calculate, from the ensemble, the probability of occurrence of the channel at that location and we denote it with p^j. Thus, we define a binary random variable, denoted $facies_j$ expressing the occurrence of the channel at the grid cell *j*.

$$facies_j = \begin{pmatrix} channel & non-channel \\ p^j & 1-p^j \end{pmatrix} \quad (1)$$

We link this random variable with the standard normal variables by the means of normal score transform (Fig. 2 right). In this way is defined a threshold β^j, so that if $X \sim N(0;1)$ then $P(x < \beta^j) = p^j$ and $P(x \geq \beta^j) = 1 - p^j$. We define on the reservoir domain a parameter field denoted ω having real values as:

$$\omega_i(j) = \begin{cases} E\left(X \,|\, X < \beta_j\right), \text{cell } j \in \text{channel} \\ E\left(X \,|\, X \geq \beta_j\right), \text{cell } j \notin \text{channel} \end{cases} \quad \text{where } X \sim N(0;1). \quad (2)$$

Here *i* represent the index for ensemble members, so we have 120 parameter fields $\left(\omega_i\right)_{i=\overline{1,120}}$

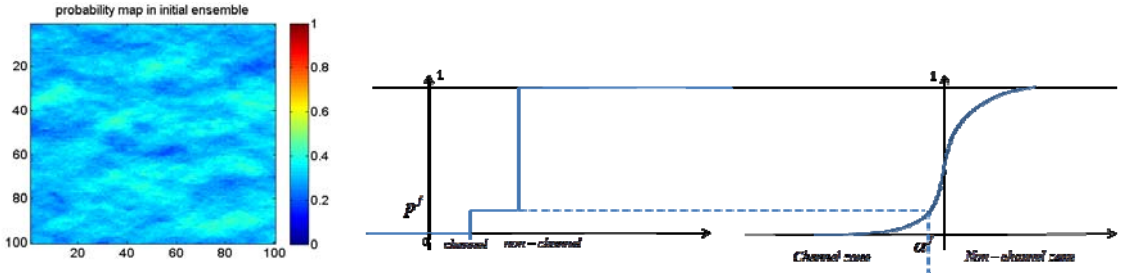

Figure 2: Probability field of channel (left) and the normal score transform (right)

An example of one parameter field is presented in Figure 4 (third picture) and represents the parametrization of the facies field from the same figure (first picture).

Let's define the rank 3 tensor T of size $100 \times 100 \times 120$, $T(:,:,i) = \omega_i, i = \overline{1,120}$. By applying the HOSVD decomposition for this tensor $T = \sum_{i=1}^{100}\sum_{j=1}^{100}\sum_{k=1}^{120} \sigma_{ijk} u_i^{(1)} \otimes u_j^{(2)} \otimes u_k^{(3)}$ where $\left(\sigma_{ijk}\right)_{i,j,k}$ is the core tensor and the sets $\left(u_i^{(1)}\right)_{i=\overline{1,100}}, \left(u_j^{(2)}\right)_{j=\overline{1,100}}$ and $\left(u_k^{(3)}\right)_{k=\overline{1,100}}$ are orthonormal vectors (basis). We aproximate the tensor T with the tensor $\overline{T} = \sum_{i=1}^{nx}\sum_{j=1}^{ny}\sum_{k=1}^{120} \sigma_{ijk} u_i^{(1)} \otimes u_j^{(2)} \otimes u_k^{(3)}$ by keeping only first n_x and n_y elements from the sets of

vectors $\left(u_i^{(1)}\right)_{i=\overline{1,100}}$ and $\left(u_j^{(2)}\right)_{j=\overline{1,100}}$ (*Ox* and *Oy* directions). Let' define new parameter fields based on tensor $\overline{T}$. For each $r \in \{1,\ldots,120\}$ we define the bi-dimensional random fields $\overline{\omega_r}$ with the relation (3). From this relation it can be observed that each new parameter field $\overline{\omega_r}$ is a linear combination of the outer products $\left(u_i^{(1)} \otimes u_j^{(2)}\right)_{i,j}$ with the coefficients $\left(\alpha_{ij}^r\right)_{i,j}$. This means that, considering fixed the set $\left(u_i^{(1)} \otimes u_j^{(2)}\right)_{i,j}$, the coefficients $\left(\alpha_{ij}^r\right)_{i,j}$ can be seen as a parameterization of the new parameter field.

$$\overline{\omega_r} = \overline{T}(:,:,r) \equiv \overline{T}(:,:,e_3^{(r)}) = \sum_{i=1}^{nx}\sum_{j=1}^{ny}\sum_{k=1}^{120}\sigma_{ijk}u_i^{(1)} \otimes u_j^{(2)} \left\langle u_k^{(3)}, e_r^{(3)} \right\rangle =$$

$$\sum_{i=1}^{nx}\sum_{j=1}^{ny}\underbrace{\left(\sum_{k=1}^{120}\sigma_{ijk}\left\langle u_k^{(3)}, e_r^{(3)}\right\rangle\right)}_{\alpha_{ij}^r} u_i^{(1)} \otimes u_j^{(2)} = \sum_{i=1}^{nx}\sum_{j=1}^{ny}\alpha_{ij}^r u_i^{(1)} \otimes u_j^{(2)} \quad (3)$$

A visual representation of the sum from eq.3 is presented in Fig.3. Here is shown an example of a particular $\overline{\omega_r}$ field and its decomposition based on eq.3 . In this example n_x=30 and n_y=15.

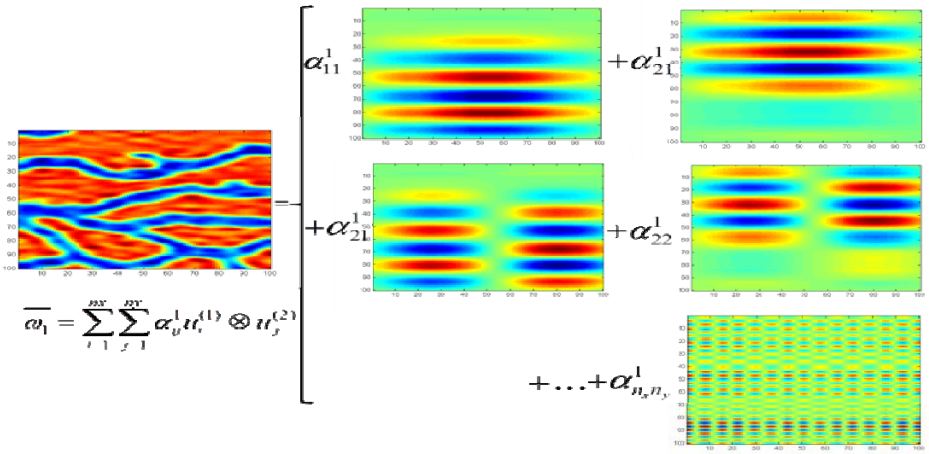

Figure 3: Truncated HOSVD decomposition

The field $\overline{\omega_r}$ is a new bi-dimensional parameter field, which approximates the original parameter field ω_r. If we truncate the field ω_r with the thresholds $\left(\beta^j\right)_{j=\overline{1,10000}}$ we obtain the original facies field (the one simulated from the training image). If we apply the same truncation to the new field $\overline{\omega_r}$ we do not obtain the original facies field if $n_x \neq 100$ or $n_y \neq 100$. However, if we set an acceptance threshold like 95% between the original facies field and the facies field obtained after the truncation of $\overline{\omega_r}$ we can consider that the plausibility from the geologic point of view is fullfilled. This criterion can be used for the setting of n_x and n_y. In our case, we observed that for n_x=30 and n_y=15 95% of the grid cell are similar between facies fields. In Figure 4 are presented both facies fields (original and

approximation) and both parameter fields (original and approximation). From this picture can be seen that the multipoint geostatistics is kept which means that with this parameterization we are keeping the prior information. In addition, if the original field has 10000 parameters to be estimated, with the new parameterization we have to estimate only $n_x \times n_y = 450$ parameters. In our example, this is a reduction of 95.5% which for large cases might be very important from the computational point of view.

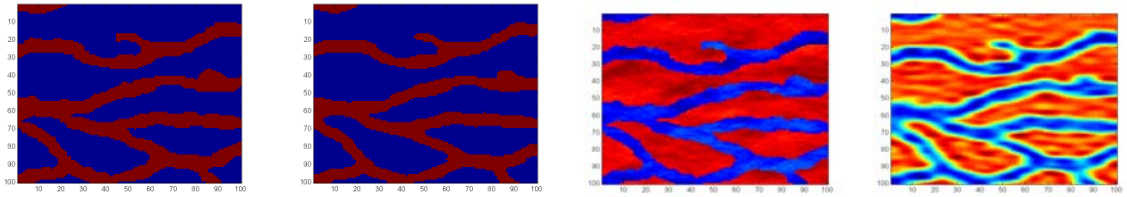

Figure 4: Trom left to right: original field, field after truncation of tensor aproximation, original parameter field, parameter field after tensor aproximation

4. Conclusions

In this paper is presented a novel parameterization of channelized reservoirs using the high order singular value decomposition (HOSVD) of tensors. Firstly, we parameterize the facies fields with random fields, marginally Gaussian, of which truncation with reliable thresholds yields the original facies fields. Secondly, we define a rank 3 tensor based on the parameter field. The parameterization consists of the first significant coefficients of HOSVD decomposition of the tensor. In this way, we reduce the dimensionality of parametrization while keeping the relevant information about the continuity of the channels.

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ON A TYPE I GROUP TRASFORMATION ALGEBRA

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Abstract: We establish the conditions under the group transformation C^*-algebra is a type I C^*-algebra. Two examples of a type I group transformation C^*-algebra are given.
Mathematics Subject Classification (2010): 46L05, 46L35, 47L65
Key words: *C**-algebras, crossed product algebras, type I algebras

1. Introduction

An involutive algebra *A* is a type I algebra if, for every representation π of *A*, the Von Neumann algebra generated by $\pi(A)$ is a type I Von Neumann algebra. According to Dixmier ([3], 9.5.9), a C^*-algebra *A* is a type I algebra if for every irreducible representation π of *A*, $\pi(A)$ contains the set of compact operators on H_π.

If $G \times X$ is a locally compact, second countable group transformation, we can associate, via left multiplication, the *C**-dynamical system $(C_0(X), G, lt)$, $lt : G \to Aut(C_0(X))$ $lt_g f(x) = f(g^{-1}x)$. We denote by $C_0(X) \times_{lt} G$ the crossed product C^*-algebra associated to dynamical system $(C_0(X), G, lt)$.

In [1], Propositon 7.29, D.P. Williams establishes in which conditions $C_0(X) \times_{lt} G$ is type I C^*-algebra. For that we recall that a group *G* is a type I group if C^*-group algebra is a type I C^*-algebra and ,in the following definitions, we state the notion of regular dynamical system. We note that, if (A, G, α) is a dynamical system, exists a natural action of *G* on *Prim*(*A*)(the set of primitive ideals of *A*), $g \cdot P := \{\alpha_g(a) / a \in P\}$.

Definition 1.1. *A dynamical system* (A, G, α) *is quasi-regular, if for every irreducible representation* π *of A, exists* $P \in Prim(A)$, *such that* $\ker \pi = \bigcap_{g \in G} g \cdot P$.

Definition 1.2. *A dynamical system* (A, G, α) *is regular, if the following conditions are simultaneously accomplished:*

a) $G \cdot P = \bigcup_{g \in G} g \cdot P$ *is locally closed in* $Prim(A)$, *which means that* $G \cdot P$ *is open in its closure* $\overline{G \cdot P}$;

b) the map $g \cdot G_P \to g \cdot P$ *defines a homeomorphism from* G/G_P *to* $G \cdot P$, *where* G_P *is the stability group of P,* $G_P = \{g \in G / g \cdot P = P\}$;

c) (A, G, α) *is quasi-regular.*

Proposition 1.3. ([1], 7.29)**.** *If* $G \times X$ *is a group transformation such that C*-dynamical system* $(C_0(X), G, lt)$ *is regular and if every stability group* G_x *is a type I group for every* $x \in X$, *the crossed product* C^**-algebra* $C_0(X) \times_{lt} G$ *is a type I .* C^**-algebra*

2. The main results and examples

Proposition 2.1. *C*-algebras* $C^*(G, X)$ *and* $C_0(X) \times_{lt} G$ *are isomorphic.*

Proof. We define $T : C_c(G \times X) \to C_c(G, C_0(X))$, $(Tf)(g)(x) = f(g, x)$ and we prove that T keeps the convolution and involution from C*-algebra structure of $C_c(G \times X)$ and $C_c(G, C_0(X))$

$$(Tf_1 * Tf_2)(g)(x) = \int_G Tf_1(h)(x) lt_h(Tf_2(h^{-1}g))(x) d\lambda(h) =$$

$$= \int_G f_1(h, x) Tf_2(h^{-1}g)(h^{-1} \cdot x) d\lambda(h) = \int_G f_1(h, x) f_2(h^{-1}g, h^{-1} \cdot x) d\lambda(h) =$$

$= (f_1 * f_2)(g, x) = T(f_1 * f_2)(g)(x)$ for every $g \in G, x \in X$, $f_1, f_2 \in C_c(G \times X)$.

$$T(f)^*(g)(x) = \Delta(g^{-1}) lt_g(T(f)(g^{-1})^*)(x) = \Delta(g^{-1}) T(f)(g^{-1})^*(g^{-1}x) =$$

$= \Delta(g^{-1}) \overline{f(g^{-1}, g^{-1}x)} = f^*(g, x) = T(f^*)(g)(x)$ for every $g \in G, x \in X$, $f \in C_c(G \times X)$.

Moreover, if $f_n, f \in C_c(G \times X)$, $f_n \to f$ in inductive limit topology, $(Tf_n)(g)(x) = f_n(g, x) \to f(g, x) = (Tf)(g)(x)$ .It follows that T is a continuous $*$-homomorphism. Since sets $C_c(G \times X)$, $C_c(G, C_0(X))$ are dense in $C^*(G, X)$, $C_0(X) \times_{lt} G$ respectively, T can be extended to a continuous $*$-homomorphism from $C^*(G, X)$ to $C_0(X) \times_{lt} G$.

Similarly, we define $U : C_c(G, C_0(X)) \to C_c(G \times X)$,

$$(Uf)(g, x) = f(g)(x), \forall f \in C_c(G, C_0(X)), g \in G, x \in X .$$

$$(Uf_1 * Uf_2)(g, x) = \int_G Uf_1(g_1, x) Uf_2(g_1^{-1}g, g_1^{-1}x) d\lambda(g_1) =$$

$$= \int_G f_1(g_1)(x) f_2(g_1^{-1}g)(g_1^{-1}x) d\lambda(g_1) = \int_G f_1(g_1)(x) lt_g(f_2(g_1^{-1}g)(x) d\lambda(g_1) =$$

$$= U(f_1 * f_2)(g, x) .$$

$$(Uf)^*(g, x) = \Delta(g^{-1}) \overline{Uf(g^{-1}, g^{-1}x)} = \Delta(g^{-1}) \overline{f(g^{-1})(g^{-1}x)} =$$

$$= \Delta(g^{-1}) lt_g(f(g^{-1})^*)(x) = f^*(g)(x) = (Uf^*)(g, x) .$$

Moreover $\int_G /Uf(g,x)/\,d\lambda(g) = \int_G / f(g)(x)/\,d\lambda(g) \le \int_G // f(g)//_\infty\, d\lambda(g) \le // f //_1$. *U* is a continuous homomorphism relative to the topology of norm $//\cdot//_1$ on $C_c(G, C_0(X))$ and inductive limit topology on $C_c(G\times X)$. Consequently, *U* can be extended to a continuous $*$-homomorphism. from $C_0(X)\times_{lt} G$ to $C^*(G,X)$. Obviously, on dense sets, *T* is the inverse operator of *U,* and conversely. In conclusion, $C^*(G,X)$ and $C_0(X)\times_{lt} G$ .are isomorphic *C**-algebras.

Corollary 2.2. *If* (G,X) *is a second countable, locally compact group transformation, the orbit space* $G\setminus X$ *is a topologically space such that different points have different closures and, if, for every* $x\in X$ *every stability group* G_x *is type I, then* $C^*(G,X)$ *is a type I. C*-algebra*

Proof According to Proposition 1.3, if, for every $x\in X$, G_x is a type I group, then the crossed product $C_0(X)\times_{lt} G$ is a type I *C**-algebra.. The conclusion of the corollary results from the isomorphism of $C^*(G,X)$ with $C_0(X)\times_{lt} G$ (Proposition 2.1)

Examples

1) We consider the aditive group $G = \boldsymbol{R}^2$ acting on the set of real numbers $\boldsymbol{R}$ in this way:

$$(s,t)\cdot r = \begin{cases} e^s r & if\ \ r>0 \\ 0 & if\ \ r=0 \\ e^t r & if\ \ r<0 \end{cases},$$

The stability groups of this action are $G_r = \{0\}\times \boldsymbol{R}$, if $r<0$, $G_r = \boldsymbol{R}\times\{0\}$ if $r>0$ and $G_0 = \boldsymbol{R}^2$. If $r\neq 0$, the stability groups G_r can be identified with the set of real numbers $\boldsymbol{R}$. Since $C^*(\boldsymbol{R})$ is isomorphic with $C_0(\boldsymbol{R})$, via Fourier transform, the aditive group of real numbers is a type I. Moreover, G_0 is a classical example of a type I group. The reunion of orbits from $\boldsymbol{R}$ is $\boldsymbol{R}$, and $\boldsymbol{R}$ is a Hausdorff space. Using Corollary 2.1, $C^*(G,\boldsymbol{R})$ is a type .I *C**-algebra. We note that in [4], it is proved that $C^*(G,\boldsymbol{R})$ is a non continuous trace *C**-algebra.

2) If we consider the multiplicative group $G = \boldsymbol{R}-\{0\}$ and the topological space $\boldsymbol{R}^2$, $r\cdot(a,b) = (\frac{a}{r},\frac{b}{r})$ defines an action of G on $\boldsymbol{R}^2$. If the pair $(a,b)\neq(0,0)$, the stability group $G_{(a,b)} = \{r\in G / r\cdot(a,b) = (a,b)\}$ contains only one element, neutral element $\{1\}$. $G_{(0,0)} = \{r\in G / r\cdot(0,0) = (0,0)\} = G$. Obviously, all the stability groups are type I, the orbit space is $\boldsymbol{R}^2$, which is a Hausdorff space. In conclusion, using Corollary 2.1 , $C^*(G,\boldsymbol{R}^2)$ is a type .I *C**-algebra.

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TRIGONOMETRIC FORMULAS FOR SOLVING EULER SPHERICAL TRIANGLES WITH MATHCAD

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Abstract: In planar geometry, to connect three points on a plane using the shortest path, we would draw straight lines and hence create a triangle. By analogy, if we wish to connect three points on the surface of a sphere using the shortest possible route, we would draw arcs of great circles and hence create a spherical triangle. Since the Earth is approximately a sphere, the study of the relations existing among the sides and angles of a spherical triangle is very practical, for example, because we live on a sphere. Manny practical applications from geodesy, astronomy, navigation, architecture require spherical triangle solutions.
Mathematics Subject Classification (2010): 51M04, 14J29.
Key words: sphere; great circle; small circle; Euler spherical triangle; solution of spherical triangle; MathCad

1. Introduction

Unlike other engineering maths software tools, many professors from a variety of disciplines choose MathCad because it features an "easy-to-use interface", allowing users to combine standard math notations, graphs, images and text into a single, interactive, professionally presented document. We have found that MathCad can fundamentally change the way we teach mathematics. A technical program like MathCad is particularly important because students need to understand the symbolism that is used to make the abstract concepts of geometry (point, lines, circles, planes and surfaces etc.) observable and measurable.

The present paper is constructed to show how MathCad can be used as an alternative teaching and learning tool to explore a chapter of Geometry, "Spherical Trigonometry".

2. Geometry of the sphere. Background notions

On beginning the work, we review briefly some of the basic concepts and notations of solid geometry theory. We hope that this short overview will be further useful even for the less experienced reader.

Spherical trigonometry is a branch of trigonometry that concerns with triangles extracted from a spherical surface and is the spherical analogue to the planar geometry. A *spherical surface* is a surface of which all the points are equidistant from a fixed interior point called the *center.* A *sphere* is a solid bounded by a spherical surface. The *radius* of a sphere is a straight line joining the center to any point of the spherical surface. The straight line in a sphere that passes through the center of the sphere and terminates both ways by the surface is the *diameter*; the diameter length is twice the length of the radius of the sphere. Two points on the sphere on opposite sides of a diameter are known as *antipodal points*.

In analytic geometry, a sphere with center denoted by $O(a, b, c)$ and radius R is the locus of all points (x, y, z) such that: $(x-a)^2+(y-b)^2+(z-c)^2=R^2$.

2.1. The sphere and its plane sections. Great and small circles on a sphere

A *plane section* of a sphere is a figure whose boundary is the intersection of a plane and the surface of the sphere.

If we consider an arbitrary plane and any sphere, there are three possible situations of intersection between them: the sphere and the plane miss each other, only one tangential contact in which case the plane is known as *tangent plane* and an infinite number of points lying in a circle. Every section of the surface of a sphere made by a plane cutting it, is a *circle*. If the cutting plane contains the center of the sphere, the plane section is called a *great circle*, and respectively, a *small circle* if the plane does not pass through the centre of the sphere. Therefore, all great circles have evidently the same center and the same radius as its sphere. As in plane geometry, an arc of a great circle is measured by the angle subtended at the center of the sphere and it is expressed in degrees, minutes, and seconds.

If we choose two different points A and B on the surface of a sphere with center O and radius R, there are two possibilities:

1) If A and B are not antipodal points, then there is only one great circle that passes through both points. The two points divide the great circle into two arcs of differing length. The length of the shorter arc is the shortest distance (on the sphere) between the two points.

2) If A and B are antipodal points, then an infinite number of great circles containing A and B can be found. The two points separate the great circles into arcs of the same length πR.

The vector equation of a great circle on a sphere with center O and radius R passing through two not antipodal points A and B is written as ([1]):

$$\overrightarrow{g(t)} = O + \cos(t)\cdot\overrightarrow{OA} + \sin(t)\cdot\vec{u}, \text{ where } \vec{u} = \frac{R}{\left\|\vec{w}\right\|}\vec{w} \text{ and } \vec{w} = \overrightarrow{OB} - \frac{\overrightarrow{OA}\cdot\overrightarrow{OB}}{R^2}\overrightarrow{OA}.$$

On a surface, the curves that minimize the path between points are called *geodesics*. Great circles become more important when we realize that the shortest distance between any two points on a sphere lies always along the great circle which joins the points; great circles define geodesics on a sphere.

A small circle always has smaller radius than its sphere, and an arc of small circle does not represent the shortest route between two points on a spherical surface.

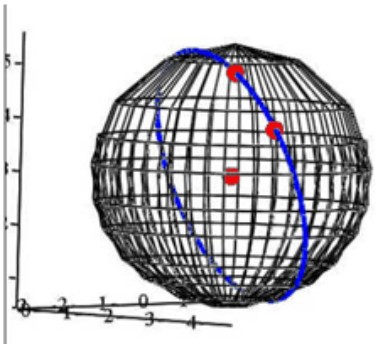

Fig. 2.1. Great circle

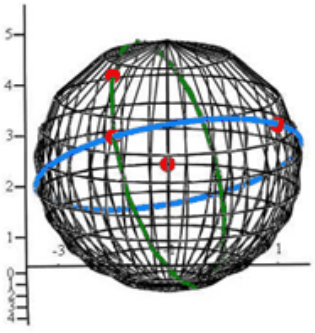

Fig. 2.2. Two great circles

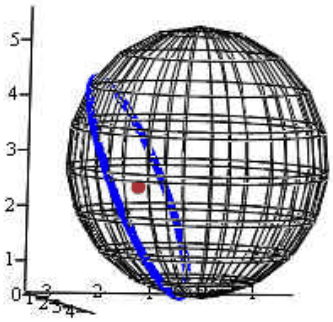

Fig. 2.3. Small circle

2.2. Spherical triangles on a sphere

A *spherical angle* on the surface of a sphere is an angle formed by the intersection of two arcs of great circles. A *spherical triangle* is a figure on the surface of a sphere comprised between three arcs of intersecting great circles. As in the case of planar triangle, the vertices (the angles) of a spherical triangle are generally denoted by the letters A, B, and C and the sides (the arcs of great circles) opposite to them by the letters a, b and c, respectively. The measure of an angle of a spherical triangle is that of the plane angle formed by the two tangents to the sides of the angle at its vertex. The magnitudes of the sides of a spherical triangle, as well as the measures of the angles, are all expressed in angular measure (degrees-minutes-seconds, radians) and not in linear measure (metres, kilometres).

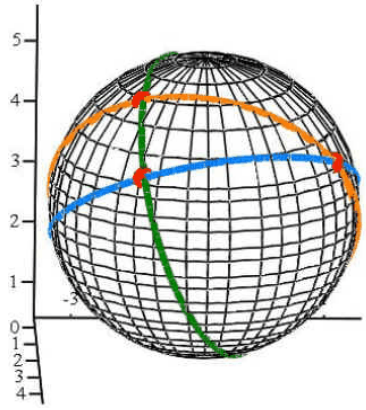

Fig. 2.4. Spherical triangle

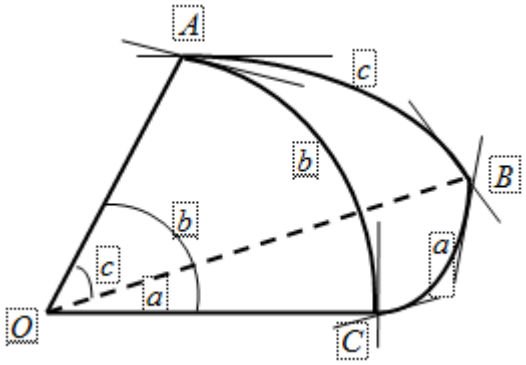

Fig. 2.5. The six elements of spherical triangle

In what follows, we shall consider only *Euler spherical triangles*, each of whose sides and each of whose angles is less than 180°.

To avoid ambiguities, a triangle formed on the surface of a sphere is a spherical triangle if are established all of the following conditions: the three sides are all arcs of great circles, any one side is less than the sum of all the others, the sum of the three angles add up to more than 180°, each individual spherical angle, as well as each side, is less than 180°.

Spherical triangles classifications:

- *right-angled triangle* (in which an angle equals to 90°); it is possible for more than one angle to be equal to 90°;
- *quadrantal triangle* (in which a side equals to 90°); it is possible for more than one side to be equal to 90°;
- *oblique triangle* (which is not a right-angled or a quadrantal one);
- *isosceles triangle* (in which is two sides and two angles are equal);
- *equilateral triangle* (in which all sides and all angles are equal).

For a Euler spherical triangle ABC on the surface of a sphere with radius R, the following important properties hold:

1) $a, b, c < 180^\circ$; $A, B, C < 180^\circ$
2) $a + b > c,\ a + c > b,\ b + c > a$
3) $0^\circ < a + b + c < 360^\circ$
4) $A + B < 180^\circ + C,\ A + C < 180^\circ + B,\ B + C < 180^\circ + A$
5) $180^\circ < A + B + C < 540^\circ$; $180^\circ < A + B + C < 360^\circ$, if the triangle is right-angled
6) If two sides are equal, the opposite angles are equal; and conversely
7) If two sides are unequal, the opposite angles are unequal, and the greater side is opposite the greater angle; and conversely
8) In the triangle is isosceles, the angles at its base are equal
9) $\text{Area}\left(\triangle ABC\right) = \left(A + B + C - 180^\circ\right) \cdot \dfrac{\pi R^2}{180^\circ}$.

Two spherical triangles are said to be *symmetrically equal* when each of the six elements of one are respectively equal to each of the six elements of the other. Because spherical triangles lie on the surface of a sphere, and are hence three dimensional, symmetrically equal does not necessarily mean congruent. Two spherical triangles are *congruent* only if they can be applied one to the other so as to coincide.

Two spherical triangles situated on the same sphere, or on equal spheres, are symmetrically equal under the conditions:

- three sides of one are correspondingly equal to the three sides of the other;
- three angles of one are respectively equal to the three angles of the other;
- two sides and their included angle of one are respectively equal to two sides and included angle of the other;
- two angles and their included side of one are respectively equal to two angles and included side of the other.

3. Solution of spherical triangles

The spherical trigonometry has for its object the study of the relations existing among the sides and angles of a spherical triangle.

When any three parts of a spherical triangle $A\,BC$ are given, the other remaining parts can be computed using various formulae of spherical trigonometry, which are given below. The solution of spherical triangle involves six cases, in which the given elements are respectively:

Case 1 (SSS): Three sides;
Case 2 (AAA): Three angles;
Case 3 (SAS): Two sides and their included angle;
Case 4 (ASA): Two angles and their included side;
Case 5 (SSA): Two sides and the angle opposite one of them;
Case 6 (AAS): Two angles and the side opposite one of them.

The Spherical Law of Cosines for the sides (are relations between the three sides and an angle, [3]):

$$\cos(a)=\cos(b)\cos(c)+\sin(b)\sin(c)\cos(A)$$

$$\cos(b)=\cos(a)\cos(c)+\sin(a)\sin(c)\cos(B)$$

$$\cos(c)=\cos(a)\cos(b)+\sin(a)\sin(b)\cos(C).$$

The Spherical Law of Cosines for the angles (are relations between the three angles and a side, [3]):

$$\cos(A)=-\cos(B)\cos(C)+\sin(B)\sin(C)\cos(a)$$

$$\cos(B)=-\cos(A)\cos(C)+\sin(A)\sin(C)\cos(b)$$

$$\cos(C)=-\cos(A)\cos(B)+\sin(A)\sin(B)\cos(c).$$

The Spherical Law of Sines (are relations between two angles and tow sides, [3]):

$$\frac{\sin(a)}{\sin(A)}=\frac{\sin(b)}{\sin(B)}=\frac{\sin(c)}{\sin(C)}.$$

Observation 3.1. When an element of a spherical triangle is found by means of the law of sines (case 5 and case 6), there is often some difficulty in determining whether the element found is of the first quadrant, or of the second quadrant, or if we have double solutions, or if we have no solution. However, the following theorems from solid geometry will often enable the computer to determine the quadrant.

Theorem 3.1. ([3]) The order of magnitude of the sides of a spherical triangle is the same as that of their respective opposite angles; or, if $a<b<c$, then $A<B<C$.

Theorem 3.2. ([3]) The sum of two sides of a spherical triangle is greater than the third.

The Cotangent four-part formulae (are relations between two angles and tow sides, [3]):

$$\sin(A)\operatorname{ctg}(B)=\sin(c)\operatorname{ctg}(b)-\cos(c)\cos(A)$$

$$\sin(A)\operatorname{ctg}(C)=\sin(b)\operatorname{ctg}(c)-\cos(b)\cos(A)$$

$$\sin(B)\operatorname{ctg}(A)=\sin(c)\operatorname{ctg}(a)-\cos(c)\cos(B)$$
$$\sin(B)\operatorname{ctg}(C)=\sin(a)\operatorname{ctg}(c)-\cos(a)\cos(B)$$
$$\sin(C)\operatorname{ctg}(A)=\sin(b)\operatorname{ctg}(a)-\cos(b)\cos(C)$$
$$\sin(C)\operatorname{ctg}(B)=\sin(a)\operatorname{ctg}(b)-\cos(a)\cos(C).$$

Delambre-Gauss six-part rules (are relations between all the six elements of the triangle and may be used as check formulas, [3]). The results obtained should always be checked:

$$\sin\left(\frac{A}{2}\right)\sin\left(\frac{b+c}{2}\right)=\sin\left(\frac{a}{2}\right)\cos\left(\frac{B-C}{2}\right) \qquad \sin\left(\frac{A}{2}\right)\cos\left(\frac{b+c}{2}\right)=\cos\left(\frac{a}{2}\right)\cos\left(\frac{B+C}{2}\right)$$

$$\cos\left(\frac{A}{2}\right)\sin\left(\frac{b-c}{2}\right)=\sin\left(\frac{a}{2}\right)\sin\left(\frac{B-C}{2}\right) \qquad \cos\left(\frac{A}{2}\right)\cos\left(\frac{b-c}{2}\right)=\cos\left(\frac{a}{2}\right)\sin\left(\frac{B+C}{2}\right)$$

$$\sin\left(\frac{B}{2}\right)\sin\left(\frac{c+a}{2}\right)=\sin\left(\frac{b}{2}\right)\cos\left(\frac{C-A}{2}\right) \qquad \sin\left(\frac{B}{2}\right)\cos\left(\frac{c+a}{2}\right)=\cos\left(\frac{b}{2}\right)\cos\left(\frac{C+A}{2}\right)$$

$$\cos\left(\frac{B}{2}\right)\sin\left(\frac{c-a}{2}\right)=\sin\left(\frac{b}{2}\right)\sin\left(\frac{C-A}{2}\right) \qquad \cos\left(\frac{B}{2}\right)\cos\left(\frac{c-a}{2}\right)=\cos\left(\frac{b}{2}\right)\sin\left(\frac{C+A}{2}\right)$$

$$\sin\left(\frac{C}{2}\right)\sin\left(\frac{a+b}{2}\right)=\sin\left(\frac{c}{2}\right)\cos\left(\frac{A-B}{2}\right) \qquad \sin\left(\frac{C}{2}\right)\cos\left(\frac{a+b}{2}\right)=\cos\left(\frac{c}{2}\right)\cos\left(\frac{A+B}{2}\right)$$

$$\cos\left(\frac{C}{2}\right)\sin\left(\frac{a-b}{2}\right)=\sin\left(\frac{c}{2}\right)\sin\left(\frac{A-B}{2}\right) \qquad \cos\left(\frac{C}{2}\right)\cos\left(\frac{a-b}{2}\right)=\cos\left(\frac{c}{2}\right)\sin\left(\frac{A+B}{2}\right).$$

4. An MathCad algorithm for solving a spherical triangle in case SAS

Example 4.1. A MathCad worksheet for solving a spherical triangle ABC in which are known the measurements for $b=44^\circ 26'20''$, $c=39^\circ 50'30''$, and $A=90^\circ$.

Solution. Case 3 (SAS). ORIGIN $\equiv 1$

The convert function for sides and angles in radians: $F(d,m,s):=\frac{\pi}{180}\cdot\left(d+\frac{m}{60}+\frac{s}{60^2}\right)$

The convert function for sides and angles in DMS:

$$M(x):=\begin{pmatrix} d\left(\operatorname{floor}\left(\frac{180}{\pi}\cdot x\right)\right) \\ \operatorname{floor}\left(60\cdot m\left(\frac{180}{\pi}\cdot x\right)\right) \\ \operatorname{round}\left(60\cdot m\left(60\cdot\left(\frac{180}{\pi}\cdot x\right)\right)\right)\end{pmatrix}$$

where $d(x)\equiv\begin{cases} x & \text{if } x>0 \\ 180+x & \text{otherwise}\end{cases}$ $\qquad m(x)\equiv x-\operatorname{floor}(x)$

$$\operatorname{round}(x)\equiv \text{if}\left(x-\operatorname{floor}(x)<0.5,\ \operatorname{floor}(x),\ \operatorname{ceil}(x)\right)$$

The magnitude of the side b in radians: $b:=F(44,26,20)$ $\qquad b=0.775$ rad

The magnitude of the side c in radians: $c:=F(39,50,30)$ $\qquad c=0.695$ rad

The measure of the angle A in radians: $A := F(90,0,0)$ $A = 1.570$ rad

Using Law of Cosines, solve $\cos(a)$:

$\cos a := \cos(b)\cdot\cos(c) + \sin(b)\cdot\sin(c)\cdot\cos(A)$ $\cos a = 0.548$

Measure of the side a in radians: $a := \operatorname{acos}(\cos a)$ $a = 0.990$ rad

Measure of the side a in DMS: $M(a) = \begin{pmatrix} 56 \\ 45 \\ 18 \end{pmatrix}$

Using Cotangent four-part Formulae, solve $\tan(B)$:

$$\tan B := \frac{\sin(A)}{\sin(c)\cdot\frac{1}{\tan(b)} - \cos(c)\cdot\cos(A)} \qquad \tan B = 1.531$$

Measure of the angle B in radians: $B := \operatorname{atan}(\tan B)$ $B = 0.992$ rad

Measure of the angle B in DMS: $M(B) = \begin{pmatrix} 56 \\ 50 \\ 30 \end{pmatrix}$

Using Cotangent four-part Formulae, solve $\tan(C)$:

$$\tan C := \frac{\sin(A)}{\sin(b)\cdot\frac{1}{\tan(c)} - \cos(b)\cdot\cos(A)} \qquad \tan C = 1.192$$

Measure of the angle C in radians: $C := \operatorname{atan}(\tan C)$ $C = 0.872$ rad

Measure of the angle C in DMS: $M(C) = \begin{pmatrix} 49 \\ 59 \\ 60 \end{pmatrix}$

Check: Law of Sines or Delambre-Gauss six-part rules may be used as check formulas:

$$\frac{\sin(a)}{\sin(A)} - \frac{\sin(b)}{\sin(B)} = 0 \qquad \frac{\sin(a)}{\sin(A)} - \frac{\sin(c)}{\sin(C)} = 0 \qquad \text{or}$$

$$\sin\left(\frac{A}{2}\right)\cdot\sin\left(\frac{b+c}{2}\right) - \sin\left(\frac{a}{2}\right)\cdot\cos\left(\frac{B-C}{2}\right) = 0\,.$$

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POCHHAMMER POLYNOMIALS OVER Q_P^TILDE

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Abstract We study certain properties of Pochhammer polynomials over the maximal closed subfield of **Q_p^tilde** isomorphic (both algebraically and topologically) to **C_p** , constructed in [1].

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THE EXCHANGE PROPERTY FOR COMODULES AND FOR GRADED MODULES

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Abstract Following the concept of an object with the exchange property in an arbitrary Grothendieck category, we study the category of comodules over a coalgebra and the category of graded modules.

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CONFIDENCE INTERVALS OF CHARACTERISTIC QUANTILES

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Abstract The paper presents a comparison of the confidence intervals of the characteristic quantiles corresponding to different mean return intervals. These quantiles and confidence intervals with a specified confidence level are considered for describing basic design parameters such as extreme wind speeds, snow depths, exterior air temperatures, wave heights, flood peaks and other annual maximum data. Asymptotic standard deviations of the quantile estimators based on the methods of moments (MOM), maximum likelihood (ML) and probability weighted moments (PWM) for different probability distributions (Log-Normal, Extreme Value, GEV, Weibull) are expressed as functions of the sample size, mean return interval and parameters. The confidence intervals of extreme air temperature quantiles estimated using climatic data recorded at some Romanian meteorological stations are also presented.

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ASSESSMENT OF SKILLS AND COMPETENCES IN LEARNING MAP THROUGH AUTHENTIC ASSESSMNETS

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Abstract Students come to higher education with different background knowledge and skills. A system concentrating on the acquisition of a predefined knowledge at a given pace with the background differences is likely to lead to a situation where a student can accomplish the given study goals, but another with equal talent might fail. To address the difficulties, which individual differences cause in higher education, some institutions have taken into use personalized learning paths, which are based on a competence-based curriculum. The assessment of skill acquisition is the core of this approach, rather than the traditional review of a fixed set of contents. The Learning Map project modernizes the way in which HEIs can provide personalized learning activities to their students through the inclusion of methodological and technological tools to design and deploy skill- or competence-based curricula (Learning Maps) and to assess the level of acquisition of those skills or competences exposing the students to real-world scenarios (Authentic Assessment). The main efforts of the project are directed to build local capacities of the partners to design, produce and deploy open courses based on the Learning Maps infrastructure and to empower them with Authentic Assessment activities to evaluate their effectiveness. The project results, being technological tools or methodological descriptions, will be released under open licenses to ease exploitation and improvement after the project.

AN INEQUALITY IN LORENTZ SPACES

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Abstract Lorentz spaces were introduced by G.G. Lorentz in the 50th and since then were intensively studied since their properties have many applications in different areas of mathematics. In this talk we discuss about an inequality related to these spaces.

BANACH SPACES OF ANALYTIC MATRICES

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Abstract In this talk we discuss about some properties of those spaces.