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CONTENT

PASSING TO THE LIMIT UNDER THE DERIVATION OPERATOR VIA WEIERSTRASS CONTINUITY PRESERVING THEOREM

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Abstract: Everybody knows how unpleasant thing is to present the proof of the theorem concerning the derivability of the limit function associated to a sequence of derivable functions.
We present here a proof coming directly from the well-known Weierstrass theorem asserting that the limit of a uniform convergent sequence of continuous functions on a topological space is also continuous.
In the last part we present a new result on differentiability at a point for the limit function of a sequence of functions which are differentiable at this point only.
Mathematics Subject Classification (2010): 28A12, 28A25.
Key words: uniform differentiability, completeness and compactness with respect to pointwise topology, thickness-regularity of a set.

1. Scalar case

In this part we consider an interval I of R and a sequence $(f_n)_n$ of real functions defined on I. To put in the light the essence of our idea of proof, we suppose that the sequence $(f_n)_n$ is uniformly convergent to a function $f: I -> R$.

With the above notations we have

Theorem 1. If the functions f_n are differentiable on I and the sequence $(f_n')_n$ is uniformly convergent to a function $g: I -> R$ then the function f is differentiable on I and

$$f'(x) = g(x) = \lim_{n\to\infty} f_n'(x) \forall x \in I \text{ or } \left(\lim_{n\to\infty} f_n\right)' = \lim_{n\to\infty} (f_n)'$$

Proof.

We fix a point $x_0 \epsilon I$ and for any $n \in N$ we denote r_n (respectively r) the function on I given by

$$r_n(x) = \begin{cases} \dfrac{f_n(x) - f_n(x_0)}{x - x_0} & if\ x \in I - \{x_0\} \\ f_n'(x_0) & if\ x = x_0 \end{cases}$$

$$\left(resp.\ r(x) = \begin{cases} \dfrac{f(x) - f(x_0)}{x - x_0} & if\ x \in I - \{x_0\} \\ g(x_0) & if\ x = x_0 \end{cases} \right)$$

a) We show that the sequence $(r_n)_n$ is uniformly Cauchy on the interval I and therefore uniformly convergent on I.

The sequence $(r_n(x_0))_n$ is a Cauchy sequence since it coincides with the sequence $f_n'(x_0)$. For the rested set $I-\{x_0\}$ we consider $\varepsilon \in R, \varepsilon > 0$ and a natural number n_ε such that

$$|f_n'(\xi) - f_m'(\xi)| < \varepsilon \ \forall n, m \in N, \forall \xi \in I; n, m \geq n_\varepsilon$$

For $x \in I - \{x_0\}$ we have

$$r_n(x) - r_m(x) = \frac{f_n\ (x) - f_n\ (x_0)}{x - x_0} - \frac{f_m\ (x) - f_m\ (x_0)}{x - x_0} =$$

$$= \frac{1}{x - x_0}\left[\left(f_n\ (x) - f_m\ (x)\ \right) - \left(f_n(x_0) - f_m\ (x_0)\right)\right]$$

Using Lagrange lemma there exists ξ between x and x_0 such that

$$\left(f_n\ (x) - f_m\ (x)\ \right) - \left(f_n(x_0) - f_m\ (x_0)\right) = (x - x_0) \cdot \left(f_n{}'(\xi) - f_m'\ (\xi)\ \right) i.e$$

$$|r_n(x) - r_m(x)| = \left|f_n{}'(\xi) - f_m'\ (\xi)\ \right| < \varepsilon\ \forall n, m \in N, n, m \geq n_\varepsilon, \forall x \in I - \{x_0\}$$

Hence the sequence $(r_n)_n$ is uniformly Cauchy on $I = (I - \{x_0\}) \cup (x_0)$

b) For any $n \in N$ the function f_n is continuous (f_n is just differentiable) and therefore the function r_n is continuous on $I - \{x_0\}$. It is also continuous at x_0 since

$$r_n\ (x_0) = f_n'\ (x_0) = \lim_{x \to x_0} \frac{f_n\ (x) - f_n\ (x_0)}{x - x_0} = \lim_{x \to x_0} r_n(x)$$

Hence the functions r_n are continuous on I.

c) Summarising a) and b) we have: the sequence of continues functions $(r_n)_n$ on I is uniformly convergent on I. But just from the definitions of r_n and r it follows that the sequence $(r_n)_n$ is simply convergent to r. Hence the sequence $(r_n)_n$ of continuous function on I converges uniformly to r on this interval. From Weierstrass continuity preserving theorem we deduce that the function r is also continuous on I. Particularly

$$r(x_0) = \lim_{x \to x_0} r(x)\ i.e\ g(x_0) = \lim_{x \to x_0} \frac{f\ (x) - f(x_0)}{x - x_0}$$

Remark 1.

The result of the previous Theorem still rest available replacing the uniform convergence of the sequence $(f_n)_n$ on the interval I by the convergence of the sequence of real numbers $\left(f_n\ (x_0)\right)_n$ for some point $x_0 \in I$.

Indeed, we consider an arbitrary bounded interval $J \subset I$ such that $x_0 \in I$ and for any $\varepsilon \in R, \varepsilon > 0$ we consider the natural number n_ε such that

$$\left|f_n{}'(\xi) - f_m'\ (\xi)\ \right| < \varepsilon, \forall \xi \in I, \forall n, m \geq n_\varepsilon$$

$$\left|f_n(x_0) - f_m\ (x_0)\right| < \varepsilon, \forall n, m \geq n_\varepsilon$$

Taking $x \in J$ and applying as above Lagrange lemma we get $\xi \in J$ such that

$$\left|\left(f_n\ (x) - f_m\ (x)\ \right) - \left(f_n(x_0) - f_m\ (x_0)\right)\right| = (x - x_0) \cdot \left(f_n{}'(\xi) - f_m'\ (\xi)\ \right)$$

$$\left|\left(f_n\ (x) - f_m\ (x)\ \right)\right| \leq \left|\left(f_n(x_0) - f_m\ (x_0)\right)\right| + (x - x_0) \cdot \left(f_n{}'(\xi) - f_m'\ (\xi)\ \right)$$

If we take $n, m \geq n_\varepsilon$ we have $\forall x \in J => \left|\left(f_n\ (x) - f_m\ (x)\ \right)\right| \leq \varepsilon + l(J)\varepsilon$

Where $l(J)$ denotes the length of J, i.e the sequence $(f_n)_n$ is uniformly Cauchy on J hence uniformly convergent on J to a function f.

By Theorem 1 applied to the sequence $(f_n)_n$ restricted to the interval J we deduce that the function f is differentiable at the point x_0 and we have

$$\left(\lim_{n\to\infty} f_n\right)'(x_0) = \lim_{n\to\infty} f_n'(x_0)$$

2. Vector case

At this point we consider an open set $D \subset R^p$ and a sequence $(f_n)_n$ of real continuous functions on D such that the sequence $(f_n(x_0))_n$ is convergent for some point $x_0 \in D$.

Theorem 2. We consider the sequence $(f_n)_n$ as above and we suppose that any function f_n is differentiable on a ball $B(x_0, r)$ and the sequence $(f_n')_n$ is uniformly convergent to a function $g: B(x_0, r) \to R^p$. Then the sequence $(f_n)_n$ is uniformly convergent to a differentiable function $f: B(x_0, r) \to R$ and we have

$$f'(x) = g(x)\ i.e \left(\lim_{n\to\infty} f_n\right)' = \lim_{n\to\infty} f_n'$$

Remark 2.

For a function $h: D -> R$ we have denoted by $h'(x_0)$ the differential of h at the point x_0 i.e the only linear functional $L: R^p \to R$ such that

$$\lim_{x\to x_0} \frac{h\ (x) - h\ (x_0) - L(x - x_0)}{||x - x_0||} = 0$$

Certainly a linear functional $L: R^p \to R$ may be identified with an element $(l_1, l_2, \dots, l_p) \in R^p$ and for any $u = (u_1, u_2, \dots, u_p)$ from R^p we have

$$L(u) = \sum_{i=1}^{p} l_i u_i$$

More precisely for any $i \in \overline{1,p}$, in our case, $l_i = \frac{\partial h}{\partial x_i}(x_0)$

A sequence $(L_n)_n$ of linear functionals on R^p is convergent to the linear functional L if the corresponding sequence $\left(l_1^n, l_2^n, \dots, l_p^n\right)_n$ from R^p converges to $\left(l_1, l_2, \dots, l_p\right)$ the vector of R^p representing the linear functional L. More precisely

$$\lim_{n\to\infty} L_n = L \Leftrightarrow \lim_{n\to\infty} ||L_n - L|| = 0 \Leftrightarrow \lim_{n\to\infty} \sqrt{\sum_{i=1}^{p} (l_i^n - l_i)^2} = 0 \Leftrightarrow$$

$$\Leftrightarrow \lim_{n\to\infty} l_i^n = l_i, i \in \overline{1,p}$$

Proof.

We start by remembering a Lagrange Lemma for several variables. If $f: B(x_0, r) \to R$ is a differentiable function then, for any $x', x'' \in B(x_0, r)$ there exists an element $c \in (x', x'')$ the open interval in R^p with the bounds x', x'' -i.e there exists $\tau \in (0,1)$ such that $c = x' + \tau(x'' - x')$

$$f(x'')\text{-f(x')=f '(c)(x"} - x')$$

Indeed, we consider the function $\lambda: [0,1] \to B(x_0, r)$ given by

$$\lambda(t) = x' + t(x" - x').$$

Obviously λ is continuous and differentiable on (0,1). We have

$$\lambda'(t)(u) = u(x" - x') \forall u \in R$$

and the function $\varphi: [0,1] \to R$ given by $\varphi(t) = f(\lambda(t))$ is continuous on [0,1], differentiable on (0,1). Hence there exists $\tau \in (0,1)$ such that

$$\varphi(1) - \varphi(0) = \varphi'(\tau)(1) = f'(\lambda(\tau))\lambda'(\tau)(1) = f'(c)(x'' - x'),$$

$$f(x'')\text{-f(x')=f '(c)(x"} - x'), c = x' + \tau(x'' - x')$$

a) The sequence $(f_n)_n$ is uniformly Cauchy on $B(x_0, r)$. Indeed, let $\varepsilon > 0$ and let $n_\varepsilon \in N$ be such that

$$\left| f_n'(x) - f_m'(x) \right| < \varepsilon, \forall n, m \geq n_\varepsilon, \forall x \in B(x_0, r)$$

Applying the above Lagrange Lemma to the function $(f_n - f_m)$ there exists $c \in (x_0, x)$ such that

$$\left(f_n(x) - f_m(x)\right) - \left(f_n(x_0) - f_m(x_0)\right) = \left(f_n'(c) - f_m'(c)\right)(x - x_0)$$

By Cauchy-Buniankoweski-Schwarz inequality we get

$$\left|\left(f_n(x) - f_m(x)\right) - \left(f_n(x_0) - f_m(x_0)\right)\right| \leq$$

$$\leq \left\|\left(f_n'(c) - f_m'(c)\right)\right\| \, \|(x - x_0)\| \leq \varepsilon r$$

for any $x \in B(x_0, r)$ and any $n, m \geq n_\varepsilon$. Since by hypotheses the sequence $\left(f_n(x_0)\right)_n$ is Cauchy, so, changing eventually n_ε we may suppose that

$$\left| f_n(x_0) - f_m(x_0) \right| < \varepsilon$$

Hence we deduce

$$\left| f_n(x) - f_m(x) \right| \leq$$

$$\leq \left| f_n(x) - f_m(x) - \left(f_n(x_0) - f_m(x_0)\right)\right| + \left|\left(f_n(x_0) - f_m(x_0)\right)\right| \leq$$

$$\leq (1 + r)\varepsilon$$

for all $x \in B(x_0, r)$ and all $n, m \geq n_\varepsilon$.

We denote by f the function on $B(x_0, r)$ given by $f(x) = \lim_{n\to\infty} f_n(x)$ and by $r(x), r_n(x)$ the functions on $B(x_0, r)$ defined by

$$r_n(x) = \begin{cases} \dfrac{f_n(x) - f_n(x_0) - f_n'(x_0)(x - x_0)}{||x - x_0||} & if\ x \neq x_0 \\ 0_{R^p} & if\ x = x_0 \end{cases}$$

$$r(x) = \begin{cases} \dfrac{f_n(x) - f_n(x_0) - g(x_0)(x - x_0)}{||x - x_0||} & if\ x \neq x_0 \\ 0 & if\ x = x_0 \end{cases}$$

b) The sequence $(r_n(x))_n$ is simply convergent on $B(x_0, r)$ since taking $x \neq x_0$ we have

$$|r_n(x) - r(x)| \leq \frac{1}{||x - x_0||} [|f_n(x) - f(x)| + |f_n(x_0) - f(x_0)| +$$

$$+|(f_n'(x_0) - g(x_0))(x - x_0)|]$$

Hence $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in B(x_0, r)$ and

$$|(f'_n(x_0) - g(x_0))(x - x_0)| \leq ||f'_n(x_0) - g(x_0)|| \cdot ||x - x_0|| \underset{n\to\infty}{\to} 0$$

c) The sequence $(r_n)_n$ is uniformly Cauchy on $B(x_0, r)$.Indeed, let $\varepsilon > 0$ and $n_\varepsilon \in N$ be such that $||f_n'(u) - f'_m(u)|| < \varepsilon$ for any $u \in B(x_0, r)$ and any $n, m \geq n_\varepsilon$. As in the point a) we have

$$\left|f_n(x) - f_m(x) - \left(f_n(x_0) - f_m(x_0)\right)\right| =$$
$$= \left|\left(f_n'(c) - f_m'(c)\right)(x - x_0)\right| \leq \varepsilon||(x - x_0)||$$

for all $x \in B(x_0, r)$ and all $n, m \geq n_\varepsilon$.

Hence for $x \neq x_0$ we have

$$|r_n(x) - r_m(x)| \leq \frac{1}{||x - x_0||} \left|\left(f_n(x) - f(x)\right) - \left(f_n(x_0) - f_m(x_0)\right)\right| +$$

$$+\frac{1}{||x - x_0||}\left|\left(f'_n(x_0) - g(x_0)\right)(x - x_0)\right| \leq \varepsilon + \left||f'_n(x_0) - g(x_0)|\right| =$$

$$= \varepsilon + \lim_{m\to\infty} \left\|f_n'(x_0) - f_m'(x_0)\right\| \leq 2\varepsilon$$

Summarizing a) b) c) we have: The sequence $(r_n)_n$ is uniformly Cauchy and simply convergent to r on the set $B(x_0, r)$.Hence $(r_n)_n$ is uniformly convergent to r.

The functions r_n being continuous on $B(x_0, r)$it follows (Weierstess result) that the function r is also continuous on $B(x_0, r)$.Particularly, r is continuous at the point x_0 i.e

$$\lim_{x\to x_0} \frac{f(x) - f(x_0) - g(x_0)(x - x_0)}{||x - x_0||} = 0$$

and therefore f is differentiable at x_0 and $f'(x_0) = g(x_0)$ i.e

$$\left(\lim_{n\to\infty} f_n\right)'(x_0) = \lim_{n\to\infty} f_n'(x_0)$$

3. Vector- vector case

At this point we consider two Banach spaces E and F an open set $D \subset R^p$ a sequence $(f_n)_n$ of continuous functions on D with values in F such that the sequence $\left(f_n(x_0)\right)_n$ is convergent in F for some point $x_0 \in D$.

Theorem 3. We consider the sequence $(f_n)_n$ as above and we suppose that any function is differentiable on a ball $B(x_0, r) \subset$ D and the sequence $(f_n')_n$ is uniformly convergent to a function $g: B(x_0, r) \to \mathcal{L}(E, F) -$ the set of all linear continuous functional from E into F endowed with the distance.

$$d\ (L', L'') = ||\ L' - L''|| = sup\{\ ||L'(x) - L''(x)||_F\ ; x \in E, ||x|| \leq 1\}$$

Then the sequence $(f_n)_n$ is uniformly convergent to a differentiable function $f: B(x_0, r) \to F$ and we have $f'(x) = g(x)$ i.e. $\left(\lim_n f_n\right)' = \lim_n (f_n')$ on $B(x_0, r)$

Proof.

We follow step by step the proof of Theorem 2, but instead of Lagrange's lemma of finite encreaments we use the well known Lagrange inequality: if D is an open set of E,f:D→F is a differentiable function and $x', x'' \in D$ are such that the closed interval $[x', x''] = \{x \in E; x = x' + tx'', 0 \leq t \leq 1\}$ is included in D then we have $||f(x') - f(x'')|| \leq M||x' - x''||$

Where $M := sup\left\{||f'(x)||;\ x \in [x', x'']\right\}$

4. Equidifferentiability. Consequences

Let E,F,D be as in the processing section.

Definition. A family $(f_\alpha)_{\alpha\in A}$ of functions $f_\alpha: D \to F$ such that any f_α is differentiable at the point $x_0 \in D$ i.e for any $\alpha \in A$ there exists a continous linear functional $f'(x_0): E \to F$ and a function $\varepsilon_\alpha: E \to F$ with $\varepsilon_\alpha(x_0) = 0$ such that

$$\underset{\alpha}{f}(x) = \underset{\alpha}{f}(x_0) + \underset{\alpha}{f'}(x_0)(x - x_0) + \varepsilon_\alpha||x - x_0||\forall x \in D$$

will be called equidifferentiable at the point x_0 if the family $(\varepsilon_\alpha)_{\alpha\in A}$ of functions on D is equicontinous at the point x_0 i.e. for any $\varepsilon \in R, \varepsilon > 0$ there exists

$\delta_\varepsilon \in R, \delta_\varepsilon > 0$ such that

$$||\varepsilon_\alpha(x)|| \leq \varepsilon\ \ \forall\, x \in D\ with\ ||x - x_0|| < \delta_\varepsilon, ,\forall\ \ \alpha \in A$$

Theorem 4. Let E, F, D as above and let $(f_n)_n$ be a sequence of functions $f_n: D \to F$ which is simply convergent to a function $f: D \to F$

If the sequence $(f_n)_n$ is equidifferentiable at the point $x_0 \in D$ and the sequence $(f_n'(x_0))_n$ is convergent in $\mathcal{L}(E,F)$ to the element $g \in \mathcal{L}(E,F)$ then the function f is differentiable at x_0 and we have

$$f'(x_0) = g, \left(\lim_{n\to\infty} f_n\right)'(x_0) = \lim_{n\to\infty} f_n'(x_0)$$

Proof.

We consider the functions $r_n, n \in N$ and r defined on D with values in F given by

$$r_n(x) = \begin{cases} \dfrac{f_n(x) - f_n(x_0) - f_n'(x_0)(x - x_0)}{||x - x_0||} & if\ x \neq x_0 \\ 0_F & if\ x = x_0 \end{cases}$$

$$r(x) = \begin{cases} \dfrac{f(x) - f(x_0) - g(x - x_0)}{||x - x_0||} & if\ x \neq x_0 \\ 0_F & if\ x = x_0 \end{cases}$$

If $x \neq x_0$, by hypotheses we have

$$\lim_{n\to\infty} \left(f_n(x) - f_n(x_0) - f_n'(x_0)(x - x_0)\right) = f(x) - f(x_0) - g(x - x_0)$$

and therefore $\lim_{n\to\infty} r_n(x) = r(x)$. The last equality obviously holds for $x = x_0$ also.

Hence $(r_n)_n$ is simply convergent to r on D.

The family $(r_n)_n \in N$ of functions on D is equicontinuous at the point x_0. Indeed, for any $n \in N$ let $\varepsilon_n: D \to F$ the functions for which we have

$$f_n(x) = f_n(x_0) + f'_n(x_0)(x - x_0) + \varepsilon_n(x)||x - x_0||\ \forall\, x \in D, \varepsilon_n(x_0) = 0$$

i.e. $r_n(x) = \varepsilon_n(x)\ \forall\, x \in D$. But the family of functions $(\varepsilon_n)_n$ is by hypothleses equicontinuous at the point x_0, hence the family $(r_n)_n$ is equicontinuous at the point x_0. The sequence $(r_n)_n$being simply convergent to r we deduce that r is also continuous at the point x_0 i.e. $\lim_{x\to x_0} r(x) = 0$. The assertion follows now from the relation

$$f(x) - f(x_0) - g(x - x_0) = r(x)||x - x_0||\ and\ \lim_{x\to x_0} r(x) = 0$$

Remark 3.

The new result asserted in Theorem 4 is stronger than the previous ones and may be derived from Theorem 4. For instance in Theorem 3 we have supposed that the sequence $(f'_n)_n$ is uniformly convergent and therefore uniformly Cauchy on $B(x_0, r)$. We deduce immediately that the family $(f_n)_n$ is equidifferentiable at the point x_0. Indeed using the notations

$$r_n(x) = \begin{cases} \dfrac{f_n(x) - f_n(x_0) - f_n'(x_0)(x - x_0)}{||x - x_0||} & if\ x \neq x_0 \\ 0_F & if\ x = x_0 \end{cases}$$

we show that the sequence $(r_n)_n$ of continuous functions is uniformly Cauchy on $B(x_0, r)$ and therefore it will be equicontinuous at any point of $B(x_0, r)$.

Let $\varepsilon > 0$ and let $n_\varepsilon \in N$ be such that

$$||f'_n(x) - f'_m(x)|| < \varepsilon \ \forall\, x \in B(x_0, r), \forall\, n, m \geq n_\varepsilon$$

Using Lagrange inequality we have

$$\left|\left|f_n(x) - f_m(x) - \left(f_n(x_0) - f_m(x_0)\right)\right|\right| \leq \varepsilon\left|\left|x - x_0\right|\right| \forall x \in B(x_0, r)$$

$$\forall n, m \geq n_\varepsilon$$

Obviously

$$\left|\left|\left(f'_n(x_0) - f'_m(x_0)\right)(x - x_0)\right|\right| \leq \left|\left|f'_n(x_0) - f'_m(x_0)\right|\right| \cdot \left|\left|x - x_0\right|\right| \leq$$

$$\leq \varepsilon||x - x_0|| \ \forall\, n, m \geq n_\varepsilon$$

and we have

$$||r_n(x) - r_m(x)|| \leq 2\varepsilon \ \forall\, x \in B(x_0, r).$$

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THE DETERMINATION COEFFICIENT AND THE SIGNIFICANCE IN LINEAR REGRESSION AND ARMA TIME SERIES MODELS

DANIEL CIUIU

Abstract. In this paper we will use first the coefficient of determination R^2 to test the significance of a linear regression. In fact, we will use the adjusted R-squared and the Snedecor—Fisher distribution.

In the case of $AR(p)$, $MA(q)$ or $ARMA(p,q)$ time series model, we expect not to obtain a Snedecor—Fisher distribution: this distribution is the distribution of the ratio of two chi square distribution, and a chi square distribution is obtained by the estimation of a variance in the case of **independent identical distributed** random variables. Nevertheless, we will compute the coefficient of determination - adjusted or not. In the $ARMA(p,q)$ we will also compute the relative AR/ relative MA coefficient of determination. For this, we compare the $ARMA(p,q)$ with the $MA(q)$, respectively $AR(p)$ time series in the same way as we do in the classical case of linear regression model vs. Y data.

Mathematics Subject Classification (2010): 62J05, 62M10, 62F03.
Key words: Coefficient of determination, ARMA time series, significance tests.

1. Introduction

The multi-linear regression that express the dependent variable Y in terms of the explanatory variables X_1,..., X_k is (see [3])

$$Y = A_0 + \sum_{i=1}^{k} A_i X_i. \tag{1.1}$$

Consider the residuals (the errors)

$$u = Y - A_0 - \sum_{i=1}^{k} A_i X_i, \tag{1.1'}$$

and denote by σ_u^2 the variance of the errors. In fact there exists a conceptual difference between errors and residuals: the errors are random iid (independent identical distributed) variables, while the residuals are the "observations", i.e. observed values of these random variables (see [3]).

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Key words and phrases. Coefficient of determination, ARMA time series, significance tests.

A goodness-of-fit measure for linear regression is the coefficient of determination, R^2. Denote first by $\widehat{Y}_t$ the estimation of Y using the linear regression (1.1), Y_t the observed values of Y and $u_t = Y_t - \widehat{Y}_t$. Denote also by σ_Y^2 and by $\sigma_{\widehat{Y}}^2$ the variances of Y and of estimated (by linear regression) Y.

If the errors are normal random variables it results that (see [3]) the residuals and Y are also normal variables. If the reasonable hypothesis of the independece between the estimations and errors is fulfilled, the variance of the dependent variable is the sum between the variances of estimations and errors. From this (see [3]) comes the coefficient of determination

$$R^2 = \frac{\sigma_{\widehat{Y}}^2}{\sigma_Y^2} = 1 - \frac{\sigma_u^2}{\sigma_Y^2}, \tag{1.2}$$

i.e. the ratio of the variance of model-estimated in variance of the observations.

Remark 1.1. *In the above formula we have estimated the variance without taking into account the number of degrees of freedom: each sum of squares is divided by the size of data, n. It means that the estimated variances are biased.*

Using the coefficient of determination R^2 by increasing the number of explanatory variables we increase also the coefficient of determination. To avoid the arbitrary increase of the number of variables, we use the adjusted coefficient of determination $\overline{R}^2$ (see [3])

$$\overline{R}^2 = 1 - \frac{\sigma_u^2}{\sigma_Y^2} \cdot \frac{n-1}{n-k-1}, \tag{1.3}$$

i.e. in the estimation of σ_u^2 and σ_Y^2 we take into account the numbers of degrees of freedom. In this case, when a new explanatory variable is not important, the adjusted coefficient of determination decrease.

For a time series X_1,..., X_n we test first the stationarity using the Dickey—Fuller unit root test (see [2]). Consider three models of linear regression:

$$\Delta X_t = \Phi \cdot X_{t-1}, \ for\ model\ 1, \tag{1.4}$$

$$\Delta X_t = \beta + \Phi \cdot X_{t-1}, \ for\ model\ 2,\ and \tag{1.4'}$$

$$\Delta X_t = \beta + \Phi \cdot X_{t-1} + \gamma \cdot t, \ for\ model\ 3. \tag{1.4''}$$

We say that according a given model the time series is stationary (see [2]) if Φ is significant and γ is not. But we can not use the Student distribution, because the assumption for this distribution is the independence, which for time series is true only for white noise.

Dickey and Fuller (see [2]) used the Monte Carlo methods to obtain quantiles for Student estimated statistics, according the model and the number of observations n.

If γ in model 3 is significant (as well as Φ) it is recomanded (see [4]) the stationarization using the moving average or exponential smooth. If γ is significant and Φ is not, we consider the linear regression of ΔX_t in terms of t. Otherwise, we have to use one of the other two models, and the stationarization is made by differentiation.

For the stationary time series we consider the $AR\,(p)$, $MA\,(q)$, or $ARMA\,(p,q)$ model (see [1, 7, 4]). To estimate the coefficients we use the Yule—Walker, the innovations', respectively the Hannan—Rissanen algorithm. The coefficient of determination, adjusted or not, is also computed in literature.

2. Methodology

We notice in (1.3) that $1-\overline{R}^2$ is the ratio between the estimated variance of the errors and the estimated variance of the model taking into account the number of degrees of freedom. Therefore

$$\frac{1}{1-\overline{R}^2} \sim F_{n-1,n-k-1} \tag{2.1}$$

(the Snedecor—Fisher distribution with $n-k-1$ and $n-1$ degrees of freedom).

Definition 2.1. We say that the linear regression (1.1) is not significant with the first degree error ε if $\frac{1}{1-\overline{R}^2} < F_{n-1,n-k-1;\varepsilon}$ (the quantile of error ε for the Snedecor—Fisher distribution with $n-1$ and $n-k-1$ degrees of freedom).

The above test is called the significance test for linear regression.

For the the AR, MA and $ARMA$ time series model only the white noise values a_t (corresponding to the errors of linear regression) are independent. That's why we can not use the Snedecor—Fisher distribution, for similar reasons we can not use the Student distribution in the Dickey—Fuller test.

Nevertheless, we compute R^2 for the $AR\,(p)$, $MA\,(q)$ and $ARMA\,(p,q)$ time series as

$$R^2 = 1 - \frac{\sigma_a^2}{\sigma_x^2}, \tag{2.2}$$

where σ_x^2 is the variance of the stationary time series X_t, and σ_a^2 is the variance of the corresponding white noise a_t. If the mentioned variances are computed as sum of squares divided by the size of data, n, we obtain the coefficient of determination, R^2. The adjusted coefficient of determination is determined dividing the above sum of squares by $n-1$ for X_t and $n-np-1$ for a_t, where np is the number of parameters: p for $AR\,(p)$, q for $MA\,(q)$ and $p+q$ for $ARMA\,(p,q)$ time series:

$$\overline{R}^2 = 1 - \frac{\sigma_a^2}{\sigma_x^2} \cdot \frac{n-1}{n-np-1}. \tag{2.2'}$$

For the $ARMA\,(p,q)$ time series we compute also

$$(2.3)\qquad \begin{cases} R^2_{MA} = 1 - \frac{\sigma_a^2}{\sigma_{a,AR}^2} \\ R^2_{AR} = 1 - \frac{\sigma_a^2}{\sigma_{a,MA}^2} \end{cases},$$

where $\sigma_{a,AR}^2$ is the variance of the corresponding white noise for the $AR\,(p)$ time series model, $\sigma_{a,MA}^2$ is the variance of the corresponding white noise for the $MA\,(q)$ time series model, and obviously σ_a^2 is the variance of the white noise in the $ARMA\,(p,q)$ case.

All these coefficients of determination become adjusted if the involved variances are computed as the sum of squares divided by the number of degrees of freedom.

We will compute also the relative R^2, adjusted or not, between p and $p+1$ in the case of AR models, and between q and $q+1$ in the case of MA models.

3. Applications

Example 3.1. Consider the CPI (Compound Price Index) from January 2007 to February 2021, monthly data (see [8]). We obtain 170 monthly data, for food, non-food and services. We want to test the significance of linear regression considering the dependent variable Y being the CPI for services, and the explanatory variables being the food/ non-food CPI.

In Table 1 we present the results for linear regression, the values of R^2 and $\overline{R^2}$, and the values for the F statistics and the corresponding error according the Snedecor—Fisher distribution. In the second column we write matrices with coefficients on the first row and the corresponding Student statistics on the second row.

Table 1. The results for linear regression of services CPI in terms of food CPI and non-food CPI

Model	Coefficients and Student	R^2	$\overline{R^2}$	F	Error
$Y=f(X_1,X_2)$	$\begin{pmatrix} 9.473413 & 0.101255 & 0.785486 \\ 3.278507 & 2.349958 & 38.492187 \end{pmatrix}$	0.981789	0.981571	54.262304	$3.152129\cdot 10^{-98}$
$Y=f(X_1)$	$\begin{pmatrix} -64.761188 & 1.600074 \\ -9.607443 & 27.685558 \end{pmatrix}$	0.820223	0.819153	5.529536	$4.640688\cdot 10^{-26}$
$Y=f(X_2)$	$\begin{pmatrix} 15.726458 & 0.828822 \\ 13.776149 & 93.606223 \end{pmatrix}$	0.981187	0.981075	52.840159	$8.284552\cdot 10^{-98}$

Example 3.2. Consider the exchange rate of Euro/ RON from January 3 2008 to April 20 2021. We have 3362 daily data without legal hollidays (see [9]). We consider for the stationarized series the $AR\,(p)$, $MA\,(q)$ and $ARMA\,(p,q)$ models. We compute the coefficients of determination, and for the $ARMA\,(p,q)$ model we compute also R^2_{MA} and R^2_{AR}, adjusted or not.

If we apply the Dickey—Fuller unit root test, model 3, we obtain $\Phi = -0.005614$, with Student -3.489836, significant 5%, and $\gamma = 1.34 \cdot 10^{-6}$ with Student statistics 2.771818, also significant 5%. We stationarize by the moving average method (see [1, 4]).

We choose, according the minimum variance of the reminder, $q = 1$. Applying again the Dickey—Fuller test for the reminder, we obtain $\phi = -1.389106$ with Student statistics -87.38592 and $\gamma = 1.38 \cdot 10^{-9}$ with Student statistics 0.016203. Therefore the reminder is stationary.

In Table 2 we present the coefficients and the values of R^2 and $\overline{R}^2$ for $AR(p)$ time series. We compute also the values of the relative R^2 and relative $\overline{R}^2$, i.e. the improvement from p to $p+1$.

TABLE 2. The results for the $AR(p)$ time series

p	Coefficients	R^2	$\overline{R}^2$	Relative R^2	Relative $\overline{R}^2$
1	-0.389155	$1.514455 \cdot 10^{-1}$	$1.511888 \cdot 10^{-1}$		
2	$\begin{pmatrix} -0.498228 & -0.282808 \end{pmatrix}$	$2.181021 \cdot 10^{-1}$	$2.176363 \cdot 10^{-1}$	$7.855747 \cdot 10^{-2}$	$7.828299 \cdot 10^{-2}$
3	$\begin{pmatrix} -0.562596 & -0.394702 & -0.229657 \end{pmatrix}$	$2.59341 \cdot 10^{-1}$	$2.586789 \cdot 10^{-1}$	$5.2742 \cdot 10^{-2}$	$5.245974 \cdot 10^{-2}$
4	$\begin{pmatrix} -0.615533 & -0.485682 & -0.359337 \\ -0.230504 \end{pmatrix}$	$2.986936 \cdot 10^{-1}$	$2.978574 \cdot 10^{-1}$	$5.313182 \cdot 10^{-2}$	$5.284959 \cdot 10^{-2}$
5	$\begin{pmatrix} -0.65744 & -0.551012 & -0.447638 \\ -0.342412 & -0.181807 \end{pmatrix}$	$3.218745 \cdot 10^{-1}$	$3.208636 \cdot 10^{-1}$	$3.305396 \cdot 10^{-2}$	$3.276566 \cdot 10^{-2}$
6	$\begin{pmatrix} -0.695443 & -0.622586 & -0.541206 \\ -0.457589 & -0.319231 & -0.209028 \end{pmatrix}$	$3.515035 \cdot 10^{-1}$	$3.50343 \cdot 10^{-1}$	$4.369249 \cdot 10^{-2}$	$4.340728 \cdot 10^{-2}$

Analogously, in Table 3 we present the similar results for $MA(q)$ time series.

TABLE 3. The results for the $MA(q)$ time series

q	Coefficients	R^2	$\overline{R}^2$	Relative R^2	Relative $\overline{R}^2$
1	0.389155	$1.514455 \cdot 10^{-1}$	$1.511888 \cdot 10^{-1}$		
2	$\begin{pmatrix} 0.458608 & 0.086393 \end{pmatrix}$	$1.85934 \cdot 10^{-1}$	$1.85449 \cdot 10^{-1}$	$4.064827 \cdot 10^{-2}$	$4.03625 \cdot 10^{-2}$
3	$\begin{pmatrix} 0.481473 & 0.101811 & 0.027452 \end{pmatrix}$	$1.982632 \cdot 10^{-1}$	$1.975465 \cdot 10^{-1}$	$1.514524 \cdot 10^{-2}$	$1.485172 \cdot 10^{-2}$
4	$\begin{pmatrix} 0.490309 & 0.106657 & 0.032352 \\ 0.031809 \end{pmatrix}$	$2.03808 \cdot 10^{-1}$	$2.028587 \cdot 10^{-1}$	$6.915951 \cdot 10^{-3}$	$6.61995 \cdot 10^{-3}$
5	$\begin{pmatrix} 0.492565 & 0.106657 & 0.033722 \\ 0.037486 & -0.026156 \end{pmatrix}$	$2.050949 \cdot 10^{-1}$	$2.039099 \cdot 10^{-1}$	$1.61638 \cdot 10^{-3}$	$1.31871 \cdot 10^{-3}$
6	$\begin{pmatrix} 0.492288 & 0.108796 & 0.034241 \\ 0.039074 & -0.030824 & 0.028794 \end{pmatrix}$	$2.058855 \cdot 10^{-1}$	$2.044645 \cdot 10^{-1}$	$9.924626 \cdot 10^{-4}$	$6.966822 \cdot 10^{-4}$

In the following table we present the coefficients of $ARMA(p,q)$ models, with $p = \overline{1,3}$ and $q \in \{1,2\}$.

Table 4. The results for the $ARMA\,(p,q)$ time series

$p\backslash q$	1	2
1	$\begin{pmatrix} 0.232785 \\ 0.79536 \end{pmatrix}$	$\begin{pmatrix} 0.331859 & \\ 0.30455 & 0.36664 \end{pmatrix}$
2	$\begin{pmatrix} 0.307044 & 0.033204 \\ 0.92399 & \end{pmatrix}$	$\begin{pmatrix} -0.629521 & 0.25141 \\ 0.02917 & 0.78674 \end{pmatrix}$
3	$\begin{pmatrix} 0.263502 & 0.015818 & 0.001509 \\ 0.92508 & & \end{pmatrix}$	$\begin{pmatrix} -0.677 & 0.30882 & 0.034308 \\ 0.02342 & 0.92166 & \end{pmatrix}$

In the above table we have written in the two rows matrives the AR coefficients on the first row and the MA coefficients on the second row. In Table 5 that follows, we present the corresponding values of R^2 and adjusted R^2 on the first row, the values of R^2_{AR} and R^2_{MA} on the second row, and the last two values in the adjusted case on the third row.

Table 5. The coefficient of determination for the $ARMA\,(p,q)$ time series

$p\backslash q$	1	2
1	$\begin{pmatrix} 1.612382\cdot 10^{-1} & 1.609884\cdot 10^{-1} \\ 1.154509\cdot 10^{-2} & 1.154509\cdot 10^{-2} \\ 1.125074\cdot 10^{-2} & 1.125074\cdot 10^{-2} \end{pmatrix}$	$\begin{pmatrix} 1.984256\cdot 10^{-1} & 1.979481\cdot 10^{-1} \\ 1.534479\cdot 10^{-2} & 5.536932\cdot 10^{-2} \\ 1.505148\cdot 10^{-2} & 5.480654\cdot 10^{-2} \end{pmatrix}$
2	$\begin{pmatrix} 4.138658\cdot 10^{-1} & 4.135166\cdot 10^{-1} \\ 3.092589\cdot 10^{-1} & 2.503699\cdot 10^{-1} \\ 3.088474\cdot 10^{-1} & 2.501466\cdot 10^{-1} \end{pmatrix}$	$\begin{pmatrix} 4.226599\cdot 10^{-1} & 4.221438\cdot 10^{-1} \\ 2.907945\cdot 10^{-1} & 2.61617\cdot 10^{-1} \\ 2.903719\cdot 10^{-1} & 2.611769\cdot 10^{-1} \end{pmatrix}$
3	$\begin{pmatrix} 4.162174\cdot 10^{-1} & 4.156955\cdot 10^{-1} \\ 3.120302\cdot 10^{-1} & 2.118065\cdot 10^{-1} \\ 3.114152\cdot 10^{-1} & 2.115717\cdot 10^{-1} \end{pmatrix}$	$\begin{pmatrix} 4.249938\cdot 10^{-1} & 4.243082\cdot 10^{-1} \\ 2.936614\cdot 10^{-1} & 2.236559\cdot 10^{-1} \\ 2.930298\cdot 10^{-1} & 2.231931\cdot 10^{-1} \end{pmatrix}$

4. Conclusions

We notice that the values of the first order error for the significance regression test are small, even in the case of F statistics equal to 5.529536017: the order is 10^{-26}. This makes the computations using C++ prohibitive: by numerical methods (for instance Simpson method) we have to use the same order for number of intervals (or closed), and same for the number of simulated Snedecor—Fisher random variables in the case of Monte Carlo methods.

In the case of Monte Carlo methods, we have to take into account that simulation of a Snedecor—Fisher with m and n degrees of freedom means to simulate $m+n$ standard normal random variables. And these numbers are also big in Table 1: 169 and 167 for both explanatory variables, respectively 169 and 168 for one explanatory variable. For instance, if we perform 10 million simulations (for errors of order at least 10^{-10}) the running time on an Intel computer 3.30 GHz and $6Gb\ RAM$, on Windows 7 Enterprise on 64 bits is nine minutes. We have compute the the last column in the above mentioned table using the Excel FDIST function.

From the correlogram of the reminder, we notice a negative AutoCorrelation Function in one, and the other values of ACF almost zero. The Partial AutoCorrelation Function decreases exponentially to zero. Therefore, the most appropiate model is $MA(1)$ (see [1, 7, 4]). We notice in Table 3 that we have less relative influences from q to $q+1$ in the last two columns (of order 10^{-3} and even 10^{-4}), opposite the $AR(p)$ case, where the order of influence is always 10^{-2} (Table 2).

For $ARMA(p,q)$ time series we do not consider $q=3$ due to the less than one roots for $\theta(L)$ in these cases. We notice that, because the values of the coefficient of determination is higher for AR models, for $p>1$ we have higher values of R^2_{AR} than R^2_{MA}, adjusted or not.

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THE TECHNIQUE OF THE AUXILIARY i-SUBLINEAR OPERATOR IN INTERVAL ANALYSIS. 2

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Abstract. The technique of the auxiliary sublinear operator, known from the vector space theory, is a method to prove the existence of a linear operator by using the Hahn-Banach existence Theorem. Interval Analysis is a branch of Functional Analysis which operates with closed intervals in an ordered vector space. It originated in the last century (R.E. Moore, 1966), having many precursors and contributors. In 2013, the first author introduced the interval-spaces (abbreviated as i-spaces) and studied (2015, 2016) the problem of the extension of some i-linear functionals. The main difficulty comes from the fact that the i-spaces are not vector spaces. In a previous paper (2020) we studied the extension of i-linear operators with values in a Dedekind complete Riesz space. By the convenient choice of the auxiliary i-sublinear operator, denoted t_1, we proved then a Hahn-Banach existence type Theorem for i-linear operators. Now, using another i-sublinear operator, called t_2, we prove a Mazur-Orlicz type Theorem and a Hahn-Banach extension type Theorem for i-linear operators.
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1. Introduction

The modern Interval Analysis starts with the observation that if we compute a member A and a rigorous bound B on the total error in A, as an approximation to some unknown number x, such that $|x-A|\le B$, then no matter how you compute A and B, we will know that certainly x lies in the interval $[A-B, A+B]$. Interval Analysis operates with intervals. Suppose that we work with *closed intervals* of *real numbers*. We will denote the interval $[\underline{a},\overline{a}]=\{x\in\mathbb{R} | \underline{a}\le x\le \overline{a}\}$ by $[a]$. In the literature, the interval $[\underline{a},\overline{a}]$ is sometimes denoted by $[a_L, a_R]$ or, in short, by $\overline{\underline{a}}$. We will denote by $I\mathbb{R}$ the set of all closed intervals of real numbers, and we will consider algebraic operations in $I\mathbb{R}$. Defining the *addition* of intervals $[a]$ and $[b]$ as the usual Minkowski addition of sets, that is, $[a]\oplus[b]=\{x+y | x\in[a],\ y\in[b]\}$, we follow the logic of the *principle of containing*: the sum of two intervals certainly contains the sums of all pairs of real numbers, one from each of two intervals. According to [1] one of the most important challenge in the Interval Analysis, is that for the previous mentioned addition of two intervals, the additive inverse (the opposite element) generally does not exist. Indeed, remarking that the neutral element for the addition of the intervals is the interval $[0,0]$, denoted by $\mathbf{0}$, let us suppose by the way of contradiction that for the order interval $[a]=[\underline{a},\overline{a}]$, with $\underline{a}<\overline{a}$ in $\mathbb{R}$, there exists an inverse $[b]=[\underline{b},\overline{b}]$. Hence $[\underline{a},\overline{a}]+[\underline{b},\overline{b}]=\mathbf{0}$, that is, $\underline{a}+\underline{b}=0$ and $\overline{a}+\overline{b}=0$. Therefore $\underline{b}=-\underline{a}$ and $\overline{b}=-\overline{a}$. But $\underline{b}\le\overline{b}$ implies $-\underline{a}\le-\overline{a}$ or, equivalently, $\overline{a}\le\underline{a}$ which contradicts that $\underline{a}<\overline{a}$. Of course, one can define the *scalar multiplication* of an interval $[a]=[\underline{a},\overline{a}]$ with a scalar α in the usual way:

$$\alpha[a]=\begin{cases}[\alpha\underline{a},\alpha\overline{a}], \text{ if } \alpha\in\mathbb{R},\ \alpha\ge 0\\ [\alpha\overline{a},\alpha\underline{a}], \text{ if } \alpha\in\mathbb{R},\ \alpha<0\end{cases}.$$

The axiom of distributivity (the so-called *second-distributive law*) $(\alpha+\beta)[a]=\alpha[a]\oplus\beta[a]$ is certainly true only when $\alpha,\beta\in\mathbb{R}$ are such that $\alpha\beta>0$.

Because the axiom of the existence of the opposite and the axiom of distributivity do *not* work (and as consequence, $I\mathbb{R}$ endowed with the previous algebraic operations, is *not a real vector space*) the *attempt to extend classical results of Functional Analysis* in the *Interval Analysis becomes difficult*. Such is the case, for example, with the extensions theorems for linear operators, such as Mazur-Orlicz Theorem and Hahn-Banach Theorem.

Like in the previous our work “The technique of the auxiliary $\mathrm{i}-$sublinear operator in Interval Analysis. 1” (see [7]) we mention that in 2013, the first author of this paper gave versions of the Hahn-Banach extension Theorem and Mazur-Orlicz Theorem for some functionals *by using closed intervals in ordered vector spaces* - see [5].

In the following as in [5] and [7], we will consider an arbitrary real ordered vector space E and will denote by IE the set of all closed intervals in E. As in the case $E=\mathbb{R}$, we will denote such an interval by $[a]=[\underline{a},\overline{a}]$, where, obviously, $\underline{a}\le\overline{a}$ in E. We say that IE is an *interval-set* or, in short, an $i-set$. Also, we will endow IE with the usual algebraic operations, defined as in $I\mathbb{R}$. We say that IE endowed with this algebraic structure is an *interval-space* or, in short, an $i-space$. Obviously, just like in the case $E=\mathbb{R}$, IE is *not a real vector space*. We mention that we can define the *subtraction* in IE:

$$[a]\ominus[b]=[a]\oplus(-[b]), \text{ where } [a],[b]\in IE \text{ and } -[b] \text{ means } (-1)[b].$$

Let us denote by $\mathcal{O}$ the set of all symmetric intervals of IE:

$$\mathcal{O}=\{[-a,a]\mid a\ge 0,\ a\in E\}.$$

We say that an interval $[-a,a]$, where $a\in E$, is a *symetric interval* and we denote such an interval by $[o]$. If $[a]=[\underline{a},\overline{a}]$ and $[b]=[\underline{b},\overline{b}]$, it follows that:

$$[a]\ominus[b]=[\underline{a},\overline{a}]\ominus[\underline{b},\overline{b}]=[\underline{a}-\overline{b},\overline{a}-\underline{b}].$$

Then $[a]\ominus[a]=[\underline{a}-\overline{a},\overline{a}-\underline{a}]$, so $[a]\ominus[a]=[-(\overline{a}-\underline{a}),\overline{a}-\underline{a}]$, that is, $[a]\ominus[a]$ is a symmetric interval. Hence, $[a]\ominus[a]\in\mathcal{O}$ for each $[a]\in IE$. Remark that if $[a]$ is a *degenerate interval*, that is, $[a]=[a,a]$, then $[a]\ominus[a]=\mathbf{0}$. For this reason, we will say that the set $\mathcal{O}$ is the *null set* of IE. Obviously, $[a]\ominus[a]\ne\mathbf{0}$ if $[a]$ is *not* a degenerate interval $[a,a]$. Thus we conclude again that IE is *not* a *real vector space*.

It is widely recognized that the birth of modern Interval Analysis was marked by the appearance of the book “Interval Analysis” by R.E. Moore (1966), see [9]. Actually, Moore had the idea in Spring 1958 and a year later he published an article about computer arithmetic. Also, the Moore's book was the outgrowth of his Ph.D. Thesis. In a book from 2003, R.E. Moore mentioned that the ideas of fuzzy sets and Interval Analysis are both connected to a general topological theory developed in the 1930s and 1940s, mainly by French mathematicians. Other contributors come from:

a) the German school, represented by U.W. Kulisch (1969);

b) the Bulgarian school, especially S. Markov (1977, 1979, 1995, 2000) and R. Anguelov (2013) – see also a paper by S. Markov and R. Anguelov from 1981;

c) the Spanish school, represented by L.G. Casado, I. García and Ya.D. Sergeyev (2002) – see for example [3].

In Romania, R-M. Dăneț is interested to extend extension theorems for linear operators from vector spaces to interval-spaces. But it turned out that this desideratum is not simple, precisely because interval-spaces are not vector spaces. In [5], [2] and [4], it has been studied how this barrier can be overcome.

2. Preliminaries

Keep in mind that an interval-space is not a vector space and that we are interested in extension theorems. For this aim we must define in the Interval Analysis the notions that would correspond to those of *linear subspace*, *linear functional*, *linear operator* from the classical Functional Analysis. These notions were introduced in [5]. In the following we consider an arbitrary real ordered vector space E and its associated interval-space IE.

Definition 1. An *interval-subspace* (in short, $i-space$) of IE is a nonempty set IS of IE, closed under the algebraic operations (this meaning that for any $[a],[b]\in IS$ and $\alpha\in\mathbb{R}$, we have $[a]\oplus[b]\in IS$ and $\alpha[a]\in IS$).

Obviously $\mathbf{0}=[0,0]\in IS$ (because for any $[a]\in IS$, taking $\alpha=0$, it follows that $\mathbf{0}=0\cdot[a]\in IS$). Also the *null set* $\mathcal{O}$ of IE is an i-subspace of IE. (Recall that $\mathcal{O}=\{[-b,b]\mid b\geq 0,\, b\in E\}$.) We can define the *null set* $\mathcal{O}_{IS}$ of IS as the set $\mathcal{O}\cap IS$. It follows that $\mathcal{O}_{IS}=\{[a]\ominus[a]\mid[a]\in IS\}$. (Indeed, note that for all $[b]\in IS$, we can write $[b]=\left[\frac{1}{2}b\right]\ominus\left[\frac{1}{2}b\right]$ and $\left[\frac{1}{2}b\right]\in IS$, because IS is an i-subspace of IE.)

Definition 2. If IS is as before, an *interval-linear functional* (in short *i-linear functional*) on IS is a real-valued function $f: IS\to\mathbb{R}$ such that:

a) $f([a]\oplus[b])=f([a])+f([b])$, for all $[a],[b]\in IS$;

b) $f(\alpha[a])=\alpha f([a])$, for all $[a]\in IS$ and $\alpha\in\mathbb{R}$.

Remark 1. (Properties of an i-linear functional)

a) If $f: IS\to\mathbb{R}$ is an *i*-linear functional, and $[o]\in\mathcal{O}_{IS}$, then $f([o])=0$ (Indeed, if $[o]=[-a,a]$, then $[o]=[-o]$ and thus $f([o])=f((-1)\cdot[o])=-f([o])$, that is, $f([o])=0$.)

b) If $f: IS\to\mathbb{R}$ is an i-linear functional, and $[a]\in IS,[o]\in\mathcal{O}_{IS}$, then $f([a]\oplus[o])=f([a])$. (Obviously, this results from "a)".)

Definition 3. If IS is like in the previous definition, an *interval-sublinear functional* (in short an *i-sublinear functional*) on IS is a real-valued function $p: IS\to\mathbb{R}$ such that:

1) $p([a]\oplus[b]) \le p([a]) + p([b])$, for all $[a],[b]$ (that is, p is an *i-subadditive functional*);

2) $p(\alpha[a]) = \alpha p([a])$, for all $[a] \in IS$ and $\alpha > 0$ (that is, p is an *i-positively homogeneous functional*);

3) $p([a]\oplus[o]) = p([a])$, for all $[a] \in IS$ and $[o] \in \mathcal{O}_{IS}$.

Remark 2. Note that we assume "3)" for an i-sublinear functional since any i-linear functional (that checks "b)" from Remark 1) must be an i-sublinear functional.

Remark 3. (Properties of an *i*-sublinear functional) Let IS be an i-subspace of an i-space IE $(IS \subseteq IE)$. Let also $p: IS \to \mathbb{R}$ be an i-sublinear functional. Then:

a) $p([o]) = 0$, for all $[o] \in \mathcal{O}_{IS}$;

b) $p(0 \cdot [a]) = 0$, for all $[a] \in IS$.

Remark 4. Note that from now on, F will be an arbitrary Dedekind complete Riesz space. Also, IE will be an interval-space and $IS \subseteq IE$ an arbitrary interval-subspace. In the Definition 2 and Definition 3 we defined what it means an *i-linear functional* $f: IS \to \mathbb{R}$ and an *i-sublinear functional* $p: IS \to \mathbb{R}$. It is obvious that we can extend these definitions, if *instead of real-valued functions*, we consider *operators with values in a Dedekind complete Riesz space* F. Thus we will define the notions of *interval-linear operator* (in short, *i-linear operator*) and *interval-sublinear operator* (in short *i-sublinear operator*), respectively. We mention not only the conditions that have defined those concepts remain the same but also, the same properties from Remark 1 and Remark 3, respectively, are valid.

In [5] the first author generalized in Interval Analysis the well-known Hahn-Banach existence Theorem, the Mazur-Orlicz Theorem and its consequence, the Hahn-Banach extension Theorem, all this for functionals. The obtained results are the following (see also [7, Theorem 1, Theorem 2 and Corollary 3]).

Theorem 1. *Let* IE *be an arbitrary i-space and* $IS \subseteq IE$ *an i-subspace. Let also* $s: IS \to \mathbb{R}$ *be an i-sublinear functional. Then there exists an i-linear functional* $l: IS \to \mathbb{R}$ *such that* $l([v]) \le s([v])$, *for all* $[v] \in IS$, *that is,* $l \le s$ *on* IS.

Theorem 2. *Let* IE, IS *and* $s: IS \to \mathbb{R}$ *be as in the previous result. Let also* A *be a nonempty arbitrary set and* $f: A \to \mathbb{R}$, $g: A \to IS$ *two arbitrary maps. Then the following are equivalent:*

i) there exists an i-linear functional $l: IS \to \mathbb{R}$, *such that:*

1) $l \le s$ *on* IS;

2) $f(a) \le l([g(a)])$, *for each* $a \in A$;

ii) the inequality $\sum_{j=1}^{n} \lambda_j f(a_j) \le s\left(\overset{n}{\underset{j=1}{\oplus}} \lambda_j [g(a_j)]\right)$ *holds for all finite subsets* $\{a_1, a_2, ..., a_n\}$ *in* A *and* $\lambda_1 \ge 0, ..., \lambda_n \ge 0$ *in* $\mathbb{R}$.

Theorem 3. *Let* IE *and* IS *be as in Theorem 1, and* $s: IE \to \mathbb{R}$. *Let also* $t: IS \to \mathbb{R}$ *be an i-linear functional. Then the following are equivalent:*

i) there exists an i-linear functional $l: IE \to \mathbb{R}$, *such that:*

a) $l \le s$ *on* IE, *and*

b) $l = t$ *on* IS, *that is,* l *is an i-linear extension of* t;

ii) $t \le s$ *on* IS.

Remark 5.(see, for example, [7, Preliminaries]) The proof of Theorem 1, like its classical version, uses the so-called "*technique of the auxiliary sublinear functional*". But unlike the classic case, to apply this technique in i-space setting we must overcome the *difficulty created by the inexistence of the opposite element for a nondegenerate closed interval.*

Definition 4. A set $K \subset IE$ is called an *i-convex set* if $\alpha[x] \oplus (1-\alpha)[y] \in K$ for all $[x],[y] \in K$ and $\alpha \in [0,1]$.

Definition 5. An operator $S : K \to F$ (where K is an *i-convex set*) is called *i-convex* (*i-concave*, respectively) if:

$$S\left(\alpha[x] \oplus (1-\alpha)[y]\right) \le \alpha S([x]) + (1-\alpha) S([y]) \text{ } (\ge, \text{ respectively}),$$

for all $[x],[y] \in K$ and $\alpha \in [0,1]$.

3. Main part

Auxiliary i-sublinear operators. A new example

It is known (see for example [7, Main part "B)"]) that in the classical case of the extension of some linear operators between (ordered) vector spaces, auxiliary sublinear operators appear in the so-called "*the technique of the auxiliary sublinear operator*". This technique is a method to prove the existence of a linear operator by using the Hahn-Banach existence Theorem (see the version of Theorem 1 in the vector spaces setting). Actually, the technique of the auxiliary sublinear operator has two steps (note that here, X will be an arbitrary vector space and F a Dedekind complete Riesz space):

1) construct a sublinear operator $U : X \to F$;

2) apply the Hahn-Banach existence Theorem obtaining a linear operator $L : X \to F$ dominated by U (i.e., $L \le U$ on X).

This technique, due to V. Pták (1956), was used by him to give a much simpler proof to an extension theorem known as the Mazur-Orlicz Theorem (see [10]).

Note that in [7] we provided an example of an *auxiliary i-sublinear operator* t_1 and used it to prove some existence and extension theorems for i-sublinear operators in Interval Analysis, two of them being the operatorial form for the Hahn-Banach existence Theorem and Hahn-Banach extension Theorem (see [7, section "C)" in "Main part"]). The previously mentioned operator t_1 appeared in the following result.

Theorem 4.([7, Theorem 4]) *Let $S : IE \to F$ be an i-sublinear operator, $K \subset IE$ a nonempty i-convex set, $P : K \to F$ an i-concave operator such that $P \le S$ on K (that is, $P([x]) \le S([x])$ for all $[x] \in K$). We define $t_1 : IE \to F$ by:*

$$t_1([x]) = \inf\left\{ S([x] \oplus \alpha[u]) - \alpha P([u]) \,\middle|\, \alpha \ge 0, [u] \in K \right\}, \; [x] \in IE. \tag{1}$$

Then t_1 is an i-sublinear operator such that $t_1 \le S$ (on IE), that is, $t_1([x]) \le S([x])$ for all $[x] \in IE$. Moreover, if $T : IE \to F$ is an i-linear operator, then the following are equivalent:

i) $T \le t_1$, on IE (that is, $T([x]) \le t_1([x])$ for all $[x] \in IE$);

ii) $T \le S$ on IE (that is, $T([x]) \le S([x])$ for all $[x] \in IE$), and $P \le T$ on K (that is, $P([x]) \le T([x])$ for all $[x] \in K$).

Now we give *a new example of an auxiliary i-sublinear operator*, namely t_2, and we use it to prove:

a) an operatorial form of the Mazur-Orlicz Theorem in i-space setting (see Theorem 5 below);

b) an operatorial form of the Hahn-Banach extension Theorem in i-space setting (see Corollary 6 below).

Note that Theorem 5 and Corollary 6 below generalize two results from [5]. Now let's define the auxiliary i-sublinear operator $t_2 : IE \to F$.

Let IE be an arbitrary i-space, and F a Dedekind complete Riesz space. Let also A be an arbitrary nonempty set and $f : A \to F$, $g : A \to IE$ two arbitrary maps. Let also $S : IE \to F$ be a given i-sublinear operator and $[x] \in IE$. The operator t_2 is defined by:

$$t_2([x]) = \inf\left\{ S\left([x] \oplus \left(\overset{n}{\underset{i=1}{\oplus}} \lambda_i [g(a_i)]\right)\right) - \sum_{i=1}^{n} \lambda_i f(a_i) \,\middle|\, n \in \mathbb{N},\ a_1,...,a_n \in A,\ \lambda_1 \geq 0,...,\lambda_n \geq 0 \text{ in } \mathbb{R} \right\} \tag{2}$$

It can be shown that t_2 is well-defined, it is an i-sublinear operator and $t_2 \leq S$, on IE. Actually in the proof of Theorem 5, t_2 will play the role of the auxiliary i-sublinear operator. The following result also generalizes the classical Mazur-Orlicz Theorem - see [8] and [6, Theorem 2.1]. (Note that we denote $[g(a_i)]$ only to remind that actually, $g(a_i) \in IE$ is an interval in E.)

Theorem 5.(the operatorial form of a Mazur-Orlicz type Theorem in i-space setting) *Let IE be an i-space and $S : IE \to F$ an i-sublinear operator. Let also A be a nonempty arbitrary set and $f : A \to F$, $g : A \to IE$ two arbitrary maps. Then the following are equivalent:*

i) there exists an i-linear operator $L : IE \to F$ such that:

a) $L \leq S$ on IE;

b) $f(a) \leq L([g(a)])$ for each $a \in A$;

ii) the inequality

$$\sum_{i=1}^{n} \lambda_i f(a_i) \leq S\left(\overset{n}{\underset{i=1}{\oplus}} \lambda_i [g(a_i)]\right)$$

holds for all finite subsets $\{a_1,...,a_n\}$ in A and $\lambda_1 \geq 0,...,\lambda_n \geq 0$ in $\mathbb{R}$.

(Here, as we mentioned before about $[g(a_i)]$, we also make known that we denoted $[g(a)]$, only to remind that, actually, $g(a)$ is an order interval from IE.)

Proof. "$i) \Rightarrow ii)$" is obvious. Indeed since L is an i-linear operator and $L \leq S$, it follows that:

$$\sum_{i=1}^{n} \lambda_i f(a_i) \leq \sum_{i=1}^{n} \lambda_i L([g(a_i)]) = L\left(\overset{n}{\underset{i=1}{\oplus}} \lambda_i [g(a_i)]\right) \leq S\left(\overset{n}{\underset{i=1}{\oplus}} \lambda_i [g(a_i)]\right).$$

"$ii) \Rightarrow i)$" First we will prove that "$ii) \Rightarrow i)\ a)$", by using the *technique of the auxiliary i-sublinear operator*.

Since $t_2 : IE \to F$ is an i-sublinear operator, by applying our Hahn-Banach existence type Theorem for i-space setting (see Theorem 1 above or [7, Theorem 5]) we deduce that there exists an i-linear operator $L : IE \to F$ such that $L \leq t_2$ on IE (that is, $L([x]) \leq t_2([x])$ for all $[x] \in IE$). By using the definition of t_2 (see (2)), and taking $\lambda_i = 0$ for all $i = 1, 2, ..., n$, it follows that $t_2([x]) \le S([x])$ for all $[x] \in IE$, and then $L \le S$ on IE. Hence we prove that "$ii) \Rightarrow i)$ $a)$".

Now we will prove that "$ii) \Rightarrow i)$ $b)$". We have to prove that $f(a) \le L([g(a)])$ for all $a \in A$. We will use the inequalities $L \le t_2$ and $t_2 \le S$, and the fact that L is an i-linear operator. Then for all $a \in A$, it follows:

$$-L([g(a)]) = L(-[g(a)]) \le t_2(-[g(a)]) \le S([g(a)]) \oplus (-[g(a)]) - f(a) =$$
$$= S([g(a)] \text{Ş} [g(a)]) - f(a) = S([o]) - f(a) = -f(a).$$

(We apply that S is an i-sublinear operator and, consequently, $S([o]) = 0$, for all $[o] \in \mathcal{O}$; here we take $[o] = [g(a)] \text{Ş} [g(a)]$.) Hence $f(a) \le L([g(a)])$ for all $a \in A$. ■

Remark 6. Note that in the previous result, since A is a nonempty arbitrary set, we can include the case when A is a set of intervals (in an arbitrary ordered vector space). If this is the case, we will write $[a_i]$ instead of a_i and $[a]$ instead of a. If moreover, $A \subseteq IE$, then g can be the inclusion and therefore we will write $[a_i]$, $f([a_i])$ and $f([a])$ instead of $[g(a_i)]$, $f(a_i)$ and $f(a)$, respectively.

Note that now we can prove our generalization for the operatorial form of the Hahn-Banach extension Theorem (see Theorem 3).

Corollary 6. (Hahn-Banach extension type Theorem in the i-space setting)
Let IE be an i-space, $IG \subseteq IE$ an i-subspace and F a Dedekind complete Riesz space. Let also $S : IE \to F$ an i-sublinear operator and $T : IG \to F$ an i-linear operator. Then the following are equivalent:

i) there exists an i-linear operator $L : IE \to F$ such that:

a) $L \le S$ on IE, and

b) $L = T$ on IG, that is, L is an i-linear extension of T.

ii) $T \le S$ on IG.

Proof. Take in the version of Theorem 5 mentioned in the Remark 6, IG instead of A, g the inclusion of IG in IE and $f = T$. Then the inequality

$$\sum_{i=1}^{n} \lambda_i f([a_i]) \le S\left(\bigoplus_{i=1}^{n} \lambda_i [g(a_i)]\right),$$

that is,

$$\sum_{i=1}^{n} \lambda_i T([a_i]) \le S\left(\bigoplus_{i=1}^{n} \lambda_i [a_i]\right)$$

becomes (by using the i-linearity of T):

$$T\left(\bigoplus_{i=1}^{n} \lambda_i [a_i]\right) \le S\left(\bigoplus_{i=1}^{n} \lambda_i [a_i]\right)$$

for all $n \in \mathbb{N}^*$, $\lambda_i \ge 0$ and $[a_i] \in IG$ for all $i = 1, 2, ..., n$.
So $T \le S$ on IG. ■

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NOTE ON THE QUADRATIC ASSIGNMENT PROBLEM

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Abstract. The quadratic assignment problem (QAP) is a NP-hard combinatorial optimization problem introduced by Koopmans and Beckmann in 1957 to model a plant location problem. This paper, which continues, to our Conference of Department of Mathematics and Computer Science, the study presented last year, gives some general remarks on the problem and discuss a new algorithm to try to find permutations $\pi \in S_n$ to approximate the exact solution of a QAP.

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1. Preliminaries

The quadratic assignment problem (QAP) is a NP-hard (nondeterministic polynomial time complete) combinatorial optimization problem introduced by Koopmans and Beckmann [9] which proposed a mathematical model of assigning a set of economic activities to a set of locations, which was presented also in [3]. However the applications of the QAP in the field of facility location problems remain very useful, there are other applications of QAP as areas as: wiring problems in electronics [12], the transportation problems [11], parallel and distributed computing, statistical data analysis, design of control panels and typewriter keyboards, chemistry, archeology, balancing of turbine runners and computer manufacturing (see [1] for more details). Moreover, a number of other combinatorial optimization problems, as for example, a series of problems in graphs can be reduced to QAP.

Consider two square matrices of order n $A = (a_{ij})$, $B = (b_{ij})$ and a permutation $\pi \in S_n$, where S_n is the symmetric group of all permutations of a set with n elements. QAP *with coefficient matrices* A *and* B, denoted QAP(A, B), can be stated as follows

$$\min_{\pi \in S_n} \sum_{i=1}^{n} \sum_{j=1}^{n} a_{\pi(i)\pi(j)} b_{ij}. \tag{1.1}$$

Denote $A^{\pi} = (a_{\pi(i)\pi(j)})$, the matrix obtained from A by permuting its rows and columns according to permutation π and

$$f(A, B, \pi) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{\pi(i)\pi(j)} b_{ij} = \mathrm{tr}(A^{\pi T} B), \tag{1.2}$$

where tr$A$ is the trace of A and A^T is the transposed matrix of A. Consider P^{π} the permutation matrix corresponding to the permutation π. Since a permutation matrix is an orthogonal matrix it follows that $A^\pi = P^{\pi T} A P^\pi$ and the coefficients of the characteristic polynomial of A are invariants with respect to any permutation π. Moreover, the set of the elements a_{ii} is also invariant with respect to any permutation $\pi \in S_n$.

Consider, as a typical example, *the Hamiltonian cycle problem*, that is to seek a Hamiltonian cycle in an arbitrary graph $G = (V, E)$ of order n. If $V = \{v_l, v_2, ..v_n\}$, define two $n \times n$ matrices $A = (a_{ij})$ and $B = (b_{ij})$ as follows:

$$a_{ij} = \begin{cases} 1, & \text{if } (v_i, v_j) \in E \\ 2, & \text{otherwise} \end{cases},$$

$$b_{ij} = \begin{cases} 1, & \text{if } j = i+1, i = 1, 2, ..., n-1, \text{ or } i = n, j = 1 \\ 0, & \text{otherwise} \end{cases}.$$

Then by considering QAP(A, B) it follows that that the optimal value of QAP is equal to n if and only if G contains a Hamiltonian cycle.

Let R_i^A, $i = 1, ..., n$, be the rows of the matrix A. Denote by $sp_{R,ij}^{A,B} =< R_i^A, R_j^B >$ the scalar product of R_i^A and R_j^B considered as vectors in $\mathbb{R}^n$ and $SP_R(A, B) = (sp_{R,ij}^{A,B})$. Then it follows that $f(A, B, \pi) = \text{tr} SP_R(A^\pi, B)$.

Similarly, if C_i^A, $i = 1, ..., n$, are the columns of the matrix A, denote $sp_{C,ij}^{A,B} =< C_i^A, C_j^B >$. Thus we obtain the matrix $SP_C(A, B) = (sp_{C,ij}^{A,B})$ and $f(A, B, \pi) = \text{tr} SP_C(A^\pi, B)$.

A useful result for estimation of scalar product of two n dimensional vectors $U = (u_1, ..., u_n)$, $V = (v_1, ..., v_n)$ having non-negative components is a theorem of Hardy, Littlewood and Pólya in [7], p. 261. Thus if $u_1 \geq u_2 \geq ... \geq u_n$ and $v_1 \leq v_2 \leq ... \leq v_n$, then, for any $\pi_1, \pi_2 \in S_n$,

$$< U, V > \leq < U^{\pi_1}, V^{\pi_2} > . \tag{1.3}$$

Remark 1.1. Assume that the elements of A are ordered such that

$$a_{ij} \leq a_{i'j'} \text{ if either } i \geq i' \text{ or } i = i' \text{ and } j \geq j' \tag{1.4}$$

and the elements of B are ordered in the following way

$$b_{ij} \geq b_{i'j'} \text{ if either } i \geq i' \text{ or } i = i' \text{ and } j \geq j'. \tag{1.5}$$

Then, by (1.3), for every $\pi \in S_n$ $f(A, B, 1) \leq f(A, B, \pi)$ and, in this case, $f(A, B, 1)$ is the exact solution of the quadratic assignment problem.

This remark cannot be use in an arbitrary case because, generally, the elements of the matrices A and B cannot be ordered as in (1.4) and (1.5) by using a permutation $\pi \in S_n$. However, by analogy, we use this idea as an algorithm to try to find permutations $\pi \in S_n$ such that, when $f(A, B, 1)$ is not the exact solution of the quadratic assignment problem, $f(A, B, \pi) < f(A, B, 1)$.

For particular cases as circulant and small bandwidth matrices, Monge and Toeplitz matrices there are a series of results on the solutions of the quadratic assignment problem. However that generally problems of size larger than 20 cannot be solved to optimality in reasonable time and problems of size larger than 15 are considered to be difficult (see [1]). Thus heuristic approaches to the QAP are often used (see [1]-[6]), [8]).

Since the results achieved by using the best existing exact algorithms are modest, the computation of lower bounds is the main subject of many papers on the QAP. A lower bound is a basic tool for testing the quality of the solutions produced by heuristics. We shortly describe Gilmore-Lawler bound (see [1] and [10]).

Consider QAP(A, B) of size n. Define a new $n \times n$ matrix $C = (c_{ij})$,

$$c_{ij} = \min_{\pi \in S_n,\, \pi(j)=i} \sum_{k=1}^{n} a_{i\pi(k)} b_{jk}.$$

By (1.3) it follows that c_{ij} can be easily computed by sorting the vectors L_i^A and L_j^B in increasing and decreasing order, respectively.

Assume that a preprocessing step has been performed for sorting the rows of both matrices A and B in increasing and decreasing order, respectively. Then, it takes $O(n^3)$ elementary operations to compute all c_{ij} and the preprocessing step takes $O(n^2 \log n)$ elementary operations. Then, by (1.2), we get, for each $\pi \in S_n$,:

$$f(A, B, \pi) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{\pi(i)\pi(j)} b_{ij} \geq \sum_{i=1}^{n} < L_{\pi(i)}^{A^\pi}, L_i^B > \geq \sum_{i=1}^{n} c_{\pi(i),i}.$$

Hence it follows that

$$\min_{\pi \in S_n} f(A, B, \pi) \geq GL,$$

where $GL := \min_{\pi \in S_n} \sum_{i=1}^{n} c_{\pi(i),i}$ is the Gilmore-Lawler bound for QAP(A,B). It is known that the computation of the matrix C can be done in $O(n^5)$ time, which is the time complexity for the computation of the GL. However the gap between the Gilmore-Lawler bound and the optimal solution increases quite fast as the size of the problem increases.

2. Algorithm description

In this section we present a new algorithm to try to find permutations $\pi \in S_n$ to approximate the exact solution of a QAP. We describe the four steps of the proposed algorithm.

1. Initialization

Since, for two arbitrary matrices A and B, the difference between $f(A, B, 1)$ and the exact solution of the corresponding QAP problem could be large, as an initial step, we replace A and B with the matrices $A^{(0)} = A^{\pi_1}$ and $B^{(0)} = B^{\pi_2}$, obtained by using two permutations $\pi_1, \pi_2 \in S_n$. Notice that by replacing a matrix A with A^π, for every i, the rows R_i^A and $R_{\pi(i)}^{A^\pi}$ contain the same elements. Thus

$$\sigma_i^A := \sum_{j=1}^{n} a_{i,j} = \sum_{j=1}^{n} a_{\pi(i),\pi(j)}. \tag{2.1}$$

Inspired by Remark 1.1 we choose π_1 such that

$$\sigma^{A^{(0)}}_{\pi_1(1)} \geq \sigma^{A^{(0)}}_{\pi_1(2)} \geq ... \geq \sigma^{A^{(0)}}_{\pi_1(n)}. \tag{2.2}$$

Similarly, the permutation π_2 is constructed such that

$$\sigma^{B^{(0)}}_{\pi_2(1)} \leq \sigma^{B^{(0)}}_{\pi_2(2)} \leq ... \leq \sigma^{B^{(0)}}_{\pi_2(n)}. \tag{2.3}$$

Since $f(A^{\pi_1}, B^{\pi_2}, 1) = f(A, B, \pi_1\pi_2^{-1})$ and S_n is a group, if $f(A^{\pi_1}, B^{\pi_2}, 1) < f(A, B, 1)$, without loss of generality, we can replace A and B with the matrices $A^{(0)}$ and $B^{(0)}$.

2. Construction of matrices of scalar products $SP_R(A^0, B^0)$ and $SP_C(A^0, B^0)$

As in Section 1 we consider $SP_R(A^0, B^0)$ (resp. $SP_C(A^0, B^0)$) , the matrix of the scalar products of the rows (resp. of the columns) of the matrices $A^{(0)}$ and $B^{(0)}$.

3. Finding a possible smaller value of $f(A, B, \pi)$

We seek a new permutation $\pi \in S_n$ such that, for every i, $\pi(j_i) = i$, and

$$\sum_{i=1}^{n} (sp^{A^{(0)},B^{(0)}}_{R,i,j_i} - sp^{A^{(0)},B^{(0)}}_{R,i,i}) < 0. \tag{2.4}$$

In this manner we construct the permutation π and by permuting the rows of $A^{(0)}$ with respect to π, by (2.4), we find the matrix $\tilde{A}^{(0)}$. Then the trace of the matrix $SP_R(\tilde{A}^{(0)}, B^{(0)})$ cannot be greater that those of $SP_R(A^{(0)}, B^{(0)})$. Since $f(A^{(0)}, B^{(0)}, 1) = \mathrm{tr} SP_R(A^{(0)}, B^{(0)})$, if $A^{(1)} = {A^{(0)}}^{\pi}$ and $f(A^{(1)}, B^{(0)}, 1) < f(A^{(0)}, B^{(0)}, 1)$, we can replace $A^{(0)}$ by $A^{(1)}$. When the form of the matrix $SP_R(A^{(0)}, B^{(0)})$ is not suitable we can use the matrix $SP_C(A^{(0)}, B^{(0)})$. In this case (2.4) is replaced by

$$\sum_{j=1}^{n} (sp^{A^{(0)},B^{(0)}}_{C,i_j,j} - sp^{A^{(0)},B^{(0)}}_{C,j,j}) < 0. \tag{2.5}$$

and, for every j, $\pi(i_j) = j$.

4. Repeat, if it is suitable, Steps 2 and 3 for the matrices $A^{(1)}$ and $B^{(0)}$

The flowchart of this algorithm is given in Fig. 1.

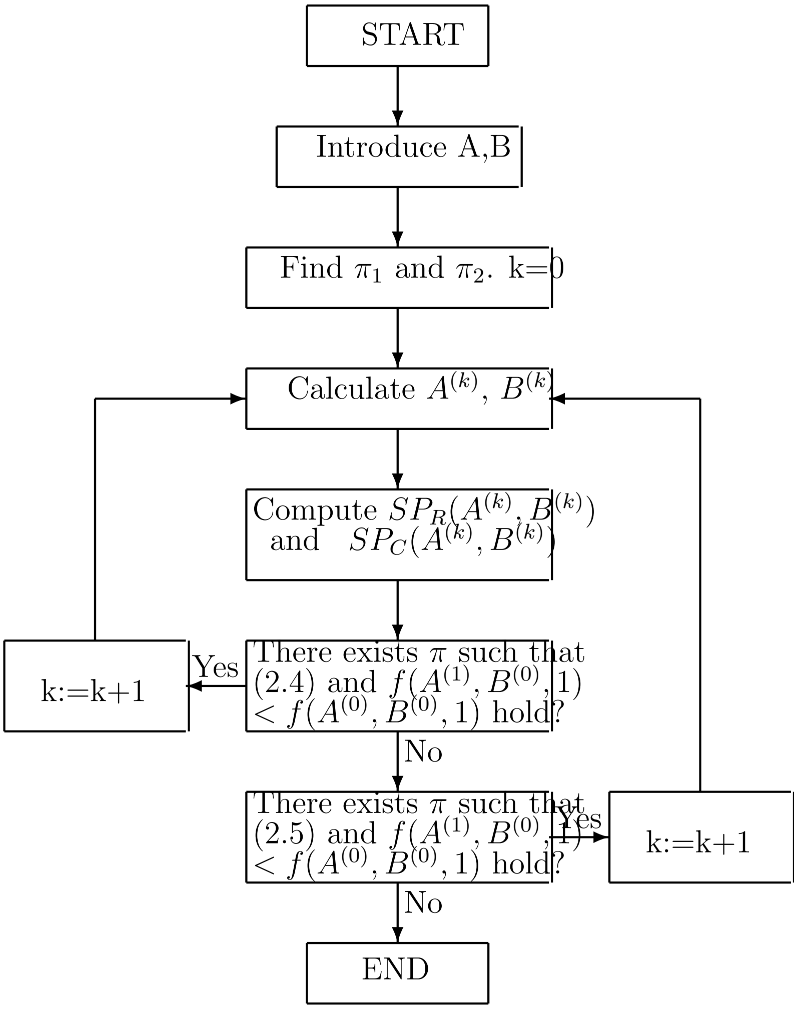

FIGURE 1.

We apply the algorithm in the following example.

Example 2.1. Consider the following matrices

$$A = \begin{bmatrix} 0 & 3 & 5 & 2 \\ 1 & 2 & 3 & 0 \\ 4 & 1 & 1 & 1 \\ 5 & 1 & 2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 1 & 3 & 1 & 2 \\ 2 & 1 & 0 & 3 \\ 3 & 2 & 1 & 1 \end{bmatrix}.$$

In this case

$$SP_R(A,B) = \begin{bmatrix} 12 & 18 & 9 & 13 \\ 7 & 10 & 4 & 10 \\ 7 & 10 & 12 & 16 \\ 10 & 12 & 14 & 20 \end{bmatrix}, \quad SP_C(A,B) = \begin{bmatrix} 24 & 17 & 6 & 19 \\ 10 & 9 & 9 & 11 \\ 16 & 14 & 15 & 16 \\ 7 & 3 & 5 & 6 \end{bmatrix}$$

and $f(A,B,1) = 54$.

Step 1.

To construct the matrices $A^{(0)}$ and $B^{(0)}$ we see that

$$\sigma_1^A = 10,\ \sigma_2^A = 6,\ \sigma_3^A = 7,\ \sigma_4^A = 9$$

and

$$\sigma_1^B = 4,\ \sigma_2^B = 7,\ \sigma_3^B = 6,\ \sigma_4^B = 7.$$

Hence we can take $\pi_1 = (2,4)$ and $\pi_2 = (2,3)$ and

$$A^{(0)} = \begin{bmatrix} 0 & 2 & 5 & 3 \\ 5 & 1 & 2 & 1 \\ 4 & 1 & 1 & 1 \\ 1 & 0 & 3 & 2 \end{bmatrix}, \quad B^{(0)} = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 2 & 0 & 1 & 3 \\ 1 & 1 & 3 & 2 \\ 3 & 1 & 2 & 1 \end{bmatrix}.$$

Then we get

$$SP_R(A^{(0)},B^{(0)}) = \begin{bmatrix} 7 & 14 & 23 & 15 \\ 8 & 15 & 14 & 21 \\ 7 & 12 & 10 & 16 \\ 3 & 11 & 14 & 11 \end{bmatrix}, \quad SP_C(A^{(0)},B^{(0)}) = \begin{bmatrix} 17 & 5 & 19 & 24 \\ 5 & 5 & 4 & 7 \\ 19 & 14 & 11 & 16 \\ 12 & 9 & 8 & 10 \end{bmatrix}$$

and $f(A^{(0)},B^{(0)},1) = 43$.

Step 2.

The matrix $SP_R(A^{(0)},B^{(0)})$ is not useful in this case because, for every j, $sp_{R,3,3}^{A^{(0)},B^{(0)}} \leq sp_{R,3,j}^{A^{(0)},B^{(0)}}$ and $sp_{R,4,4}^{A^{(0)},B^{(0)}} \leq sp_{R,4,j}^{A^{(0)}}$ imply $\pi(3) = 3$ and $\pi(4) = 4$. Hence it follows that we cannot find a suitable π in this case. From $SP_C(A^{(0)},B^{(0)})$, by using (2.5), we see that $\pi = (132)$ is suitable. Thus

$$A^{(1)} = \begin{bmatrix} 1 & 2 & 5 & 1 \\ 1 & 1 & 4 & 1 \\ 2 & 5 & 0 & 3 \\ 0 & 3 & 1 & 2 \end{bmatrix}.$$

and $f(A^{(1)},B^{(0)},1) = 35$ and this is the exact solution of the QAP(A,B).

Notice that

$$SP_R(A^{(1)},B^{(0)}) = \begin{bmatrix} 5 & 4 & 7 & 8 \\ 18 & 12 & 22 & 18 \\ 16 & 11 & 6 & 18 \\ 12 & 7 & 14 & 12 \end{bmatrix}, \quad SP_C(A^{(1)},B^{(0)}) = \begin{bmatrix} 6 & 10 & 20 & 16 \\ 4 & 9 & 16 & 13 \\ 15 & 13 & 13 & 14 \\ 8 & 7 & 10 & 7 \end{bmatrix}.$$

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A NOTE ON THE HEAVISIDE STEP FUNCTION

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Abstract. The Heaviside step function plays a fundamental role not only in control theory and signal processing, but also in network reliability. In this short note we prove that the Bernstein polynomial and the full-Hermite interpolation polynomial are identical for the unit step function.

Mathematics Subject Classification (2010): 41A05, 41A10, 05C31, 68Rxx
Key words: Heaviside step function, networks, reliability polynomial, Bernstein polynomial, full Hermite interpolation polynomial

1. The Heaviside step function and the reliability of a network

A network is a probabilistic graph, $\mathbf{N} = (V, E)$, where V is the set of nodes (vertices) and E is the set of (undirected) edges [3]. We assume that each edge is operational (closed) with probability p and fails with probability $q = 1 - p$ (nodes are always operational). We also consider in the set of nodes V two special nodes, S and T, called terminals (source/input and terminus/output respectively). The two-terminal reliability of the network $\mathbf{N}$, denoted by $h(p)$, is the probability that there exists a path (a sequence of adjacent edges) made of operational edges between S and T. Thus, the reliability $h(p)$ is a polynomial function in p whose expression depends on the number of *pathsets* of the network. A *pathset* is a subset of edges that contains at least one path between the S and T. Let $\mathcal{P}$ be the set of all the pathsets of $\mathbf{N}$. If $n = |E|$ is the size of the graph and N_i is the number of pathsets with exactly i edges, $i = 1, 2, \ldots, n$, then the two-terminal reliability of the network $\mathbf{N}$ can be expressed as (see [3]):

$$h(p) = \sum_{P \in \mathcal{P}} p^{|P|} q^{n-|P|} = \sum_{i=1}^{n} N_i p^i (1-p)^{n-i}. \tag{1.1}$$

Finding the exact expression of the reliability polynomial is a very difficult task, especially for complex networks. It belongs to the class of $\#P$-complete problems, a class of computationally equivalent counting problems that are at least as difficult as the NP-complete problems [3]. For a special type of networks - namely for *hammock networks* [12] - the exact reliability polynomials were computed in [4] for relatively "small" dimensions (up to 5×5 edges). For hammock networks of "large" dimensions, different methods of approximation were employed - like the full Hermite interpolation polynomials (see [7]) or the Markov chain method (see [10]). A detailed description of the hammock networks and their duality properties can be found in [6].

In "real life" the networks are used either for communication, or for computation. The "ideal" reliability is very different in the two cases. Thus, as shown in [1], an ideal network used for *communication* would have the reliability $h(p) = 1$ for any $p \in (0, 1]$, while the reliability of a network used for *computation* should tend to the shifted Heaviside step function (restricted to the interval $[0, 1]$ since p is a *probability*):

$$\eta_a(p) = \begin{cases} 0, & p \in [0, a) \\ 1/2, & p = a \\ 1, & p \in (a, 1], \end{cases} \tag{1.2}$$

where a is a fixed number between 0 and 1. We shall use the value $a = 1/2$ because we study symmetric (square) networks mostly.

2. The full Hermite interpolation polynomial

Theorem 2.1. *Given $n+1$ distinct points in the interval $[a, b]$, $x_0 < x_1 < \ldots < x_n$, $n+1$ positive integers $k_0, k_1, \ldots, k_n > 0$ and the function $f : [a, b] \to \mathbb{R}$ of class $C^k[a, b]$, where $k = \max_{j=\overline{0,n}} k_j$, there exists a unique polynomial $H_{\mathbf{k}}(x)$, $\mathbf{k} = (k_0, k_1, \ldots, k_n)$, of degree at most $N = \sum_{j=0}^{n} k_j - 1$ such that*

$$H_{\mathbf{k}}^{(i)}(x_j) = f^{(i)}(x_j)\,, \text{ for all } i = 0, 1, \ldots, k_j - 1,\ j = 0, 1, \ldots, n.$$

This theorem was firstly proved in 1878 by Charles Hermite [9], so the polynomial $H_{\mathbf{k}}(x)$ is called the *full Hermite interpolation polynomial.* Its expression is

$$H_{\mathbf{k}}(x) = \sum_{j=0}^{n} \sum_{i=0}^{k_j-1} f_j^{(i)} l_{i,j}(x)\,, \tag{2.1}$$

where, for every $i = 0, 1, \ldots, k_j - 1$ and $j = 0, 1, \ldots, n$,

$$l_{i,j}(x) = u_j(x) \frac{(x - x_j)^i}{i!} \sum_{r=0}^{k_j-i-1} \frac{1}{r!} v_j^{(r)}(x_j)(x - x_j)^r, \tag{2.2}$$

$$u_j(x) = \prod_{\substack{t=0\\ t\neq j}}^{n} (x - x_t)^{k_t}, \quad v_j(x) = \frac{1}{u_j(x)}.$$

Suppose that $n = 1$, so we have two distinct points, which may be the limits of the interval: $x_0 = a$, $x_1 = b$. We use the notations $k_0 = k$, $k_1 = m$ and denote by $H_{k,m}(x)$ the full Hermite interpolation polynomial of order (k, m) for the function $f \in C^{\max(k,m)}[a, b]$. Thus, $H_{k,m}(x)$ is the (unique) polynomial of degree at most $k + m - 1$ such that

$$f^{(i)}(a) = H_{k,m}^{(i)}(a)\,,\ i = 0, 1, \ldots, k - 1,$$

and

$$f^{(i)}(b) = H_{k,m}^{(i)}(b)\,,\ i = 0, 1, \ldots, m - 1.$$

The expression of this polynomial is:

$$H_{k,m}(x) = \sum_{i=0}^{k-1} f^{(i)}(a)A_i(x) + \sum_{i=0}^{m-1} f^{(i)}(b)B_i(x)\,,$$

where

$$A_i(x) = \left(\frac{b-x}{b-a}\right)^m \frac{(x-a)^i}{i!} \sum_{r=0}^{k-i-1} \binom{m+r-1}{m-1} \left(\frac{x-a}{b-a}\right)^r,$$

for $i = 0, 1, \dots, k-1$, and

$$B_i(x) = \left(\frac{x-a}{b-a}\right)^k \frac{(x-b)^i}{i!} \sum_{r=0}^{m-i-1} \binom{k+r-1}{k-1} \left(\frac{b-x}{b-a}\right)^r,$$

for $i = 0, 1, \dots, m-1$.

The full Hermite interpolation polynomial is a useful tool in the approximation of functions. It can be used for instance to approximate the reliability of a network (as in [7]), or the solution of a boundary value problem (see [11]).

Let us find the expression of the full Hermite interpolation polynomial of order (n, n) for the Heaviside step function $\eta_{1/2}(p)$ (at the points $a = 0$, $b = 1$). Since all the derivatives of the function $\eta_{1/2}(p)$ are equal to 0, it follows that the full Hermite interpolation polynomial $H_{n,n}(p))$ is written:

$$H_{n,n}(p) = p^n \cdot \sum_{r=0}^{n-1} \binom{n+r-1}{n-1} (1-p)^r. \tag{2.3}$$

Note that (2.3) represents the full Hermite interpolation polynomial of any function $f : [0, 1] \to \mathbb{R}$ such that

$$f(0) = 0, f(1) = 1 \;\; \textit{and} \;\; f^{(k)}(0) = f^{(k)}(1) = 0 \;\; \textit{for} \;\; k = 1, 2, \dots, n-1, \tag{2.4}$$

and the reliability polynomial of a hammock network of dimensions $n \times n$ satisfies (2.4) (see [6],[7]).

3. Bernstein polynomials

In 1912, in his two-page paper "Démonstration du théorème de Weierstrass fondée sur le calcul des probabilités", Serghei Bernstein introduced, for a given function $f : [0, 1] \to \mathbb{R}$, the sequence of polynomials

$$B_n(x) = \sum_{k=0}^{n} f\left(\tfrac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k}, n = 1, 2, \dots, \tag{3.1}$$

and he proved that this sequence uniformly converges to f on $[0, 1]$ if f is a continuous function [2]. The polynomials (3.1) used in his elegant constructive proof (called now the *Bernstein polynomials* of the function f) provide simultaneous approximation of the function and its derivatives: if f is a function of class $\mathcal{C}^p[0,1]$, then $B_n^{(p)}(f;x)$ uniformly converges to $f^{(p)}$ on $[0,1]$ (see [5], Theorem 6.3.2).

Moreover, if the function f is not continuous on the whole interval $[0,1]$, the sequence of Bernstein polynomials $B_n(x)$ is still (simply) convergent to $f(x)$ at any point x where the function is continuous ([5], Theorem 6.2.2).

The Bernstein polynomials have remarkable "shape-preserving" properties: for any $n = 1, 2, \ldots$ $B_n(0) = f(0)$, $B_n(1) = f(1)$; if f is continuous, then $B_n(x)$ lies between the extreme values of the function; if f is monotonic or convex, then $B_n(x)$ is correspondingly monotonic or convex (see [5], Theorem 6.3.3). But the price paid for these beautiful approximation properties is the slowness of the convergence (see [5]).

We remark that the polynomials used in the expression (3.1),

$$b_{k,n}(x) = \binom{n}{k} x^k (1-x)^{n-k}\ , \quad k = 0, 1, \ldots, n, \tag{3.2}$$

are the well-known binomial probabilities. If $x \in [0,1]$ is the probability of a certain event E, then $b_{k,n}(x)$ is the probability that E will occur exactly k times in n independent trials.

It is easy to see that the $n+1$ polynomials of degree n from (3.2) form a basis in the space of polynomials of degree at most n. The expression of the reliability polynomial (1.1) using the Bernstein basis (3.2) is:

$$h(p) = \sum_{i=0}^{n} \mu_i b_{i,n}(p)\ , \tag{3.3}$$

where

$$\mu_i = \frac{N_i}{\binom{n}{k}},\ i = 0, 1, \ldots, n \tag{3.4}$$

are positive coefficients, less than 1 (since $N_i \leq \binom{n}{k}$ is the number of pathsets with exactly i edges - see also [8]).

4. The main result

According to the formula (3.1), the Bernstein polynomial $B_{2n}(p)$ for the function $\eta_{1/2}(p)$ is written:

$$B_{2n}(p) = \sum_{k=0}^{2n} \eta_{1/2}\left(k/2n\right) \binom{2n}{k} p^k (1-p)^{2n-k}$$

$$= \frac{1}{2}\binom{2n}{n} p^n (1-p)^n + p^n \sum_{k=1}^{n} \binom{2n}{n+k} p^k (1-p)^{n-k}.$$

Theorem 4.1. *For the Heaviside step function*

$$\eta_{1/2}(p) = \begin{cases} 0, & p \in [0, 1/2) \\ 1/2, & p = 1/2 \\ 1, & p \in (1/2, 1], \end{cases}$$

the Bernstein polynomial,

$$B_{2n}(p) = \frac{1}{2}\binom{2n}{n} p^n (1-p)^n + p^n \sum_{k=1}^{n} \binom{2n}{n-k} p^k (1-p)^{n-k}, \tag{4.1}$$

and the full Hermite interpolation polynomial,

$$H_{n,n}(p) = p^n \cdot \sum_{k=0}^{n-1} \binom{n+k-1}{n-1} (1-p)^k, \tag{4.2}$$

are identical, for all $n = 1, 2, \dots$.

Proof. In order to prove that the polynomials $B_{2n}(p)$ and $H_{n,n}(p)$ are identical we have to show that

$$\sum_{i=0}^{n-1} \binom{n+k-1}{k} (1-p)^k \equiv \frac{1}{2}\binom{2n}{n} (1-p)^n + \sum_{i=1}^{n} \binom{2n}{n-i} p^i (1-p)^{n-i},$$

or, equivalently (denoting $q = 1-p$ and $j = n-i$),

$$\sum_{i=0}^{n-1} \binom{n+k-1}{k} q^k \equiv \frac{1}{2}\binom{2n}{n} q^n + \sum_{j=0}^{n-1} \binom{2n}{j} (1-q)^{n-j} q^j \tag{4.3}$$

The coefficient of q^n in the right side of (4.3) is

$$\frac{1}{2}\binom{2n}{n} + (-1)^n \left(\binom{2n}{0} - \binom{2n}{1} + \ldots + (-1)^{n-1} \binom{2n}{n-1} \right)$$

$$= \frac{(-1)^n}{2} \left[\binom{2n}{0} - \binom{2n}{1} + \ldots + (-1)^{n-1} \binom{2n}{n-1} + (-1)^n \binom{2n}{n} \right.$$

$$\left. +(-1)^{n+1} \binom{2n}{n+1} + \ldots - \binom{2n}{2n-1} + \binom{2n}{2n} \right] = \frac{(-1)^n}{2} (1-1)^{2n} = 0,$$

so the degree of the polynomial in the right side of (4.3) is $n-1$, the same as in the left side.

Now, we have to prove that, for each $k = 0, 1, \ldots, n-1$, the coefficient of q^k in the right side of (4.3) is equal to $\binom{n+k-1}{k}$:

$$\sum_{i=0}^{k} (-1)^{k-i} \binom{2n}{i} \binom{n-i}{k-i} = \binom{n+k-1}{k}. \tag{4.4}$$

We prove, by mathematical induction on n, a more general statement:

$P(n)$: For any $n \geq 1$, for any $m > n$ and $k = 0, 1, \ldots, n$,

$$\sum_{i=0}^{k} (-1)^{k-i} \binom{m}{i} \binom{n-i}{k-i} = \binom{m-n+k-1}{k}. \tag{4.5}$$

If $n = 1$ the statement $P(1)$ is obviously true.

Suppose that $P(n)$ is true and prove that $P(n+1)$ is true. Let $m > n+1$ and $k \in \{0, 1, \ldots, n\}$. By the induction hypothesis we can write:

$$(-1)^k \sum_{i=0}^{k} (-1)^i \binom{m}{i} \binom{n-i}{k-i} = \binom{m-n+k-1}{k}$$

$$(-1)^{k-1} \sum_{i=0}^{k-1} (-1)^i \binom{m}{i} \binom{n-i}{k-1-i} = \binom{m-n+k-2}{k-1}$$

We subtract the two relations and obtain

$$(-1)^k \sum_{i=0}^{k} (-1)^i \binom{m}{i} \binom{n+1-i}{k-i} = \binom{m-n+k-1}{k} - \binom{m-n+k-2}{k-1}$$

$$= \binom{m-n+k-2}{k}.$$

For $k = n+1$, we have to prove that

$$(-1)^{n+1} \sum_{i=0}^{n+1} (-1)^i \binom{m}{i} \cdot 1 = \binom{m-1}{n+1}.$$

By the induction hypothesis (for $k = n$), we know that

$$(-1)^n \sum_{i=0}^{n} (-1)^i \binom{m}{i} \cdot 1 = \binom{m-1}{n}.$$

Hence,

$$(-1)^{n+1} \sum_{i=0}^{n+1} (-1)^i \binom{m}{i} \cdot 1 = -\binom{m-1}{n} + \binom{m}{n+1} = \binom{m-1}{n+1},$$

and $P(n+1)$ is true.

If we write (4.5) for $m = 2n$ we obtain the equality (4.4) and the theorem is proved.

□

As can be readily seen, $B_{2n}(1/2) = 1/2$ for any $n = 1, 2, \ldots$, so it follows that the sequence of polynomials $H_{n,n}(p) \equiv B_{2n}(p)$ simply converges to the function $\eta_{1/2}(p)$, for all $p \in [0, 1]$. To illustrate this convergence, we represent in Figure 1 the graphs of the polynomials $H_{5,5}(p), H_{9,9}(p), H_{20,20}(p), H_{100,100}(p)$ and $H_{500,500}(p)$.

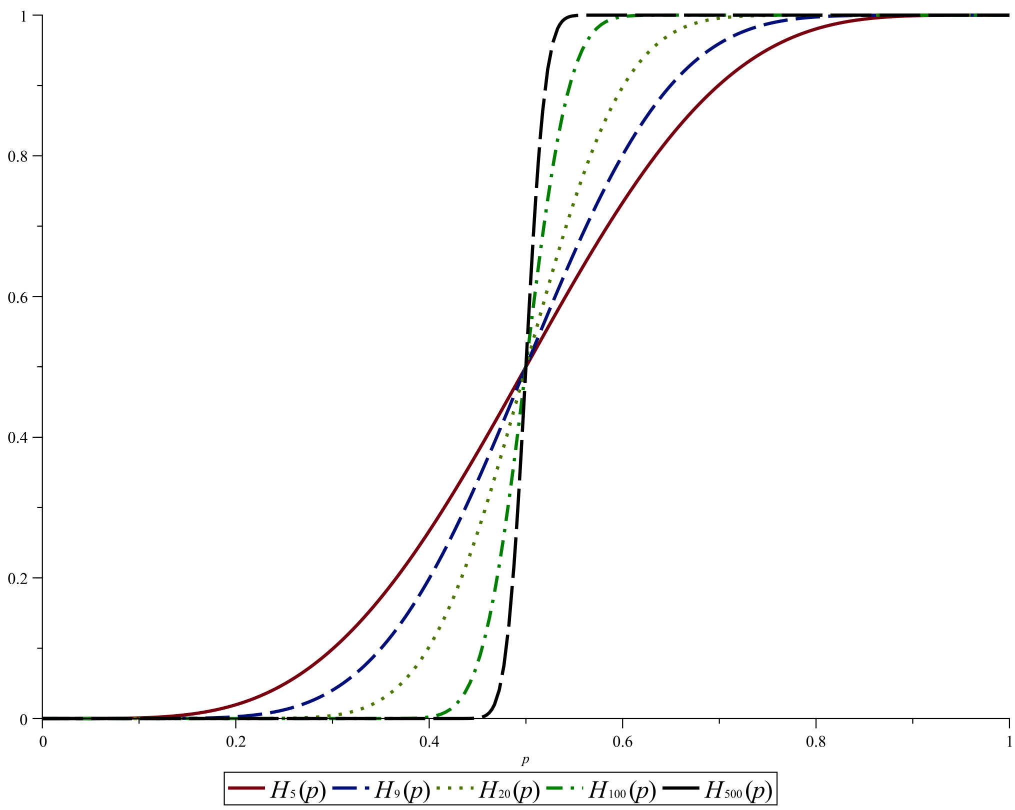

Figure 1

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AN APPLICATION OF THE INTRINSIC PROPERTIES OF THE SPHERE

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Abstract The properties of a surface of a 3-dimensional Euclidean space which depend only on the coefficients of the first fundamental form and their derivatives are called intrinsic properties. Roughly speaking, intrinsic properties are properties of a surface that do not depend on the way the surface is immersed in the ambient space, whereas extrinsic properties depend on the properties of the ambient space. We give an application in real life of some known intrinsic properties of the sphere.
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Key words: curves on the sphere, intrinsic properties.

1. Preliminaries

One considers the parametrized surface
$f: \boldsymbol{R^2} \to \boldsymbol{E_3}, f(x^1, x^2) = (cosx^1cosx^2, sinx^1cosx^2, sinx^2)$,
where $\boldsymbol{E_3}$ is the *3*-dimensional Euclidean space with the standard Euclidean inner product; obviously, $S = Imf$ is the sphere centered at the origin *O(0,0,0)* and radius *r=1.*
We define the following curves on the sphere:
$c = f \circ x, x: \boldsymbol{R} \to \boldsymbol{R^2}, x(t) = \left(x^1(t), x^2(t)\right) = (t^2, t),$
$\bar{c} = f \circ \bar{x}, \bar{x}: \boldsymbol{R} \to \boldsymbol{R^2}, \bar{x}(\bar{t}) = \left(\bar{x}^1(\bar{t}), \bar{x}^2(\bar{t})\right) = (-\bar{t}^2, \bar{t}),$
$\bar{\bar{c}} = f \circ \bar{\bar{x}}, \bar{\bar{x}}: \boldsymbol{R} \to \boldsymbol{R^2}, \bar{\bar{x}}(\bar{\bar{t}}) = \left(\bar{\bar{x}}^1(\bar{\bar{t}}), \bar{\bar{x}}^2(\bar{\bar{t}})\right) = (\bar{\bar{t}}, 1).$
For the curvilinear triangle $\Delta A'B'C'$ on the sphere S we will determine the perimeter, the angles and the area; the points are:
$A = x \cap \bar{x}, A' = f(A)$,
$B = x \cap \bar{\bar{x}}, B' = f(B),$
$C = \bar{x} \cap \bar{\bar{x}}, C' = f(C),$
where we denoted also by $x, \bar{x}$ and $\bar{\bar{x}}$ the images of the curves.
We recall next the theoretical notions and formulae which we will use in our calculations.
In order to calculate the coefficients of the first fundamental form, we will use the following formulae:
$g_{11} =< f_{x^1}, f_{x^1} >,$
$g_{12} =< f_{x^1}, f_{x^2} > = g_{21},$
$g_{22} =< f_{x^2}, f_{x^2} >,$
where $f_{x^1} = \frac{\partial f}{\partial x^1}$ and $f_{x^2} = \frac{\partial f}{\partial x^2}$.
We also recall:

1. The formula for the *length of the curve arch*:

$$(1) \quad \mathcal{L}\left(c|_{[a,b]}\right) = \int_a^b \sqrt{g_{ij}(x(t))\dot{x}^i(t)\dot{x}^j(t)}\, dt,$$

which results from $\mathcal{L}(c|_{[a,b]}) = \int_a^b ||\dot{c}(t)||dt$, $\dot{c}(t)$ being the usual notation for the first derivative $c'(t)$.

2. The formula for the *angle between the curves c and* $\bar{c}$ at their common point $c(t_0) = \bar{c}(\bar{t}_0)$ or, to be more specific, the angle formed by the tangents to the two curves in their common point:

$$(2)\quad \cos u = \frac{<\dot{c}(t_0),\dot{\bar{c}}(\overline{t_0})>}{||\dot{c}(t_0)||||\dot{\bar{c}}(\overline{t_0})||} = \frac{g_{ij}(x(t_0))\dot{x}^i(t_0)\dot{\bar{x}}^j(t_0)}{\sqrt{g_{rs}(x(t_0))\dot{x}^r(t_0)\dot{x}^s(t_0)}\sqrt{g_{pq}(x(t_0))\dot{\bar{x}}^p(\overline{t_0})\dot{\bar{x}}^q(\overline{t_0})}}.$$

3. The *area* of Δ is given by

$$(3)\quad \Delta = \iint_D \sqrt{\det\left(g_{ij}(x(t))\right)}\, dx^i\, dx^j,$$

where $\Delta = f(D)$ represents the portion of the surface and, also, the duplication of the subscripts of each indexed term, represents Einstein's summation convention.

Definition. The properties of a surface from $\boldsymbol{E_3}$ which only depend on the coefficients of the first fundamental form and their derivatives are called *intrinsic properties* (intrinsic geometry is not affected by the choice of the coordinate system).

Remark. The following properties are intrinsic for a surface:

- The length of a curve arch on a surface, given by the 1st formula;
- The angle between two curves on a surface, given by the 2nd formula;
- The area of a portion of a surface, given by the 3rd formula.

2. Practical application and concrete calculations

We present a practical application of our calculations, as a motivation of study of our theoretical problem. More precisely, the background of this mathematical problem is related to the construction field. For example, the situation in which some tubes or pillars must be placed on a round surface uses the steps described in this article (see the Figure 1).

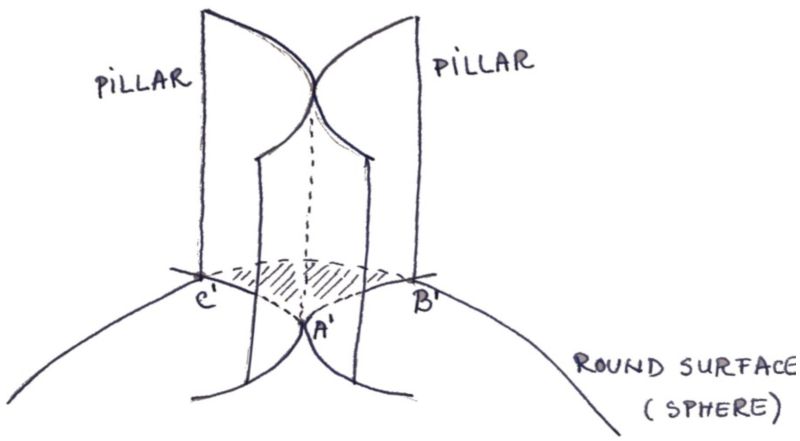

Figure 1

We will start solving the mathematical problem. Recall the expression of the surface:
$f(x^1, x^2) = (cosx^1cosx^2, sinx^1cosx^2, sinx^2)$.

Then

$f_{x^1} = (-sinx^1 cosx^2, cosx^1 cosx^2, 0)$,

$f_{x^2} = (-cosx^1\, sinx^2, -sinx^1\, sinx^2, cosx^2)$,

$g_{11} =< f_{x^1}, f_{x^1} >= sin^2 x^1 \, cos^2 x^2 + cos^2 x^1 \, cos^2 x^2 = cos^2 x^2$,

$g_{12} = g_{21} =< f_{x^1}, f_{x^2} >= sin\, x^1 \, sin\, x^2 \, cos\, x^1 \, cos\, x^2 - sin\, x^1 \, sin\, x^2 \, cos\, x^1 \, cos\, x^2 = 0$,

$g_{22} =< f_{x^2}, f_{x^2} >= sin^2 x^2 \, cos^2 x^1 + sin^2 x^1 \, sin^2 x^2 + cos^2 x^2 = 1$,

which implies

$$(g_{ij})_{i,j=1,2}(x^1, x^2) = \begin{pmatrix} cos^2 x^2 & 0 \\ 0 & 1 \end{pmatrix}.$$

In order to determine the coordinates of the points *A*, *B* and *C*, we will do the following steps:

$$A = x \cap \bar{x} \Rightarrow \begin{cases} t^2 = -\bar{t}^2 \\ t = \bar{t} \end{cases} \Rightarrow t = \bar{t} = 0 \Rightarrow A(0,0),$$

$$B = x \cap \bar{\bar{x}} \Rightarrow \begin{cases} t^2 = \bar{\bar{t}} \\ t = 1 \end{cases} \Rightarrow t = \bar{\bar{t}} = 1 \Rightarrow B(1,1),$$

$$C = \bar{x} \cap \bar{\bar{x}} \Rightarrow \begin{cases} -\bar{t} = \bar{\bar{t}} \\ \bar{t} = 1 \end{cases} \Rightarrow \bar{t} = 1, \bar{\bar{t}} = -1 \Rightarrow C(-1,1).$$

One can see the planar projection in the following figure:

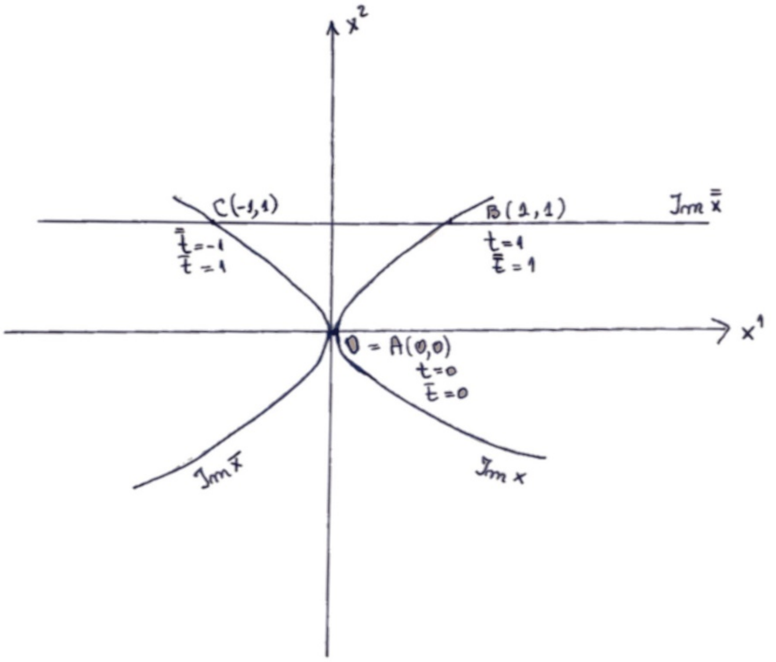

Figure 2

$$\text{Also, we calculate: } \begin{cases} \dot{x}(t) = (\dot{x}^1(t), \dot{x}^2(t)) = (2t, 1) \\ \dot{\bar{x}}(\bar{t}) = (\dot{\bar{x}}^1(\bar{t}), \dot{\bar{x}}^2(\bar{t})) = (-2\bar{t}, 1) \\ \dot{\bar{\bar{x}}}(\bar{\bar{t}}) = (\dot{\bar{\bar{x}}}^1(\bar{\bar{t}}), \dot{\bar{\bar{x}}}^2(\bar{\bar{t}})) = (1, 0) \end{cases}.$$

In order to calculate the length of each side of the curvilinear triangle *ΔABC*, the 1st formula is used. Then, by numerical approximation of some integrals, we obtain

$$AB = \mathcal{L}(c|_{[0,1]}) = \int_0^1 \sqrt{g_{ij}x(t)\dot{x}^i(t)\dot{x}^j(t)}\, dt =$$

$$= \int_0^1 \sqrt{g_{11}(x(t))(\dot{x}^1(t))^2 + 2g_{12}(x(t))\dot{x}^1(t)\dot{x}^2(t) + g_{22}(x(t))(\dot{x}^2(t))^2}\, dt =$$

$$= \int_0^1 \sqrt{(\cos^2 t)(2t)^2 + 1}\, dt \simeq 1.29,$$

$$AC = \mathcal{L}(\bar{c}|_{[0,1]}) = \int_0^1 \sqrt{(\cos^2 \bar{t})(4\bar{t}^2) + 1}\, d\bar{t} \simeq 1.29,$$

$$BC = \mathcal{L}(\bar{\bar{c}}|_{[-1,1]}) = \int_0^1 \sqrt{\cos^2 \bar{\bar{t}}}\ d\bar{\bar{t}} = \int_0^1 \cos \bar{\bar{t}}\, d\bar{\bar{t}} = \sin 1 \ \simeq 0.84.$$

Therefore, the *perimeter* is: $P \simeq 1.29 + 1.29 + 0.84 \simeq 3.42$.

Next we will calculate the *angles* of the curvilinear triangle *ΔA'B'C'*.

$A' = f(A) = f(0,0) = (\cos 0 \cos 0, \sin 0 \cos 0, \sin 0) = (1,0,0)$,

$B' = f(B) = f(1,1) = ((\cos 1)^2, \cos 1 \sin 1, \sin 1)$,

$C' = f(C) = f(-1,1) = ((\cos 1)^2, -\sin 1 \cos 1, \sin 1)$.

Remark. In order to calculate the lengths of the arches (the sides of the curvilinear triangle) it was enough to write the *ΔABC* triangle (the sides are not affected by their position on the surface, in our case, on the sphere); but, to find the values of the angles, it is necessary to be taken into consideration *ΔA'B'C'*.

We denote by

$u_{A'}$ − the angle between the curves c and $\bar{c}$ at their common point

$$c(0) = \bar{c}(0) = f(0,0) = f(A) = A';$$

$u_{B'}$ − the angle between the curves c and $\bar{\bar{c}}$ at their common point

$$c(1) = \bar{\bar{c}}(1) = f(1,1) = f(B) = B';$$

$u_{C'}$ − the angle between the curves $\bar{c}$ and $\bar{\bar{c}}$ at their common point

$$\bar{c}(1) = \bar{\bar{c}}(-1) = f(-1,1) = f(C) = C'.$$

To calculate the angles, we will use the 2nd formula. Then

$$\cos u_{A'} = \frac{g_{ij}(0,0)\dot{x}^i(0)\dot{\bar{x}}^j(0)}{\sqrt{g_{rs}(0,0)\dot{x}^r(0)\dot{x}^s(0)}\sqrt{g_{pq}(0,0)\dot{\bar{x}}^p(0)\dot{\bar{x}}^q(0)}}.$$

We have

$$(g_{ij})_{i,j=1,2}(0,0) = \begin{pmatrix} \cos^2 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\dot{x}(0) = (\dot{x}^1(0), \dot{x}^2(0)) = (0,1),$$

$$\dot{\bar{x}}(0) = (\dot{\bar{x}}^1(0), \dot{\bar{x}}^2(0)) = (0,1).$$

It follows that:

$$\cos u_{A'} = \frac{g_{11}(0,0)(\dot{x}^1(0))(\dot{\bar{x}}^1(0)) + g_{22}(0,0)(\dot{x}^2(0))(\dot{\bar{x}}^2(0))}{\sqrt{g_{22}(0,0)(\dot{x}^2(0))^2}\sqrt{g_{22}(0,0)(\dot{\bar{x}}^2(0))^2}} = 1 \Rightarrow u_{A'} = 0,$$

i.e. the curves c and $\bar{c}$ are tangent in the point A'.

Similarly,

$$\cos u_{B'} = \frac{2\cos 1}{\sqrt{4\cos^2 1 + 1}} \simeq 0.734 \text{ (numerical approximation)},$$

because $\left(g_{ij}\right)(1,1) = \begin{pmatrix} cos^2 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{cases} \dot{x}(1) = \left(\dot{x}^1(1), \dot{x}^2(1)\right) = (2,1) \\ \dot{\bar{x}}(1) = \left(\dot{\bar{x}}^1(1), \dot{\bar{x}}^2(1)\right) = (1,0) \end{cases}.$

Also,
$\cos u_{C'} = \frac{-2\cos 1}{\sqrt{4\cos^2 1+1}} \simeq -0.734$ (numerical approximation),
because $(g_{ij})_{i,j=1,2}(-1,1) = \begin{pmatrix} cos^2 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{cases} \dot{\bar{x}}(1) = \left(\dot{\bar{x}}^1(1), \dot{\bar{x}}^2(1)\right) = (-2,1) \\ \dot{\bar{\bar{x}}}(-1) = \left(\dot{\bar{\bar{x}}}^1(-1), \dot{\bar{\bar{x}}}^2(-1)\right) = (1,0) \end{cases}.$

We remark that $cos\, u_{B'} = \frac{2\,cos\,1}{\sqrt{4\,cos^2\,1+1}} = -\,cos\,u_{C'}$, which implies that $u_{B'}$ and $u_{C'}$ are supplementary angles and, in addition, we obtained that the sum of the angles of the curvilinear triangle angles is 180°. But this result was expected, because the triangle is in fact obtained by an isometry of a planar triangle.
More precisely, this result is not in contradiction with the statement of the Gauss-Bonnet Theorem (which says that the sum of the angles of a geodesic triangle on the sphere is greater than 180°), because the triangle *ΔA'B'C'* is not geodesic, its sides are not arches of great circles!

In order to calculate the *area* of the curvilinear triangle, we will use the 3rd formula; then we have

$$\begin{aligned} A = & \int_{-1}^{0} \left(\int_{\sqrt{-x^1}}^{1} \sqrt{det\, g}\ dx^2 \right) dx^1 \\ & + \int_{0}^{1} \int_{\sqrt{x^1}}^{1} \left(\sqrt{det\, g}\ dx^2 \right) dx^1 = \\ & = 2\int_0^1 \left(\int_{\sqrt{x^1}}^{1} \sqrt{cos^2\, x^2}\, dx^2 \right) dx^1 = \\ & = 2\int_0^1 \left(\int_{\sqrt{x^1}}^{1} cos\, x^2\, dx^2 \right) dx^1 = \\ & = 2\int_0^1 \left(\left(sin\, x^2\right) \Big|_{\sqrt{x^1}}^{1} \right) dx^1 = \ 2\int_0^1 \left(sin\, 1 - sin\sqrt{x^1} \right) dx^1 = \\ & = 2\, sin\, 1 \int_0^1 dx^1 - 2\int_0^1 sin\sqrt{x^1}\, dx^1. \end{aligned}$$

We denote

$$\int_0^1 sin\sqrt{x^1}\, dx^1 = I$$

and then $A = 2\,sin\,1 - 2I$. By using the change $\sqrt{x^1} = u$, we obtain $I = -\,cos\,1 + sin\,1$. Then the *area* is $A = 2cos1 \simeq 1.08$.

Aknowledgements: The author thanks Professors Adela Mihai and Daniel Tudor for checking the calculations and for their constant encouragement and support.

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ONLINE MATHEMATICS TEACHING TRENDS IN HIGHER EDUCATION

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Abstract: Mathematics plays one of the most important roles in developments of our modern and technology-centered society. Additionally, it lays the basis for technical studies, but is also needed e.g. in economics and life science. Excellent problem solving skills are needed from the engineers. These skills are based on good knowledge of technology and good competencies in math and natural sciences. Engineering studies are a wide ranging high-tech field which requires mathematical and natural science skills. In fact, good mathematical skills are crucial for science and economy. Unfortunately, various studies have shown that mathematical competence in Europe has weakened in recent decades. The lack of mathematical proficiency is already causing problems in engineering mathematics' and other courses in European HEIs. In fact, this seems to be a global problem, and e.g. the learning outcomes of Eastern European countries have been weaker than expected, especially in mathematics, after they moved to the Western European model of education (e.g. SEFI 2002). Compounding the issues, the resources allocated to teaching have been decreased so that there are fewer resources for teaching and the development of teaching.

In this paper, we will present the online learning resources developed within the EU projects and how materials developed within this projects are used by students in our universities and their positive influence in the process of online teaching and learning mathematics.

Mathematics Subject Classification (2010): 97Q60
Key words: teaching methods, digitalization, engineering mathematics education

1. Introduction

Mathematical skills are a prerequisite in technical studies and mathematics lay the basis for understanding different engineering disciplines. Thus, the students' poor skills in mathematics slow down or even prevent their studies. In principle, an engineer must be able to think analytically and to be capable of logical reasoning. In addition, an engineer needs to understand mathematics, which allows them to deal with and understand technical problems. Overall, mathematics penetrates deep into the engineering professional field, affecting the opportunities to absorb and learn engineering subjects. Thus, for example to be able to make new technological innovations, the understanding and skills of mathematics are crucial.

Unfortunately, the lack of basic skills and knowledge of mathematics among the European engineering students complicates and in the worst case, even prevents future technological development in Europe. In order to maintain the competitiveness of Europe, the basic level of mathematical proficiency needs urgently to be increased on a large scale. Based on above described situation, the proposed project aims to improve the mathematical proficiency of European engineering students by developing methods and best practices to learn, teach and assess mathematics effectively. Since the objectives of the project are international, the best

results can be achieved with transnational co-operation. Based on the results of a survey collected in our universities, students expect more digital learning possibilities and utilizations of ubiquitous technology in mathematics' studies. This is very natural as the whole of society is changing. Big data, open data, cloud services, digitalization, IoT etc. affect society and social activities on a large scale. As working life is constantly changing, its expectations and requirements have become more diverse. The 21st century skills, such as collective thinking, collaboration, creativity and shared problem solving skills are key components in modern working life and therefore the university teaching and learning should also train these skills.
The EU projects aims to respond to the requirements of modern society and to make mathematics' learning and teaching more digitalized, effective and accessible. Additionally, the aim is to explore and develop the most motivational, learner centered methods, techniques and resources for engineering mathematics learning and teaching with the help of technology. All the learning resources developed in the project will be made available for free under the idea of Open Source or Open Educational Resource (OER).
Overall, the project respects and enables i.e. collective thinking, collaboration and shared problem solving skills. The project aims to develop and improve technology-enhanced methods and resources to teach, learn and study engineering mathematics under the themes such as collaboration, peer instruction and assessment, mostly based on approaches of e-learning 2.0 and 21st century skills.

2. Trends in Higher Education

The NMC Horizon Report: "Higher Education Edition" examines emerging technologies for their potential impact on and use in teaching, learning, and creative inquiry within the environment of higher education. Over the decade of the NMC Horizon Project research, more than 850 internationally recognized practitioners and experts had participated in the creation of every annual report. The discussions of trends and technologies which present the adoption of technology in higher education are organized into three categories due to the temporal significance [1]:

- fast trend - driving changes in higher education over the next one to two years;
- mid-range trend - driving changes in higher education within three to five years;
- long-range trend - driving changes in higher education in five or more years.

Creative Classroom Research Model is a directive for each of the six trends examined bellow.

2.1. Elements of the Creative Classroom Research Model

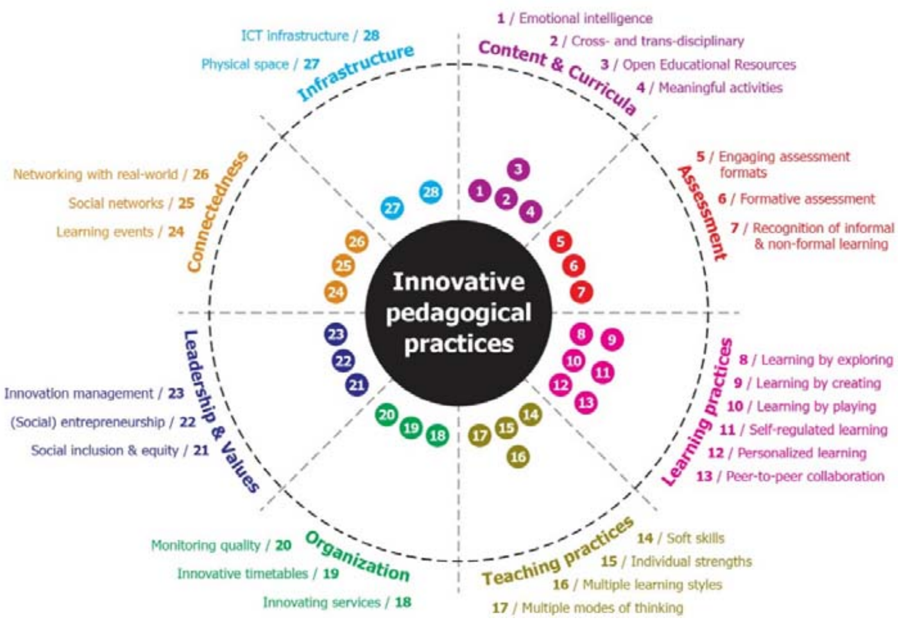


Figure 1 Elements of Creative Classroom Research Model

Up-Scaling Creative Classrooms (CCR) project, developed by the European Commission Institute for Prospective Technological Studies (IPTS) and pictured in the chart on figure 1, is used to identify implications for policy, leadership, and practice [2] that are related to the trends. The CCR multi-dimensional concept (8 key dimensions and 28 reference parameters) intends to capture the essential elements of Creative Classrooms which can be seen as 'live ecosystems' that constantly evolve over the time, mainly depending on the context and the culture to which they pertain. Focus is on the innovative pedagogical practices and on the systemic approach that is needed for the sustainable implementation and progressively scaling up of ICT-enabled innovative learning environments.

2.2. Fast trend

2.2.1. Growing ubiquity of social media

"The top 25 social media platforms worldwide share 6.3 billion accounts among them" the Horizon report says. "Educators, students, alumni, and the general public routinely use social media to share news about scientific and other developments. The impact of these changes in scholarly communication and on the credibility of information remains to be seen, but it is clear that social media has found significant traction in almost every education sector" [1, p. 7].

For educational institutions, social media enable "two way dialogues between students, prospective students, educators, and the institution that are less formal than with other media", it continues, adding that educators are using them "as professional communities of practice, as learning communities, and as a platform to share interesting stories about topics students are studying in class" [1, p. 7].

2.2.2. Integration of online, hybrid, and collaborative learning

According to Horizon Report: 2014 Higher Education Edition, "education paradigms are shifting to include more online learning, blended and hybrid learning, and collaborative models" [1, p. 10]. The introduction of more online learning platforms through courses make dynamic, flexible and accessible content. "To encourage collaboration and reinforce real world skills, universities are experimenting with policies that allow for more freedom in interactions between students when working on projects and assessments" [1, p. 10].

2.3. Mid-range trend

2.3.1. Rise of data-driven learning and assessment

"There is a growing interest in using new sources of data for personalizing the learning experience and for performance measurement," the Horizon report says. "As learners participate in online activities, they leave an increasingly clear trail of analytics data that can be mined for insights" [1, p. 12]. Using platforms generate more information, which leads to the creation of models for training type library content and algorithms for fast data transfer.

2.3.2. Shift from students as consumers to students as creators

Students rather than consumers of knowledge become creative thinkers by taking control of the development of research and analysis of research by including more practical experience. "University departments in areas that have not traditionally had lab or hands-on components are shifting to incorporate hands-on learning experiences as an integral part of the curriculum. Courses and degree plans across all disciplines at institutions are in the process of changing to reflect the importance of media creation, design, and entrepreneurship" [1, p. 14].

2.4. Long-range trend

2.4.1. Agile approaches to change

According to the Horizon report, there is "a growing consensus among many higher education thought leaders" that institutional leadership and curricula could benefit from "agile startup models". Such models "use technology as a catalyst for promoting a culture of innovation in a more widespread, cost-effective manner" [1, p. 16]. Involving employers in the planning process of learning, searching for “real world experience”, is structuring learning activities in the way that educators are able to experiment with new technologies and approaches before implementing them in courses and they have the opportunity to evaluate them and make improvements to teaching models.

2.4.2. Evolution of online learning

Integration of some forms of face-to-face learning in the process of education is becoming a part of the perception of online learning. According to the 56-strong panel of experts that were consulted for the report, the advent of voice and video tools is increasing the number of interactive activities between online instructors and students and improving their quality.

3. Online Mathematics Teaching

In the current public health crisis, we are all working quickly to move our classes out of the classroom. Fortunately, even if online teaching and learning are new, there is a lot of experience to draw on.

3.1. System for Teaching and Assessment - STACK

STACK - System for Teaching and Assessment using a Computer algebra Kernel it’s an Open-source system functions in online education platforms, based on Maxima commands and LaTeX, Automatic assessment of the mathematical exercises.

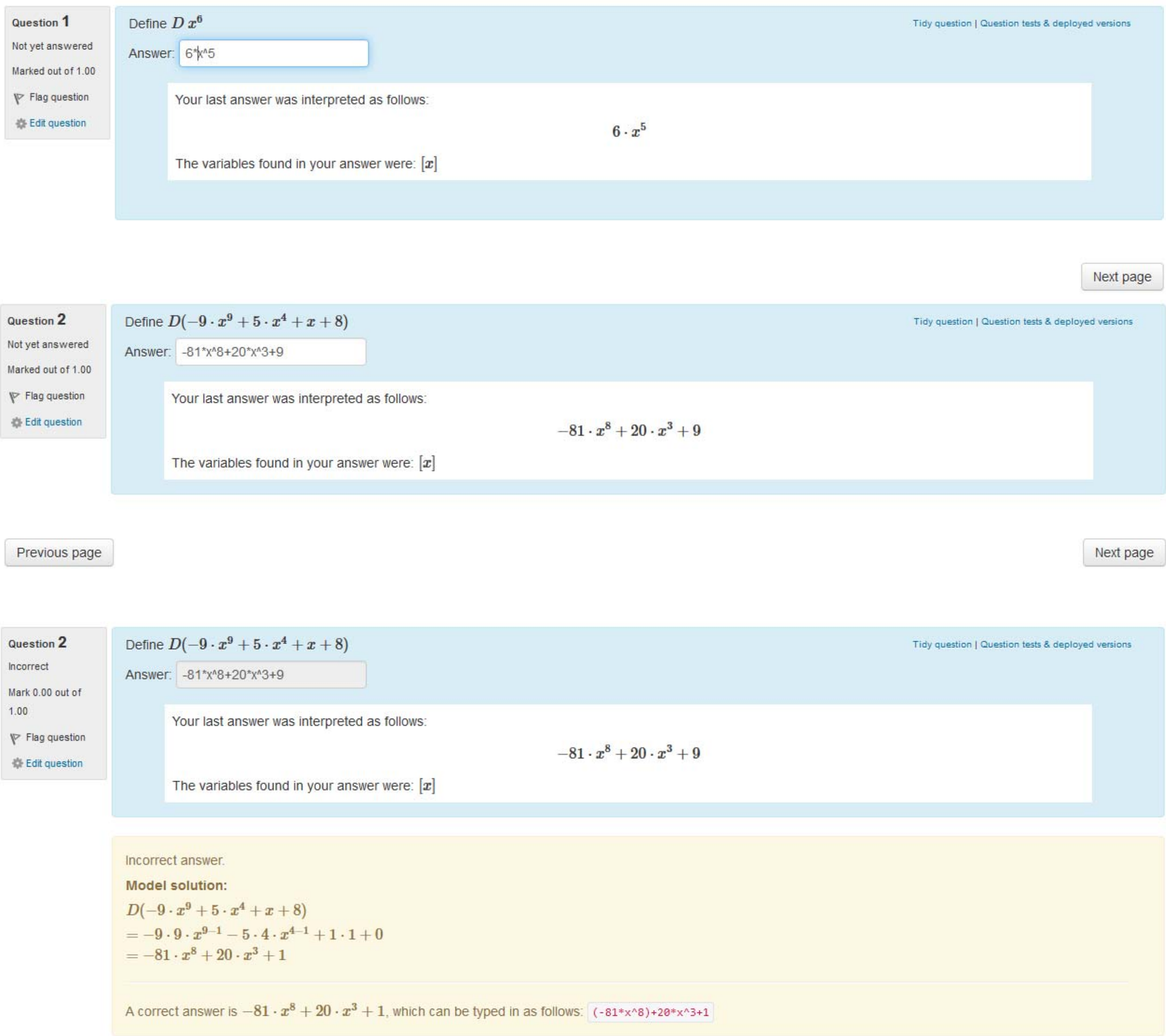
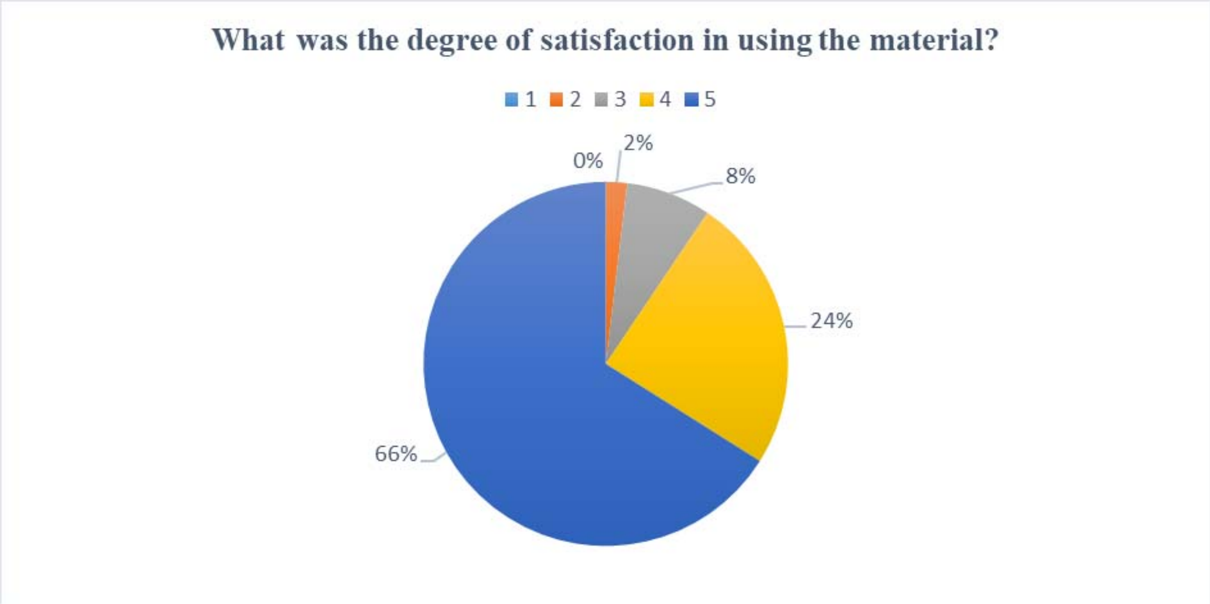


Figure 2 MLP STACK exercices

4. Case study, applied in UTCB

During the applied computer science classes, I presented to the students how to use and asked them to learn some subjects using System for Teaching and Assessment - STAK. The material was given directly to the students. All the students are push to use the materials and to learn using this materials. In this way, we test our System for Teaching and Assessment - STAK.

In the end, we give to the students a questionnaire to provide a feedback from our STAK, at the question “What was the degree of satisfaction in using the material? Mark down numbers 1-5. (1= not at all, 5= very high)”



5. Conclusion

The students from our universities, needs more online materials, they like the idea of learning using STAK, using modern tools. We hope that our projects, our STAK will be a good start for them and also will help them a lot!

Mathematical logic is easily identified as an important topic of mathematics. It plays an important role in the field of engineering. The skills developed are important for creative and innovative problem solving, and help create a skilled future work force a competitive and economically expanding Europe.

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ABOUT THE ROLE OF COUNTEREXAMPLES IN TEACHING CALCULUS

LUCIAN NIŢĂ, ŞTEFANIA CONSTANTINESCU AND ALICE ANGHELESCU

Abstract. In this paper we try to emphasize the important role of counterexamples for the deeply understanding of calculus theorems. First, we consider a contraction in an incomplete metric space, which has not fixed points. Then, we give an example of function of two variables to compare the concepts of continuity and partial derivatives. The last example provides a Cauchy problem without the unicity of solution, this being a consequence of the lack of local-Lipschitz property of the vector field which defines the differential equation. For more details, one can consult [1], [2], and [3].

Mathematics Subject Classification (2010): 26B05, 39A05, 54C05, 54E50
Key words: contraction principle, partial derivatives, continuity, Cauchy problem, local-Lipschitz function

1. The contraction theorem

Let (X, d) be a metric space.

Definition 1.1. *We say that (X, d) is complete if any Cauchy sequence is convergent.*

Definition 1.2. *Let (X, d), (Y, δ) metric spaces and the function $f : X \to Y$. We say that f is* a contraction *if there exists $C \in [0, 1)$ such that, for any $x, y \in X$, we have: $\delta(f(x), f(y)) \leq C \cdot d(x, y)$.*

Definition 1.3. *Let X be a non-empty set, $f : X \to X, x \in X$. We say that x is a* fixed point *for f if $f(x) = x$.*

Theorem 1.4 (The contraction theorem, due to Banach and Picard)**.** *If (X, d) is a complete metric space and $f : X \to X$ is a contraction, then f has an unique fixed point.*

Question: What happens if (X, d) is not complete?

Example 1.5. *Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \begin{cases} -\frac{1}{2}x + \frac{1}{4}, & \text{if } x \leq -1, \\ \frac{x^2}{4} + \frac{1}{2}, & \text{if } x \in (-1, 1). \\ \frac{1}{2}x + \frac{1}{4}, & \text{if } x \geq 1 \end{cases}$ It is easy to prove that f is derivable on $\mathbb{R}$ and $\sup\limits_{x \in \mathbb{R}} |f'(x)| = \frac{1}{2}$. Then, using the Lagrange Theorem, for any $x, y \in \mathbb{R}, x < y$, we can find a number $c \in (x, y)$ such that: $f(x) - f(y) = f'(c)(x - y)$, hence $|f(x) - f(y)| = |f'(c)||x - y| \leq \frac{1}{2}|x - y|$. We deduce that f is a contraction.*

We denote by g the restriction of f to $\mathbb{Q}$, $g : \mathbb{Q} \to \mathbb{Q}, g(x) = f(x), \forall x \in \mathbb{Q}$. Then, g is also a contraction (on $\mathbb{Q}$). We will prove that g has not fixed points. Let us consider the equation $g(x) = x$.

i) For $x \leq -1 \Longrightarrow -\frac{1}{2}x + \frac{1}{4} = x \Longrightarrow x = \frac{1}{6}$, *that is not an acceptable solution* ($\frac{1}{6} \notin (-\infty, -1]$)*;*

ii) For $x \geq 1 \Longrightarrow \frac{1}{2}x + \frac{1}{4} = x \Longrightarrow x = \frac{1}{2}$, *but* $\frac{1}{2} \notin [1, \infty)$*;*

iii) For $x \in (-1, 1) \Longrightarrow \frac{x^2}{4} + \frac{1}{2} = x \Longrightarrow x^2 - 4x + 2 = 0 \Longrightarrow x = 2 \pm \sqrt{2}$. *The number* $2 - \sqrt{2}$ *belongs to* $(-1, 1)$, *but it is not a rational number. So, the equation* $g(x) = x$ *has not solutions, that implies* g *has not fixed points.*

2. Continuity versus partial derivatives for functions of two or more variables

Definition 2.1. *Let* $f : D \subset \mathbb{R}^n \to \mathbb{R}, a \in D$. *We say that* f is continuous at a *if for any sequence* $(a_k)_k \subset D$ *such that* $\lim\limits_{k\to\infty} a_k = a$, *we have* $\lim\limits_{k\to\infty} f(a_k) = f(a)$.

Example 2.2. *We consider the function* $f : \mathbb{R}^2 \to \mathbb{R}, f(x, y) = \begin{cases} y + \sin^2 x, & \text{if } y \in \mathbb{Q} \\ \sqrt{y^2 + 2y + 2} + \sin^4 x, & \text{if } y \notin \mathbb{Q} \end{cases}$

1°) *We prove that* f *is not continuous at any point. Let us suppose that* $\exists (a, b) \in \mathbb{R}^2$ *such that* f *is continuous at* (a, b). *Let* $(x_n, y_n) \to (a, b)$, *with* $y_n \in \mathbb{Q}$*; We have:*

$$f(x_n, y_n) = y_n + \sin^2 x_n \to b + \sin^2 a. \tag{2.1}$$

Now let $(z_n, t_n) \to (a, b)$ *with* $t_n \in \mathbb{R}\backslash\mathbb{Q}$*;*

$$f(z_n, t_n) = \sqrt{t_n^2 + 2t_n + 2} + \sin^4 z_n \to \sqrt{b^2 + 2b + 2} + \sin^4 a. \tag{2.2}$$

If f *is continuous at* (a, b), *from 2.1 and 2.2, we deduce that* $b + \sin^2 a = \sqrt{b^2 + 2b + 2} + \sin^4 a$, *or, equivalent:*

$$\sqrt{b^2 + 2b + 2} - b = \sin^2 a - \sin^4 a. \tag{2.3}$$

But:

$$\sin^2 a - \sin^4 a = \sin^2 a \cos^2 a = \frac{1}{4}\sin^2 2a \leq \frac{1}{4}, \tag{2.4}$$

$$\sqrt{b^2 + 2b + 2} - b \geq \sqrt{(b+1)^2} - b = |b + 1| - b \geq 1. \tag{2.5}$$

From 2.4 and 2.5, we conclude that 2.3 cannot hold. So, f *is not continuous at* (a, b).

2°) *We prove that for any* $(a, b) \in \mathbb{R}^2, \exists \frac{\partial f}{\partial x}(a, b)$.

i) Let $b \in \mathbb{Q}$. *We have:*

$$\lim_{x\to a} \frac{f(x, b) - f(a, b)}{x - a} = \lim_{x\to a} \frac{b + \sin^2 x - \big(b + \sin^2 a\big)}{x - a} = \lim_{x\to a} \frac{\sin^2 x - \sin^2 a}{x - a} = \sin 2a,$$

hence $\frac{\partial f}{\partial x}(a, b) = \sin 2a$.

ii) If $b \in \mathbb{R}\backslash\mathbb{Q}$, *we can write*

$$\lim_{x\to a} \frac{f(x, b) - f(a, b)}{x - a} = \lim_{x\to a} \frac{\sqrt{b^2 + 2b + 2} + \sin^4 x - \big(\sqrt{b^2 + 2b + 2} + \sin^4 a\big)}{x - a} =$$

$$= \lim_{x\to a} \frac{\sin^4 x - \sin^4 a}{x - a} = 4\sin^3 a \cos a \Longrightarrow \frac{\partial f}{\partial x}(a, b) = 4\sin^3 a \cos a.$$

3°) *We prove that* $\nexists \frac{\partial f}{\partial y}(0,0)$. *Let* $y_n \to 0$, *such that* $\forall n \geq 1, y_n \in \mathbb{Q}$.

$$\lim_{n\to\infty} \frac{f(0,y_n)-f(0,0)}{y_n} = \lim_{n\to\infty} \frac{y_n}{y_n} = 1.$$

Let now $(z_n)_n \subset \mathbb{R}_+ \backslash \mathbb{Q}$ *such that* $z_n \to 0$. *We can write:*

$$\lim_{n\to\infty} \frac{f(0,z_n)-f(0,0)}{z_n} = \lim_{n\to\infty} \frac{\sqrt{z_n^2+2z_n+2}}{z_n} = \frac{\sqrt{2}}{0_+} = +\infty.$$

Hence, $\nexists \lim\limits_{y\to 0} \frac{f(0,y)-f(0,0)}{y}$, *that is:* $\nexists \frac{\partial f}{\partial y}(0,0)$.

(Similarly, it can be proved that $\nexists \frac{\partial f}{\partial y}(a,b), \forall (a,b) \in \mathbb{R}^2$ *with* $b \in \mathbb{Q}$.*)*

4°) *Now, we study the problem: is* $(0,0)$ *is extremum point for* f*?*

Obviously, $f(0,0) = 0$ *and if* $y \in \mathbb{R}\backslash\mathbb{Q}, f(x,y) > 0$.

Hence, for any $y \in \mathbb{R}\backslash\mathbb{Q}, f(x,y) - f(0,0) > 0$. *But, in any neighbourhood of* $(0,0)$, *we find points of type* $\left(\frac{1}{\sqrt{2n}}, -\frac{1}{n}\right)$ *for* $n \in \mathbb{N}^*$, *large enough. We have*

$$f\left(\frac{1}{\sqrt{2n}}, -\frac{1}{n}\right) - f(0,0) = \sin^2\frac{1}{\sqrt{2n}} - \frac{1}{n} \leq \left(\frac{1}{\sqrt{2n}}\right)^2 - \frac{1}{n} < 0.$$

We deduce that in any neighbourhood of $(0,0)$, *the difference* $f(x,y) - f(0,0)$ *has variable sign. We conclude that* $(0,0)$ *is not an extremum point for* f.

Remark 2.3. *This exemple shows us some important things:*

a) Unlike the functions of only of variable, where derivability implies continuity, for functions of two or more variables, the existence of partial derivatives does not imply continuity.

b)Sometimes, we can deduce that if a point is or not an extremum point for a function without using "the tool" of partial derivatives, but studying the behaviour of the function in the neighbourhood of that point. [In fact, in our case, the partial derivative $\frac{\partial f}{\partial y}(0,0)$ *doesn't exist !]*

3. The Cauchy-Lipschitz Theorem

Notation: For $a \in \mathbb{R}^n (n \geq 1)$, we denote by $\mathcal{V}(a)$ the set of all neighbourhoods of a.

Definition 3.1. *We say that the function* $f : D \subset \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ *is* locally Lipschitz with respect to the second argument *at* $(t_0, x_0) \in D$ *if* $\exists D_0 \in \mathcal{V}(t_0, x_0)$ *and* $L \geq 0$ *such that*

$$\|f(t,x_1) - f(t,x_2)\| \leq L\|x_1 - x_2\|, \forall (t,x_1),(t,x_2) \in D_0 \cap D.$$

Theorem 3.2 (Cauchy-Lipschitz)**.** *Let* $f : D \subset \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ *(D being an open set) be a continuous function and local Lipschitz with respect to the second argument, such that f defines the differential equation:* $\frac{dx}{dt} = f(t,x)$. *Then,* $\forall (t_0, x_0) \in D$, *the Cauchy problem* $\{\frac{dx}{dt} = f(t,x), x(t_0) = x_0\}$ *has the property of local existence and unicity.*

(this means that: $\forall \varphi_i : I_i \in \mathcal{V}(t_0) \to \mathbb{R}^n, i \in 1,2$, *solutions of the Cauchy problem,* $\exists I_0 \in \mathcal{V}(t_0)$ *such that* $\varphi_1|_{I_0\cap(I_1\cap I_2)} = \varphi_2|_{I_0\cap(I_1\cap I_2)}$*).*

Example 3.3. *Let* $n \in \mathbb{N}, n \geq 1$ *an odd number and* $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}, f(t,x) = nx^{\frac{n-1}{n}}$. *Let* $t_0 \in \mathbb{R}$, *arbitrarily, fixed. We will prove that* f *is not a local Lipschitz function with respect to the second argument at* $(t_0, 0)$. *In fact, we will prove that* $\forall L > 0, \forall \varepsilon > 0$, *(abitrarily, fixed numbers)* $\exists x, y \in (-\varepsilon, \varepsilon)$ *and* $t \in \mathbb{R}$ *such that* $|f(t,x) - f(t,y)| > L|x - y|$. *For* $m \in \mathbb{N}^*$ *we take* $x_m = \frac{1}{m^n}, y_m = \frac{3^n}{m^n}$. *We can write:*

$$|f(t,x_m)-f(t,y_m)| > L|x_m-y_m| \iff n\left|\frac{1}{m^{n-1}}-\frac{3^{n-1}}{m^{n-1}}\right|> L\left|\frac{1}{m^n}-\frac{3^n}{m^n}\right| \iff mn\frac{3^{n-1}-1}{3^n-1} > L,$$

that is true for m *large enough.*

Let us consider now the Cauchy problem:

$$\begin{cases} x' = nx^{\frac{n-1}{n}} \\ x(t_0) = 0. \end{cases}$$

The equation $x' = nx^{\frac{n-1}{n}}$ *has the stationary solution* $x_0 \equiv 0$, *that satisfies the condition* $x(t_0) = 0$. *On the other hand, the equation has also the solution* $x_1(t) = (t - t_0)^n$ *and* $x_1(t_0) = 0$. *The solutions* x_0 *and* x_1 *coincide only for* $t = t_0$, *hence, this Cauchy problem has not the property of local unicity.*

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ON THE CONVERGENCE DEGREE OF SOME SPECIAL SEQUENCES

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Abstract: In this note we introduce the convergence degree of a sequence and use it to estimate the rate of convergence for some particular sequences.
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Def. 1: We say that $\alpha > 0$ is the ***convergence degree*** (***CD***) of a real sequence $\{a_n\}$, with $a_n \in \mathbb{R}, n = 0,1,\ldots$, if the following limit exists as a real number
$l_\alpha = \lim\limits_{n\to\infty} n^\alpha (a_{n+1} - a_n)$ and $l_\alpha \neq 0$. In this case we write CD$\{a_n\}=\alpha$ and $L\{a_n\} = l_\alpha$.

Prop. 1: Assume that CD$\{a_n\}$ exists. Then it is unique.
Proof: Let us assume that $\alpha < \beta$ such that l_α and $l_\beta \in \mathbb{R}$ and $l_\alpha \neq 0, l_\beta \neq 0$.
Then $l_\beta = \lim\limits_{n\to\infty} n^\beta (a_{n+1} - a_n) = l_\alpha \cdot \lim\limits_{n\to\infty} n^{\beta-\alpha} \notin \mathbb{R}$, a contradiction.
So, $\alpha = \beta$.

Example 1: Find CD$\{a_n\}$ for $a_n = \frac{n+2}{n+3}, n \geq 0$.

We calculate $l_\alpha = \lim\limits_{n\to\infty} n^\alpha \left(a_{n+1} - a_n\right) = \lim\limits_{n\to\infty} n^\alpha \left(\frac{n+3}{n+4} - \frac{n+2}{n+3}\right) =$

$= \lim\limits_{n\to\infty} n^\alpha \left(\frac{n^2+6n+9-n^2-6n-8}{(n+3)(n+4)}\right) = 1$, for $\alpha = 2$, so $l_\alpha=1$ and CD$\{a_n\}$ = 2.

Example 2: Let us consider the known Euler's sequence,

$$c_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} - \ln n.$$

Thus $c_{n+1} - c_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} + \frac{1}{n+1} - ln(n+1) - \left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} - ln\, n\right) = \frac{1}{n+1} + ln\frac{n}{n+1}$ and so, we can evaluate:

(1) $\lim_{n\to\infty} n^{\alpha}\left[\frac{1}{n+1} + ln\frac{n}{n+1}\right]$

In $l_\alpha = \lim_{n\to\infty} n^{\alpha}\left[\frac{1}{n+1} + ln\frac{n}{n+1}\right]$ we put $x = \frac{1}{n}$ and obtain $l_\alpha = \lim_{x\to 0}\frac{1}{x^{\alpha}}\left[\frac{1}{\frac{1}{x}+1} + \ln\frac{\frac{1}{x}}{\frac{1}{x}+1}\right] =$

$\lim_{x\to 0}\frac{\frac{x}{x+1} + ln\frac{1}{x+1}}{x^{\alpha}}$ after applying l'Hospital

$l_\alpha = \lim_{x\to 0}\frac{\frac{x+1-x}{(x+1)^2} + (x+1)\left(-\frac{1}{(x+1)^2}\right)}{\alpha\cdot x^{\alpha-1}} = \lim_{x\to 0}\frac{\frac{-x}{(x+1)^2}}{\alpha\cdot x^{\alpha-1}}$ once we apply l'Hospital

$l_\alpha = \lim_{x\to 0}\frac{\frac{x-1}{(x+1)^3}}{\alpha\cdot(\alpha-1)\cdot x^{\alpha-2}} = -\frac{1}{2}$ for $\alpha = 2$.

Thus CD$\{c_n\}$= 2 and $L\{c_n\} = l_2 = -\frac{1}{2}$.

We see in the above computation that $\alpha = 0$ or 1 implies $l_\alpha = 0$, so the CD is not defined in these last cases.

Example 3: It is easy to see that $\left\{a_n = \frac{1}{ln\, n}\right\}$ and $\{b_n = e^{-n}\}$ are convergent sequences which have no CD defined.

Prop. 2: Let $\{b_n\}$ be a strictly monotonous sequence of real numbers which is convergent to $b \in \mathbb{R}$ and let $\{a_n\}$ be an arbitrary sequence such that

$$(3)\quad \lim_{n\to\infty}\frac{a_{n+1}-a_n}{b_{n+1}-b_n} = l \in \mathbb{R}$$

Then $\{a_n\}$ is a convergent sequence, say to a, and

$$(4)\quad \lim_{n\to\infty}\frac{a_n-a}{b_n-b} \text{ exists and it is equal to } l.$$

Proof: Let us assume that $b_n \downarrow b$, i.e., $\{b_n\}$ is a decreasing to b sequence (the case $b_n \uparrow b$, may be discussed in the same manner). Let $\varepsilon > 0$ be an arbitrary positive real number. From (3) we can find $N = N_\varepsilon \in \mathbb{N}$ (where $\mathbb{N}$ is the set of natural number) such that :

(5) $(l-\varepsilon)(b_n - b_{n+1}) < a_n - a_{n+1} < (l+\varepsilon)(b_n - b_{n+1})$ for any $n \geq N$.

Let us fix a $k \geq N$ and take $n = k, k+1, \ldots, k+m-1$ for an arbitrary $m \in \mathbb{N}$.

By adding the obtained inequalities in (5) we get :

(6) $(l-\varepsilon)(b_k - b_{k+m-1}) < a_k - a_{k+m-1} < (l+\varepsilon)(b_k - b_{k+m-1})$ for all $m \in \mathbb{N}$.

Since $\{b_n\}$ is a Cauchy sequence, from (6) we see that $\{a_n\}$ is also a Cauchy sequence, i.e., $a_n \to a \in \mathbb{R}$.

Let us make $m \to \infty$ in (6) and get:

(7) $(l-\varepsilon)(b_k-b) \le a_k - a \le (l+\varepsilon)(b_k-b)$.
So $\frac{a_k-a}{b_k-b} \to l$, whenever $k \to \infty$.

Prop. 3: Let $\{b_n\}$ be a strictly monotonous sequence of real numbers which is convergent to $b \in \mathbb{R}$ and let $\{a_n\}$ be another sequence convergent to a, such that

$$(8)\quad \lim_{n\to\infty} \frac{a_{n+1}-a_n}{b_{n+1}-b_n} = l = \pm\infty \in \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}.$$

Then $\quad (9) \quad \lim_{n\to\infty} \frac{a_n-a}{b_n-b} = l$.

Proof: Let us assume $l = +\infty$ and that $\{b_n\}$ is an increasing to b sequence. Take $M > 0$ be a positive real number and find from (8) $N = N_\varepsilon \in \mathbb{N}$ such that :

(10) $\quad a_{n+1} - a_n > M(b_{n+1} - b_n)$ for all $n \ge N$.

Let us fix a $k \ge N$ and make $n = k, k+1, \dots, k+m-1$ in (10) for any $m \in \mathbb{N}$.
By adding all the obtained inequalities, we get :

(11) $\quad a_{n+m} - a_n > M(b_{n+m} - b_n)$ for all $m \in \mathbb{N}$.

Take now $m \to \infty$ in (11) and find:

(12) $\quad \frac{a-a_n}{b-b_n} > M$, for all $n \ge N$, i.e., $\lim_{n\to\infty} \frac{a_n-a}{b_n-b} = l = +\infty$.

For $l = -\infty$ the proof is similar.

Corollary 1: Let $\alpha > 0$ and $\left\{b_n = \frac{1}{n^\alpha}\right\}$, *n=1,2,... ,* and let $\{a_n\}$ be an arbitrary sequence such that

(13) $\lim_{n\to\infty} n^{\alpha+1}(a_{n+1} - a_n) = \alpha \cdot l \in \mathbb{R}$, i.e., such that CD$\{a_n\}$=$\alpha + 1$ and $l_{\alpha+1} = \alpha \cdot l$. Then $\{a_n\}$ is convergent, say to $a \in \mathbb{R}$ and

(14) $\lim_{n\to\infty} n^\alpha (a - a_n) = l$.

Proof: We apply ***Prop.2*** and find:

$$\lim_{n\to\infty} \frac{a_{n+1}-a_n}{\frac{1}{(n+1)^\alpha} - \frac{1}{n^\alpha}} = -\lim_{n\to\infty} \frac{n^\alpha (n+1)^\alpha (a_{n+1}-a_n)}{(n+1)^\alpha - n^\alpha} =$$

$$= -\lim_{n\to\infty} \frac{(n+1)^\alpha (a_{n+1}-a_n)}{\left(1+\frac{1}{n}\right)^\alpha - 1} == -\lim_{n\to\infty} \frac{n^{\alpha+1}(a_{n+1}-a_n)}{\alpha} = -l$$

So $\lim_{n\to\infty} n^\alpha (a - a_n) = l$.

Corollary 2:

$$\text{Let} \quad a_n = \sum_{k=1}^{n-1} \frac{1}{k^{\alpha+1}}$$

in *Corollary 1*.

Thus $a_n \to a = \zeta\ (\alpha+1)$ the Riemann Zeta function at $\alpha+1$. So

(15) $$\lim_{n\to\infty} n^{\alpha} \sum_{k=n}^{\infty} \frac{1}{k^{\propto+1}} = \frac{1}{\propto}$$

Remark 1:

So, the rate of convergence of $\zeta_{\ n}(\alpha+1)$:

$$\zeta_{\ n} = \sum_{k=1}^{n-1} \frac{1}{k^{\propto+1}}$$

to $\zeta\ (\alpha+1)$ is $\frac{1}{\alpha} \cdot \frac{1}{n^{\alpha}}$.

This is a weaker result of a famous theorem of Hardy-Littlewood to the estimation of the reminder $\zeta(\alpha+1) - \zeta_{\ n}(\alpha+1)$ for $\alpha > 0$ (see [1]).

Theorem 1:

Let $\{a_n\}$ be a real sequence such that CD$\{a_n\}$=α >1 and $L\{a_n\} = l_\alpha \in \mathbb{R}$ $(l = l_\alpha \neq 0)$. Then $\{a_n\}$ is convergent, say to $a \in \mathbb{R}$ and

(16) $\lim_{n\to\infty} n^{\alpha-1}(a - a_n) = \frac{l}{\alpha-1}$.

Proof: Let $\varepsilon > 0$ be an arbitrary positive real number and let $N = N_\varepsilon \in \mathbb{N}$ such that :

(17) $(l-\varepsilon)\frac{1}{n^\alpha} < a_{n+1} - a_n < (l+\varepsilon)\frac{1}{n^\alpha}$ for all $n \geq N$.

Let us fix $k \geq N$ and write (17) for $n = k, k+1, \ldots, k+m-1$ where $m \in \mathbb{N}$.

Now we add all these inequalities and get:

(18) $$(l-\varepsilon) \sum_{j=k}^{k+m-1} \frac{1}{j^{\propto}} < a_{k+m} - a_k < (l+\varepsilon) \sum_{j=k}^{k+m-1} \frac{1}{j^{\propto}}$$

for any $n \geq N$ and for all $m \in \mathbb{N}$.

Since

$$\sum_{j=1}^{m} \frac{1}{j^{\propto}}$$

is convergent (Riemann Zeta function $\zeta(\alpha), \alpha > 1$) we see that $\{a_n\}$ is a Cauchy sequence i.e. convergent to a.

Let us make $m \to \infty$ in (18) and get:

(19) $$(l-\varepsilon) \sum_{j=k}^{\infty} \frac{1}{j^{\propto}} < a - a_k < (l+\varepsilon) \sum_{j=k}^{\infty} \frac{1}{j^{\propto}}$$

Since

$$\lim_{k\to\infty} k^{\alpha-1} \sum_{j=k}^{\infty} \frac{1}{j^{\propto}} = \frac{1}{\alpha-1}$$

(see (15)), we get that:

(20) $(l-\varepsilon)\frac{1}{\alpha-1} \leq \lim_{k\to\infty} k^{\alpha-1}(a-a_k) \leq (l+\varepsilon)\frac{1}{\alpha-1}$.

Since $\varepsilon > 0$ be an arbitrary positive real number, we see that the limit exists and it is equal to $\frac{l}{\alpha-1}$.

q.e.d.

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APPLICATION OF DIFFERENTIAL EQUATIONS IN ENGINEERING

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Abstract: Differential equations are very important tools in engineering but also in mechanics, biology, economy. In this article, we will present how to introduce students the solution of first order homogeneous and non-homogeneous differential equation. We will apply these notions to solve the problem of heat transffering problems in a fluid environment.
Mathematics Subject Classification (2010):
Key words: differential equation

1. Introduction

Differential equations have wide applications in various engineering and science disciplines. In general, modeling variations of a physical quantity, such as temperature, pressure, displacement, velocity or concentration of a pollutant, with the change of time t or location, would require differential equations. Similarly, studying the variation of a physical quantity on other physical quantities would lead to differential equations.

A first order differential equations is an equation that contain only first derivative. In some of the applications that are in mathematics, a first order differential equation plays a vital role in physics that includes a temperature problem which requires the use of Newton’s law of cooling of a particular substance. Differential equations generally fall into two categories, ordinary differential equation and partial differentials equations, the distinction being that ODEs involve unknown functions of one independent variable while PDEs involve unknown functions of more than one independent variable. In this paper we will focus on first order ordinary differential equation.

Hassan A. and Zakari Y. studied the first-order ordinary differential equation and discovered that it has many applications in temperature problems which lead to use Newton’s cooling law and obtain the solution of some practical problems that arise from the 1^{st}order ODEs [2].

In this article we divided the first order differential equation in two parts, the equation in which the coefficient is constant known as homogeneous differential equation and the equation in which the coefficient is function or polynomial is known as the non-homogeneous differential equation. The aim of this paper is to know the type of first order ordinary differential equation, which one is homogeneous and which one is non-homogeneous differential equation. In this paper we will discuss the solution method of first order linear homogeneous and non-homogeneous differential equation and apply it to solve the heat transferring problems especially heat convection in fluid problems that required the use of Newton’s cooling law.

2. First order differential equations

1. A separable variables equation [3] has the form

$$f_1(x)\cdot g_1(y)\cdot y' + f_2(x)\cdot g_2(y) = 0 \quad (1)$$

where f_1, f_2:I→ ℝ are continuous functions, $f_1 \neq 0$ on I, g_1, g_2:J→ ℝ are continuous functions, $g_2 \neq 0$ on J, I and J are intervals.

The variables are separated, through division by $f_1(x)\cdot g_2(y)$. The equation becomes

$$\frac{g_1(y)}{g_2(y)}dy = -\frac{f_2(x)}{f_1(x)}dx.$$

By integration, we get

$$\int \frac{g_1(y)}{g_2(y)}dy = -\int \frac{f_2(x)}{f_1(x)}dx + C, C \in \mathbb{R}.$$

Thus, we obtain the implicit general solution.

2. A homogeneous differential equation [3] is as follows

$$y' = f\left(\frac{y}{x}\right), \quad (2)$$

where f is continuous on I, $0 \notin I$.

Using the change of variables $u = \frac{y}{x}$, $u = u(x)$, $y = y(x)$, we get

$$u'\cdot x + u = f(u).$$

3. First order linear differential equations are of the following form [3]

$$y' + P(x)\cdot y = Q(x), \quad (3)$$

where P, Q are continuous functions on $I \subset \mathbb{R}$.

The solution is obtain in two steps.

Step 1. The homogeneous equation is resolved

$$y' + P(x)\cdot y = 0.$$

This is an equation with separable variables with the solution

$$y = C\cdot e^{-\int P(x)dx}, C \in \mathbb{R}.$$

Step 2. Using Lagrange constant variation method, a particular solution of the nonhomogenuous equation is obtained as follows

$$y_p = C(x)\cdot e^{-\int P(x)dx},$$

where C is a C^1 class function on I. From the condition that y_p is a solution of the nonhomogenuous equation, there is obtained

$$C(x) = \int Q(x)\cdot e^{\int P(x)dx}dx + k,\ \ k \in \mathbb{R}.$$

The general solution is

$$y = e^{-\int P(x)dx}\left(k + \int Q(x)\cdot e^{\int P(x)dx}dx\right). \quad (4)$$

3. Applications of first order differential equation in temperature problems

Heat flows in the solid by conduction which can be determined by Fourier law in the solid heat always flow from high to low temperature. But in fluid heat flowing by convection which can be calculated by Newton's cooling law. This states that the rate of heat loss of a body is directly proportional to the temperature difference between the body and its surroundings, provided that the temperature difference is small and the nature of the radiation surface remains the same.

The law of Newton's convection cooling is a reaffirmation of the differential equation given by Fourier's law:

$$\frac{dT(t)}{dt} = -\frac{hA}{C}[T(t) - T_{env}]. \quad (5)$$

where T is the surface temperature and interior of the object (because they are the same in this approximation), C is the heat capacitance, h is the coefficient of heat transfer (assumed independent of T here) (W/(m2K)), A is the heat transfer surface (m2), T_{env} is the temperature of the medium.
The solution of (5) is

$$T(t) = (T_0 - T_{env}) \cdot e^{-\frac{hA}{C}t} + T_{env}, \tag{6}$$

where T_0 is the initial temperature.
The coefficient of heat transfer h depends on the physical properties of the fluid and the physical situation in which convection occurs. Therefore, a single usable heat transfer coefficient (one which does not vary significantly between the temperature ranges covered during cooling and heating) must be derived or found experimentally for each system that can be analysed using the presumption that Newton's law is valid.

Application 1.

A composite has the initial temperature of 20^0C. We put it in an oven of 200^0C for an hour and the temperature of the composite becomes 100^0C. Find out the temperature after two hours and how much time will it take that it reaches 170^0C.
Solution. T_0 is 20^0C, T_{0env} is 200^0C. Solving (5), we obtain solution (6), which for t=1 returns

$$100^0 = (20^0 - 200^0) \cdot e^{-\frac{hA}{C}} + 200^0$$

and therefore $-\frac{hA}{C} = -0.58778$. The temperature after two hours, using (6) is

$$T(2) = (T_0 - T_{env}) \cdot e^{-\frac{hA}{C}2} + T_{env} = (20^0 - 200^0) \cdot e^{-0.58778 \cdot 2} + 200^0 \approx 144.44370^0.$$

For the second part of the problem, we want to find t so that $T(t) = 170^0$.
Using (6), we get

$$170^0 = (20^0 - 200^0) \cdot e^{-0.58778 \cdot t} + 200^0.$$

And therefore, $t \approx 3.04835$ hours.

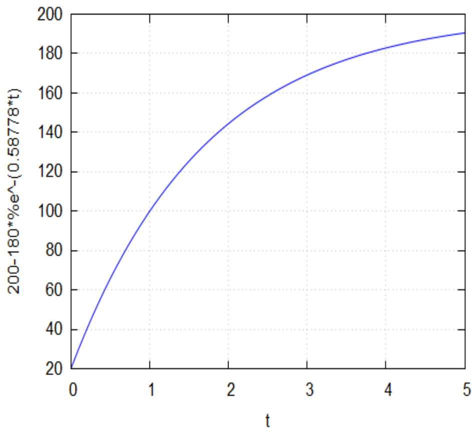

Fig.1 The graph of *T(t)* in MAXIMA

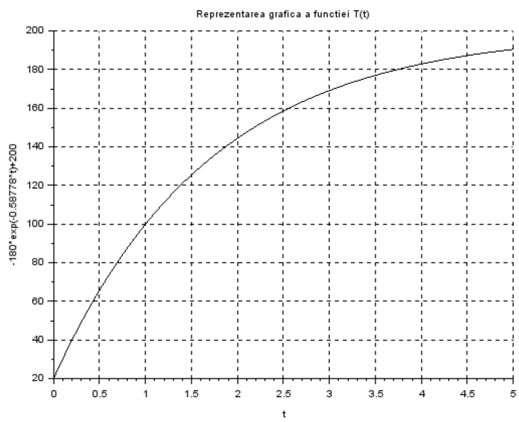

Fig.2 The graph of *T(t)* in SCILAB

Application 2.

A boiling (100^0C) solution is set on a desk where the room temperature is assumed to be constant at 25^0C, the solution cooled to 50^0C after ten minutes.

(i) Find a formula for the temperature *T(t)* of the solution, *t* minutes after it is placed on the desk.

(ii) Determine how long it will take for the solution to cool to 30^0C.

Solution.

(i)We will use the Newton's law of cooling $\frac{dT(t)}{dt} = -\frac{hA}{C}[T(t) - T_{env}]$
To simplify the calculations we will make the following substitution

$$-\frac{hA}{C} = k$$

In our problem we have $T_{env} = 25$ and the Newton's law becomes $\frac{dT(t)}{dt} = k[T(t) - 25]$.
Solving the equation using variable separable we have

$$\frac{dT(t)}{[T(t) - 25]} = kdt$$

Integrate both sides of the equations

$$\int \frac{dT(t)}{[T(t) - 25]} = \int kdt$$

We will get it

$$\ln(T(t) - 25) = kt + c$$

$$T(t) - 25 = Be^{kt}, \quad \text{where} \quad B = e^{c}$$

Substituting *T(t)*=100, *t*=0

$$100 - 25 = Be^{k\cdot 0}, B = 75$$

So we have

$$T(t) = 75e^{kt} + 25$$

Substituting *T(t)*=50, *t*=10 we've to find $k = -\frac{ln3}{10} = -0.1098$.
Finally

$$T(t) = 75e^{-0.1098t} + 25$$

(ii)We want to find out how long will it take such that T(t) is 30^0C, using the last equation.

$$30 = 75e^{-0.1098t} + 25$$

And therefore, $t \approx 25 \; minutes$.

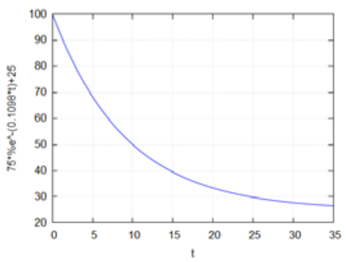

Fig.3 The graph of *T(t)* in MAXIMA

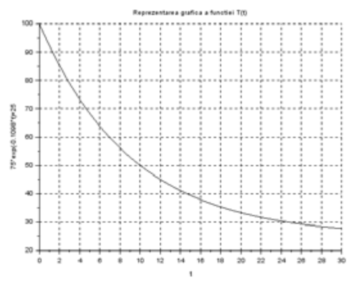

Fig.4 The graph of *T(t)* in SCILAB

4. Conclusions

The main aim of this paper is to show that the application of first order differential equation in temperature problems are useful in mathematics and physics for instance in analyzing problems involving temperature problems which requires the use of Newton's law of cooling. Using a software tool like MAXIMA and SCILAB, the students can check the solutions of more complex equations, such as EDO. Using the method of separating variables into problems that have practical applications can be a good way to teach this useful tool.

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BOUNDED DERIVATIVES WHICH ARE NOT RIEMANN INTEGRABLE

DANIEL TUDOR, AND DAN CARAGHEORGHEOPOL

Abstract. In this paper, we will give an example of a bounded derivative which are not Riemann integrable.

Mathematics Subject Classification (2010):26A42
Key words: derivative, bounded, Riemann integrable

1. Cantor's set

The Cantor set is a set of points lying on a line segment created by taking some interval, for instance [0,1], and removing the middle third $\left(\frac{1}{3}, \frac{2}{3}\right)$, then removing the middle third of each of the two remaining sections $\left(\frac{1}{9}, \frac{2}{9}\right)$ and $\left(\frac{7}{9}, \frac{8}{9}\right)$, then removing the middle third of the remaining four sections, and so on ad infinitum. Cantor sets are uncountable despite not containing any intervals, may have 0 or positive Lebesgue measures, and are nowhere dense.
The Cantor set is constructed by removing increasingly small subintervals from [0,1]. In the first step, remove $\left(\frac{1}{3}, \frac{2}{3}\right)$ from $[0,1]$. In the second step, remove $\left(\frac{1}{9}, \frac{2}{9}\right) \cup \left(\frac{7}{9}, \frac{8}{9}\right)$ from what remains after the first step. In general, on the n$^{\text{th}}$ step, remove $\left(\frac{1}{3^n}, \frac{2}{3^n}\right) \cup \left(\frac{4}{3^n}, \frac{5}{3^n}\right) \cup \dots \cup \left(\frac{3^n-2}{3^n}, \frac{3^n-1}{3^n}\right)$ from what remains after the $(n-1)^{\text{th}}$ step. After all $\mathbb{N}$ steps have been taken, what remains is the Cantor set $\mathcal{C}$.
This construction can be formalized as follows. Let

$$\mathcal{C}_0 = [0,1]$$

Then, after the first step, what remains is $\mathcal{C}_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$. After the second step, what remains is

$$\mathcal{C}_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right].$$

Continuing in this manner, one obtains an infinite collection of sets such that $\mathcal{C}_i \subset \mathcal{C}_{i-1}$ for all $i \geq 1$. Then, one defines the Cantor set to be $\mathcal{C} = \bigcap_{i=0}^{\infty} \mathcal{C}_i$.

2. Volterra' function

Consider the function

$$f(x) = \begin{cases} x^2 sin\frac{1}{x},\ if\ x \neq 0 \\ \\ 0\ if\ x = 0 \end{cases}$$

The derivative of the function is:

$$f'(x) = \begin{cases} 2xsin\frac{1}{x} - cos\frac{1}{x},\ if\ x \neq 0 \\ \\ 0\ if\ x = 0 \end{cases}$$

Note that f' is bounded on a finite interval containing the origin, but f' is not continuous at 0. Now recall the Cantor set, which is constructed by taking the interval [0, 1] and removing the middle third, then removing the middle third of the remaining intervals, and so on. The resulting set has measure zero, so we will use a modified version of the Cantor set that begins with removing the middle fourth of the interval [0, 1], and then the middle forth of the remaining intervals and so on. Note that this so-called thick Cantor set does not have measure zero (the sum total of the lengths of intervals removed is 1/2). Now, construct a function h as follows. At each step of the creation of our thick Cantor set, we will place two copies of f in each deleted interval, so that h' has discontinuities at each of their endpoints. For example, the first removed interval has length 1/4, so find the largest value of x in $[0, 1/8]$ such that $f'(x) = 0$ and call it $x0$. Define the functions:

$$a(x) = \begin{cases} x^2 sin\frac{1}{x},\ if\ 0 \leq x \leq x_0 \\ \\ x_0^2 sin\frac{1}{x_0},\ if\ x_0 < x \leq \frac{1}{8} \\ \\ 0\ if\ x = 0 \end{cases}$$

and define the reflection of a across $x = 1/8$:

$$b(x) = \begin{cases} x_0^2 sin\frac{1}{x_0},\ if\ \frac{1}{8} \leq x \leq \frac{1}{8} + x_0 \\ \\ (\frac{1}{4} - x)^2 sin\frac{1}{\frac{1}{4}-x},\ if\ \frac{1}{8} + x_0 < x < \frac{1}{4} \\ \\ 0\ if\ x = 0 \end{cases}$$

Finally, define the first term in our sequence f_1.

$$f_1(x) = \begin{cases} 0, \; if \; 0 \leq x \leq \frac{3}{8} \\ a(x - \frac{3}{8}), \; if \; \frac{3}{8} < x \leq \frac{1}{2} \\ b(x - \frac{3}{8}), \; if \; \frac{1}{2} < x \leq \frac{5}{8} \\ 0, \; if \; \frac{5}{8} < x \leq 1 \end{cases}$$

The final function provides a clearer picture of what Volterra's function begins to look like. Note that f_1 is differentiable on $(0, 1)$, f_1' is bounded, and is discontinuous at $3/8$ and $5/8$. We continue this process for each step of the construction of the thick Cantor set, obtaining a limit function f with infinitely many discontinuities. As it turns out, f' is discontinuous at every point of the thick Cantor set, which, as we noted, does not have measure zero. Thus, the function f' is a bounded derivative which is not Riemann integrable.

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BASIC PROPERTIES OF PREDECOMPOSABLE OPERATORS

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Abstract: In this paper we try to study several basic properties of predecomposable operators on Banach spaces, as a generalization of decomposable operators.
Mathematics Subject Classification (2010): 47B47, 47B40.
Key words: maximal spectral space, decomposable, predecomposable; spectral precapacity.

1. Introduction

The aim of the present paper is to give some basic results of the theory of predecomposable operators in complex Banach spaces in a systematic way.
The first section is dedicated to the preliminaries. We recall several notations, definitions and results from the specialized literature that are used in the sequel.
We shall denote by X a complex Banach space and by $\mathbf{C}$ the complex plane.
Let $\mathbf{B}(X)$ be the Banach algebra of all linear bounded operators acting on X and let $\mathbf{S}(X)$ denote the family of all linear closed subspaces of X.
If $Y \subseteq X$ is a linear closed subspace invariant to an operator $T \in \mathbf{B}(X)$, then $T \mid Y$ is the restriction operator of T to Y and $\dot{T}$ is the quotient operator induced by T on the quotient space $\dot{X} = X / Y$.
For $T \in \mathbf{B}(X)$, we also denote by $\rho(T)$ the resolvent set of T and by $\sigma(T) = \mathbf{C} \setminus \rho(T)$ the spectrum of T.

Definition 1.1. ([5], [6]) An operator $T \in \mathbf{B}(X)$ has the *single-valued extension property* if for any analytic function $f : D \to X$ ($D \subset \mathbf{C}$ open set), with $(\lambda I - T) f(\lambda) \equiv 0$, it follows that $f(\lambda) \equiv 0$.
If $T \in \mathbf{B}(X)$ has the single-valued extension property, then for any $x \in X$, $\rho_T(x)$ denotes the maximal domain of existence of the X-valued analytic function $\lambda \to x_T(\lambda)$ which verifies $(\lambda I - T)\, x_T(\lambda) \equiv x$. The open set $\rho_T(x)$ is said to be the *local resolvent set of* x *with respect to* T and $\sigma_T(x) = \mathbf{C} \setminus \rho_T(x)$ is the *local spectrum of* x *with respect to* T. We also have $\rho(T) \subset \rho_T(x)$, $\sigma_T(x) \subset \sigma(T)$ and we denote by

$$X_T(F) = \{x \in X \,;\, \sigma_T(x) \subset F,\ F \subset \mathbf{C}\}.$$

Definition 1.2. ([5], [7]) A closed linear subspace Y of X is a *spectral maximal space* of $T \in \mathbf{B}(X)$ if Y is invariant to T and if Z is another closed linear subspace of X, also invariant to T, such that $\sigma(T \mid Z) \subseteq \sigma(T \mid Y)$, we have $Z \subseteq Y$.

Definition 1.3. ([5], [7]) An operator $T \in \mathbf{B}(X)$ is called *decomposable* if for any finite open covering $\{G_i\}_{i=1}^{n}$ of the spectrum $\sigma(T)$, there is a system $\{Y_i\}_{i=1}^{n}$ of spectral maximal spaces of T such that following two conditions are verified:

1) $\sigma(T \mid Y_i) \subseteq G_i$, for all $i = 1, 2, ..., n$

2) $\sum_{i=1}^{n} Y_i = X$.

Proposition 1.1. Let $T \in \mathbf{B}(X)$ be an operator having the single-valued extension property. Then the following assertions are verified:

1) $F_1 \subset F_2 \subseteq \mathbf{C}$ implies $X_T(F_1) \subset X_T(F_2)$ ([5], 1.1.2)

2) $X_T(F) = X_T(F \cap \sigma(T))$ is a linear subspace of X invariant to T and $\sigma(T \mid X_T(F)) \subseteq F \cap \sigma(T)$, for every $F \subseteq \mathbf{C}$ ([1], I.2.3)

3) if $X_T(F)$ is closed, for $F \subseteq \mathbf{C}$ closed, then $X_T(F)$ is a spectral maximal space of T ([4], Lemma 5)

4) $T \mid Y \in \mathbf{B}(Y)$ has the single-valued extension property and $\sigma_T(x) \subseteq \sigma_{T|Y}(x)$, for any subspace Y of X invariant to T and $x \in X$ (respectively, $\sigma_T(x) = \sigma_{T|Y}(x)$, for any spectral maximal space Y of T) ([1], I.4.3)

5) $\dot{T} \in \mathbf{B}(\dot{X})$ has the single-valued extension property, for any spectral maximal space Y of T, where $\dot{X} = X / Y$ ([2], 1.1.10, [3]).

Theorem 1.1. ([7], 2.1, 2.3) Let $T \in \mathbf{B}(X)$ be a decomposable operator. Then the following assertions are verified:

1) T has the single-valued extension property

2) $X_T(F)$ is a spectral maximal space of T and $\sigma(T \mid X_T(F)) \subseteq F \cap \sigma(T)$, for every $F \subseteq \mathbf{C}$ closed

3) Every spectral maximal space Y of T can be written as $Y = X_T(\sigma(T \mid Y))$.

In Section 2, we present several basic results which characterize the class of predecomposable operators.

2. Basic properties of predecomposable operators

Definition 2.1. An operator $T \in \mathbf{B}(X)$ having the single-valued extension property is called *predecomposable* if for any finite open covering $\{G_i\}_{i=1}^{n}$ of the spectrum $\sigma(T)$, there is a system $\{Y_i\}_{i=1}^{n}$ of semiclosed subspaces of X such that the following conditions are established:

1) $Y_i \subseteq X_T(\overline{G_i})$, for all $i = 1, 2, ..., n$

2) $\sum_{i=1}^{n} Y_i = X$.

$T \in \mathbf{B}(X)$ is a *strongly predecomposable operator* if for any finite system $\{G_i\}_{i=1}^{n}$ of open sets covering $\sigma(T)$, there is a system $\{Y_i\}_{i=1}^{n}$ of semiclosed subspaces of X such that:

3) $Y_i \subseteq X_T\left(\overline{G_i}\right)$, for all $i = 1, 2, ..., n$

4) $\sum_{i=1}^{n} (Y_i \cap Y) = Y$, for any semiclosed subspace Y of X invariant to T.

$T \in \mathbf{B}(X)$ is a *weakly predecomposable operator* if for any finite open covering $\{G_i\}_{i=1}^{n}$ of $\sigma(T)$, there is a system $\{Y_i\}_{i=1}^{n}$ of semiclosed subspaces of X such that:

5) $Y_i \subseteq X_T\left(\overline{G_i}\right)$, for all $i = 1, 2, ..., n$

6) $\overline{\sum_{i=1}^{n} Y_i} = X$.

Definition 2.2. For the complex plane **C**, let $\mathcal{F}(\mathbf{C})$ be the family of all closed subsets of **C**. An application $E : \mathcal{F}(\mathbf{C}) \to \mathbf{S}(X)$ is called a *spectral precapacity* if the following conditions are verified:

1) $E(\varnothing) = \{0\}$ and $E(\mathbf{C}) = X$

2) $E\left(\bigcap_{n=1}^{\infty} F_n\right) = \bigcap_{n=1}^{\infty} E(F_n)$, for any family $\{F_n\}_{n\in\mathbb{N}} \subset \mathcal{F}(\mathbf{C})$

3) for any open finite covering $\{G_i\}_{i=1}^{m}$ of **C**, there is a system $\{Y_i\}_{i=1}^{m}$ of semiclosed subspaces of X with the properties:

a) $Y_i \subseteq E(G_i)$, for all $i = 1, 2, ..., m$

b) $\sum_{i=1}^{m} Y_i = X$.

Definition 2.3. An operator $T \in \mathbf{B}(X)$ has a spectral precapacity $E : \mathcal{F}(\mathbf{C}) \to \mathbf{S}(X)$ if the following inclusions hold:

1) $T E(F) \subseteq E(F)$, for any $F \in \mathcal{F}(\mathbf{C})$

2) $\sigma(T \mid E(F)) \subset F$, for any $F \in \mathcal{F}(\mathbf{C})$.

Proposition 2.1. If $T \in \mathbf{B}(X)$ is a predecomposable operator, then T has a spectral precapacity E. Conversely, an operator $T \in \mathbf{B}(X)$ having the single-valued extension property and which admits a spectral precapacity is a predecomposable operator.

Proof. "$\Rightarrow$" Assume first that the operator T is predecomposable. Then T has the single-valued extension property (Definition 2.1).
Putting the notation $E(F) = X_T(F)$, $F \in \mathcal{F}(\mathbf{C})$, we deduce easily that the mapping $E : \mathcal{F}(\mathbf{C}) \to \mathbf{S}(X)$ is a spectral precapacity of T.
"$\Leftarrow$" Conversely, let us assume that the operator T has the single-valued extension property and has a spectral precapacity E.
Since from $x \in E(F)$, it is easily seen that $\sigma_T(x) \subseteq \sigma_{T|E(F)}(x) \subseteq \sigma(T \mid E(F))$ (see Proposition 1.1), then $x \in X_T(\sigma(T \mid E(F)))$, therefore

$$E(F) \subseteq X_T(\sigma(T \mid E(F)))$$

$$\sigma(T \mid X_T(\sigma(T \mid E(F)))) \subseteq \sigma(T \mid E(F)) \subseteq F,\ F \in \mathcal{F}(\mathbf{C}).$$

From Definition 2.2 and by using some properties of the space $X_T(F)$, we deduce that the operator T is predecomposable.

Theorem 2.1. Let $T \in \mathbf{B}(X)$ be a decomposable operator and let Y be a spectral maximal space of T. Then the quotient operator $\dot{T}$ induced by T on the quotient space $\dot{X} = X / Y$, defined by $\dot{T}\,\dot{x} = \overbrace{T x}^{\cdot}$, $x \in X$, is predecomposable.

Proof. The operator T being decomposable, the quotient operator $\dot{T}$ has the single-valued extension property (Proposition 1.1) and we have

$$\overbrace{X_T(F)}^{\cdot} \subseteq \dot{X}_{\dot{T}}(F),\ F \in \mathcal{F}(\mathbf{C}). \quad \textbf{(1)}$$

Let us consider $\{G_i\}_{i=1}^{n}$ a finite arbitrary covering of the spectrum $\sigma(\dot{T})$ which corresponds to a finite covering $\{H_i\}_{i=1}^{n}$ of $\sigma(T)$ (since $\sigma(\dot{T}) \subseteq \sigma(T)$) such that

$$\overline{H_i} \cap \sigma(\dot{T}) = \overline{G_i} \cap \sigma(\dot{T}),\ i = 1, 2, ..., n.$$

Because $T \in \mathbf{B}(X)$ is decomposable, there is a system $\{Y_i\}_{i=1}^{n}$ of spectral maximal spaces of T, written as $Y_i = X_T(\sigma(T \mid Y_i))$ (see Theorem 1.1), with the properties

$$\sigma(T \mid Y_i) \subseteq \overline{H_i},\ \text{for all } i = 1, 2, ..., n \text{ and } \sum_{i=1}^{n} Y_i = X. \quad \textbf{(2)}$$

Moreover, $\dot{T}$ having the single-valued extension property, according to Proposition 1.1 and from assertions (1) and (2), it follows that

$$\dot{Y}_i = \overbrace{X_T(\sigma(T \mid Y_i))}^{\cdot} \subseteq \dot{X}_{\dot{T}}(\sigma(T \mid Y_i)) \subseteq \dot{X}_{\dot{T}}(\overline{H_i}) \text{ and}$$

$$\dot{X}_{\dot{T}}(\overline{H_i}) = \dot{X}_{\dot{T}}(\overline{H_i} \cap \sigma(\dot{T})) = \dot{X}_{\dot{T}}(\overline{G_i} \cap \sigma(\dot{T})) = \dot{X}_{\dot{T}}(\overline{G_i}),\ i = 1, 2, ..., n.$$

Therefore, we obtain the assertions

$$\dot{Y}_i \subseteq \dot{X}_{\dot{T}}\left(\overline{G_i}\right), \text{ for all } i = 1, 2, ..., n \text{ and } \sum_{i=1}^{n} \dot{Y}_i = \dot{X}.$$

Obviously, the spaces $\dot{Y}_i$, $i = 1, 2, ..., n$, are semiclosed, hence by Definition 2.1, it results that $\dot{T}$ is predecomposable.

Corollary 2.1. Let X and Y be two arbitrary Banach spaces, $T \in \mathbf{B}(X)$ be a decomposable operator, and $A \in \mathbf{B}(X, Y)$ be a linear bounded operator such that $AX = Y$. If the kernel of A, denoted by $\mathrm{Ker}\, A$, is a spectral maximal space of T, then the operator $S \in \mathbf{B}(Y)$, defined by $S(Ax) = A(Tx)$, $x \in X$, is predecomposable.

Corollary 2.2. Let X and Y be two arbitrary Banach spaces, $T \in \mathbf{B}(X)$ be a decomposable operator, $S \in \mathbf{B}(Y)$ be an arbitrary linear bounded operator, and $A \in \mathbf{B}(X, Y)$ be an injective linear bounded operator with the properties $\overline{AX} = Y$ and $AT = SA$. Then the operator $S \in \mathbf{B}(Y)$ is weakly predecomposable.

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SOME FUNCTIONAL INEQUALITIES

ANCA-NICOLETA MARCOCI

Abstract. The aim of this talk is to present, in the setting of Lorentz-Sobolev spaces, a general inequality for a particular type of improved Sobolev inequalities.

Mathematics Subject Classification (2010): 46A30, 46E35
Key words: Lorentz-Sobolev spaces, Arino-Muckenhoupt weights.

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MATRICEAL BLOCH SPACES VIA SCHUR MULTIPLIERS

LIVIU-GABRIEL MARCOCI

Abstract. It has been known for a long time that there is a formal relation between classical harmonic analysis and the theory of infinite matrices. However, still many challenging problems in this new theory of matriceal harmonic analysis need to be investigated before we get a complete theory. In this talk, we present some duality results concerning matriceal Bloch spaces obtained using Schur multipliers.

Mathematics Subject Classification (2010): 47L10, 47A08
Key words: Solid spaces, Abel and Köthe duality.

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