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<u>EDITORS</u>:

Ion MIERLUŞ-MAZILU- Head of Department of Mathematics and Computer Science

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Mariana ZAMFIR

<u>Organizing Committee</u>:

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Mariana NICULCEANU

Daniel TUDOR

Mariana ZAMFIR

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CONTENT

ON KY – FAN THEOREM AND SOME APPLICATIONS

ILEANA BUCUR

Abstract. In this communication we present an elementary proof of a fundamental theorem in the theory of critical points in convexity or game theory.
Our proof is done in a particular case but it contains all difficulties of the general case. Among the consequences of this Ky – Fan's results we mention the Nash Theorem of equilibrum as well as Von Neumann minimax theorem.

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Key words: convergence, fixed points theory, Von Neumann minimax

1. The Ky – Fan Theorem

One of the most powerful tool in the theory of Fixed Points is the Ky – Fan Theorem.
In the particular case of real line this assertion may be stated like that:

Theorem 1.1. *Let $g : [0,1] \times [0,1] \to \mathbb{R}$ be a function such that it is concave with respect to the first variable x and it is lower semicontinuous with respect to the second variable y.*
Then there exists $\bar{y} \in [0,1]$ such that

$$\sup_{x\in[0,1]} g(x,\bar{y}) \le \sup_{z\in[0,1]} g(z,z).$$

For the proof we need the following almost trivial lemma.

Lemma 1.2. *If $h : [0,1] \to \mathbb{R}$ is a concave function and if n and p are two points in the interval $[0,1]$ such that $h(n) \le 0$ and $h(p) \ge 0$, then we have:*
(1) $n < p \Longrightarrow h(x) \le 0, \forall x \in [0,n]$
(2) $p < n \Longrightarrow h(x) \le 0, \forall x \in [n,1]$

Proof. Indeed, let x be such that $x < n < p$ and let $\alpha \in (0,1)$ such that $n = \alpha x + (1-\alpha)p$.
Since h is concave and $h(n) \le 0, h(p) \ge 0$ we have

$$0 \ge h(n) \ge \alpha h(x) + (1-\alpha)h(p), \quad \alpha h(x) \le -(1-\alpha)h(p) < 0.$$

In the same way the assertion 2) may be justified. □

Proof of the theorem.
We may suppose that

$$M := \sup_{z\in[0,1]} g(z,z) < \infty$$

and we may consider the function $f : [0,1]^2 \to \mathbb{R}$ defined by

$$f(x,y) = g(x,y) - M.$$

Obviously, the new function f satisfies the properties of g and moreover we have

$$\sup_{z\in[0,1]} f(z,z) \le 0, \quad f(z,z) \le 0, \forall z \in [0,1].$$

We have to show that there exists $\bar{y} \in [0,1]$ such that $f(x,\bar{y}) \leq 0$ for all $x \in [0,1]$.
We suppose the contrary: for any $y \in [0,1]$ there exists at least one point $p \in [0,1]$ such that $f(p,y) > 0$. We fix one of them and we denote it by p_y.
We denote also by $\mathcal{L}$ and $\mathcal{R}$ the sets

$$\mathcal{L} = \{y \in [0,1]; p_y < y\}, \quad \mathcal{R} = \{y \in [0,1]; y < p_y\}.$$

Obviously, $\mathcal{L} \cup \mathcal{R} = [0,1], \quad \mathcal{L} \cap \mathcal{R} = \emptyset, \quad p_0 \in \mathcal{R}, p_1 \in \mathcal{L}$, i.e. $\mathcal{L} \neq \emptyset \neq \mathcal{R}$.
Since $[0,1]$ is a connex set, the sets $\mathcal{L}$ and $\mathcal{R}$ cannot be simultaneously closed.
If $\mathcal{L}$ is not a closed set there is an accumulation point $\bar{y}$ of $\mathcal{L}$ such that $\bar{y} \in \mathcal{R}$.
In this case we may consider a sequence $(y_k)_k$ in $\mathcal{L}$ which is strictly decreasing or strictly increasing to $\bar{y}$.
Let us suppose $(y_k)_k$ strictly increasing to $\bar{y}$.
Since $y_k \in \mathcal{L}, p_{y_k} < y_k$ and therefore, using Lemma 1.2, 2) where $p = p_{y_k}$ and $n = y_k$, we have $f(x,y_k) \leq 0, \forall x \in [y_k,1]$.
If we take $x > \bar{y}$ we have $x > y_k$ for all $k \in \mathbb{N}$ and therefore $f(x,y_k) \leq 0$. Using the fact that f is lower semicontinuous in the second variable we deduce

$$f(x,\bar{y}) \leq \liminf_{n\to\infty} f(x,y_k) \leq 0, \quad \forall x \in [\bar{y},1]. \tag{1.1}$$

On the other hand, since $\bar{y} \in \mathcal{R}$ using again Lemma 1.2, 1) we have taking $n = \bar{y}, p = p_{\bar{y}}$,

$$\bar{y} < p_{\bar{y}}, \quad f(x,\bar{y}) \leq 0, \quad \forall x \in [0,\bar{y}]. \tag{1.2}$$

From (1.1) and (1.2) we deduce $f(x,\bar{y}) \leq 0$ for all $x \in [0,1]$.
If the sequence $(y_k)_k$ is decreasing to $\bar{y}$ and $y_k \in \mathcal{L}$ we have $p_{y_k} < y_k$ and therefore (using Lemma 1.2, 2)) we have $f(x,y_k) \leq 0$ for all $x \in [y_k,1]$. Let now $x \in [y,1]$ and let $m_0 \in \mathbb{N}$ be such that $y_{m_0} < x, \quad y_k < x$ for all $k \geq m_0$. From the relations

$$f(x,y_k) \leq 0, \quad \forall k \geq m_0$$

and passing to $liminf$ we get

$$f(x,\bar{y}) \leq \liminf_{k\to\infty} f(x,y_k) \leq 0, \quad f(x,\bar{y}) \leq 0, \quad \forall x \in [\bar{y},1].$$

Since $\bar{y} \in \mathcal{R}$ we have from Lemma 1.2,

$$f(x,\bar{y}) \leq 0, \quad \forall x \in [0,\bar{y}]$$

and finally $f(x,\bar{y}) \leq 0$ for all $x \in [0,1]$.
Similar arguments are valid if we suppose that the set $\mathcal{R}$ is not closed and we consider an accumulation point $\bar{y}$ of $\mathcal{R}$ belonging to $\mathcal{L}$.
Again we arrive to the relation

$$f(x,\bar{y}) \leq 0, \quad \forall x \in [0,1].$$

This contradictory relation obtained both in the case $\mathcal{L} \neq \bar{\mathcal{L}}$ or in the case $\mathcal{R} \neq \bar{\mathcal{R}}$ shows that the hypothesis made in the begining of the proof is false.

2. Conclusions

The general frame Ky – Fan assertion is stated in the form of Banach spaces or just locally convex Hausdorff spaces.
If K is a compact, convex subset of a locally convex space and $g : K \times K \to \mathbb{R}$ is a function which is concave with respect to the first variable and lower semicontinuous with respect to the second variable, then there is a point $\bar{y} \in K$ such that

$$g(x,\bar{y}) \leq \sup_{z\in K} g(z,z).$$

The proof of this general assertion follows the central result in Fixed Point Theory, namely the Brouwer Theorem.
On the other hand by Ky–Fan theorem offers an efficient tool in proving the famous Nash equilibrium

assertion as well as Banach minimax theorem.
This was the motivation of my communication.

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Department of Mathematics and Computer Science, Technical University of Civil Engineering Bucharest, Bucharest, Romania
Email address: bucurileana@yahoo.com

BAYES SIGNIFICANCE TESTS FOR NON-NORMAL DISTRIBUTIONS

DANIEL CIUIU

Abstract. In this paper we will provide Bayesian signification two-sided tests for non-normal distributions, namely exponential and Poisson distributions.

As in the normal case [2], we use cdfs with jumps in $\theta = \theta_0$ (according the null hypothesis), and the decision of accepting/ rejecting H_0 depends on the posterior probability to fullfill the null hypothesis.

This posterior probability uses prior and posterior cumulative distribution functions for the involved parameter θ with jump in the value θ_0 according the null hypothesis. If the posterior jump is large enough we accept the null hypothesis.

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Key words: Bayes, significance tests, cumulative distribution function with jump

1. Introduction

In the case of the Bayesian inference, the current distribution deppending on one or several parameters is considered a conditional distribution: the involved parameters are considered random variables. Because the (possible vectorial) parameter $\theta \in \Theta$ is a continuous random variable, we consider first [8, 1, 5] a probability density function for θ, denoted by $g(\theta)$. This pdf is called the prior pdf of θ. The posterior pdf of θ is [8, 5]

$$g(\theta_0 | X = x_i) = \frac{g(\theta_0) \cdot P(X = x_i | \theta = \theta_0)}{\int_\Theta g(\theta) \cdot P(X = x_i | \theta) d\theta} \tag{1.1}$$

if X is a discrete random variable, respectively

$$g(\theta_0 | x) = \frac{g(\theta_0) \cdot f(x | \theta_0)}{\int_\Theta g(\theta) \cdot f(x | \theta) d\theta} \tag{1.1'}$$

if X is a continuous random variable with the conditional pdf $f(x|\theta)$.

Usually, instead of X we use a statistics $\widetilde{X}(X_1, ..., X_n)$, where $X_1, X_2, ..., X_n$ is a sample from a population characterised by the random variable X. In fact for normal distribution with known variance or in the exponential distribution case, $\widetilde{X} = \widehat{X}$ (the sample expectation), and in the discrete distributions cases (Poisson, binomial or geometric distribution) we have $\widetilde{X} = S$, i.e. the sum of X_i [5].

In the following, we present some definitions [5, 8] for choosing the prior distribution.

Definition 1.1. *The non-informative prior distribution is the prior distribution of the parameter such that the posterior distribution $g(\theta|x)$ is $g(x|\theta)$. Otherwise, the prior distribution is informative.*

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Definition 1.2. *The prior distribution according maximum entropy principle is the prior distribution with an eventual given prior information, with the maximum Shannon entropy.*

Definition 1.3. *A family of prior distribution $\mathcal{P}$ is called conjugated prior distribution for the pdfs family $\mathcal{F} = \{ f(x|\theta) | \theta \in \Theta \}$ if $(\forall g \in \mathcal{P})$ $(\forall f \in \mathcal{F})$ we have $g(\theta|x) \in \mathcal{P}$.*

A special case of conjugated family of prior distribution is that when $\mathcal{P}$ and $\mathcal{F}$ are identical to the normal family [8]: if $X \sim N(\theta, \sigma^2)$ with known σ^2, and $\theta \sim N(\mu, \tau^2)$ then

$$\theta | X \sim N\left(\frac{\tau^2 \cdot X + \sigma^2 \cdot \mu}{\sigma^2 + \tau^2}, \frac{\sigma^2 \cdot \tau^2}{\sigma^2 + \tau^2}\right). \tag{1.2}$$

We notice that in the above case, the non-informative prior distribution is the limit of $N(\mu, \tau^2)$ for $\tau^2 \to \infty$. As we have mentioned before, we use instead of X the conditioned variable $\overline{X} | \theta \sim N\left(\theta, \frac{\sigma^2}{n}\right)$.

For $\exp(\lambda)$ we consider [1, 5] as in the normal case the conditional distribution $\overline{X} | \lambda$. This distribution is also identical to the conditional distribution of the sample, and the common distribution is Erlang $E_{n;n\cdot\overline{X}}$. If we consider the prior distribution $\Gamma(a, b)$, we obtain

$$\lambda | \overline{X} \sim \Gamma\left(n + a, \frac{b}{n \cdot b \cdot \overline{X} + 1}\right), \tag{1.3}$$

with the non-informative case $a = 1$ and $b \to \infty$.

Consider now $X \sim Po(\lambda)$. Because the sum of n independent Poisson variables $Po(\lambda_i)$ is $Po\left(\sum_{i=1}^{n} \lambda_i\right)$, it results that $S | \lambda \propto e^{-n\cdot\lambda} \cdot \lambda^S$. If we consider the prior distribution $\Gamma(a, b)$, we obtain [1, 5]

$$\lambda | S \propto \lambda^{S+a-1} \cdot e^{-\frac{n\cdot b+1}{b}\cdot\lambda}, \tag{1.4}$$

i.e. $\Gamma\left(S + a, \frac{b}{n\cdot b+1}\right)$. The non-informative prior distribution is the limit of $\Gamma(1, b)$ for $b \to \infty$. We notice that the non-informative prior distribution is the same as in the exponential distribution case. This is another connection between the exponential and Poisson distributions, as the well-known connection that establishes that if the time T of an event is $\exp(\lambda)$, the number of events in unit time is $Po(\lambda)$.

In [8] there is presented a two-sided Bayesian significance test for normal distribution. The null hypothesis is $H_0 : \theta = \theta_0$ and the alternative hypothesis is of course $H_1 : \theta \neq \theta_0$, where θ is the expectation of the normal distribution if the variance is known. For this purpose there are considered the probability p_0, and a normal prior cdf with jump in θ_0 equal to p_0 (the difference between $P(\theta \leq \theta_0)$ and $P(\theta < \theta_0)$ is equal to p_0, and the only jump is in θ_0). p_0 is the prior probability to have $\theta = \theta_0$, and if we have no prior information we apply [8] the maximum entropy principle and we take $p_0 = 0.5$. The posterior probability to have $\theta = \theta_0$ is

$$P(\theta = \theta_0 | X) = \frac{p_0 \cdot f(X|\theta_0)}{p_0 \cdot f(X|\theta_0) + (1 - p_0) \int_{-\infty}^{\infty} f(X|\theta)\, g(\theta) d\theta}, \tag{1.5}$$

where $f(X|\theta)$ is the normal pdf with expectation θ an the above known variance, and $g(\theta)$ is the prior normal pdf without the above mentioned jump.

The two-sided Bayesian significance test tests the null hypothesis $H_0 : \theta = \theta_0$ against the alternative hypothesis $H_1 : \theta \neq \theta_0$ with the first degree error ε. We accept the null hypothesis H_0 if the above posterior probability from (1.5) is greater than $1 - \varepsilon$.

2. Methodology

For the Bayesian significance test we amplify (1.5) by $\frac{g(\theta_0)}{\int_{-\infty}^{\infty} f(X|\theta)g(\theta)d\theta}$, and we obtain

$$P\left(\theta=\theta_0|\, X\right)=\frac{p_0\cdot g\left(\theta_0|\, X\right)}{p_0\cdot g\left(\theta_0|\, X\right)+\left(1-p_0\right)g\left(\theta_0\right)}. \tag{2.1}$$

In the above formula X is a random variable conditioned by the parameter θ having the pdf $f\left(x|\,\theta\right)$. It is possible (and generally we use this possibility) to consider X a statistics depending on the sample X_1,..., X_n conditioned by θ. For instance, in the normal case X is the sample expectation $\overline{X}$. The cdfs $g\left(\theta_0\right)$ and $g\left(\theta_0|\, X\right)$ are the prior, respectively posterior pdf of the parameter θ. This formula is provided in [2] in the normal case. In the following we consider the non-normal case.

If we denote by R the ratio between the prior and posterior pdf of θ, the condition for accepting H_0 becomes (Ciuiu, 2013 for normal case)

$$R<\frac{p_0}{1-p_0}\cdot\frac{\varepsilon}{1-\varepsilon}=\gamma_1. \tag{2.2}$$

As for the method of maximum likelihood, denoting by $\gamma=\ln\gamma_1$, the condition (2.2) becomes

$$\ln R<\gamma. \tag{2.3}$$

In the case of exponential distribution we obtain

$$\ln R=\sum_{i=0}^{n-1}\ln\left(a+i\right)+n\ln b-\left(n+a\right)\ln\left(n\cdot b\cdot\overline{X}+1\right)-n\ln\lambda_0+n\cdot\lambda_0\cdot\overline{X}. \tag{2.4}$$

If we differentiate $\ln R$ with respect to $\overline{X}$ we obtain

$$\frac{\partial\ln R}{\partial\overline{X}}=n\lambda_0-\frac{n\cdot b\cdot(n+a)}{n\cdot b\cdot\overline{X}+1}, \tag{2.4'}$$

which becomes zero if $\overline{X}$ is such that $\lambda_0=\lambda_{expectation}=\frac{(n+a)b}{n\cdot b\cdot\overline{X}+1}$. If we differentiate again with respect to $\overline{X}$ we conclude that the above value $\overline{X}=\overline{X}_0$ is the minimum value for fixed a and b, and $\ln R$ is first increasing and next decreasing. The values a and b are chosen such that we have $\ln R<\gamma$ for $\lambda_0=\lambda_{expectation}$ and for $\lambda_0=\lambda_{mode}=\frac{(n+a-1)b}{n\cdot b\cdot\overline{X}+1}$, and for $\overline{X}=0$ and $\overline{X}\to\infty$ we have $\ln R>\gamma$.

We notice also that for fixed a and b, for $\overline{X}$ such that $\lambda_0=\lambda_{mode}$ we obtain a higher value for $\ln R$ than that for $\overline{X}$ such that $\lambda_0=\lambda_{expectation}$. Moreover, if we want to choose a and b in order to exist a value of $\overline{X}$ such that $\lambda_0=\lambda_{expectation}=\frac{(n+a)b}{n\cdot b\cdot\overline{X}+1}$ we must set $n\cdot b\cdot\overline{X}+1=\frac{(n+a)b}{\lambda_0}>1$. Because for $\lambda_0=\lambda_{mode}$ we obtain analogously $n\cdot b\cdot\overline{X}+1=\frac{(n+a-1)b}{\lambda_0}>1$ and $\frac{(n+a)b}{\lambda_0}>\frac{(n+a-1)b}{\lambda_0}$, it results that if a and b are chosen such that we accept the null hypothesis for $\overline{X}$ such that $\lambda_0=\lambda_{mode}$, the null hypothesis is accepted also for $\overline{X}$ such that $\lambda_0=\lambda_{expectation}$. In this case, the condition to have a value $\overline{X}$ such that $\lambda_0=\lambda_{mode}$ is $(n+a-1)\,b>\lambda_0$. If we set $\lambda_0=\lambda_{mode}$, we obtain

$$\ln R=\sum_{i=0}^{n-2}\ln\left(a+i\right)+n+a-1-\left(n+a-1\right)\ln\left(n+a-1\right)+a\cdot\ln\left(\frac{\lambda_0}{b}\right)-\frac{\lambda_0}{b}. \tag{2.5}$$

If we denote first by $c=\frac{\lambda_0}{b}$, it results that the acceptance of null hypothesis for fixed a does not depend only on b, but on this more general value of c. By differentiating on c we obtain the maximum for $c=a$. For this value of c we have

(2.5') $$\ln R=\sum_{i=0}^{n-2}\ln\left(\frac{a+i}{n+a-1}\right)+n-1+a\cdot\ln a,$$

which is increasing on a. Because $\lim\limits_{a\to 0}\ln R=-\infty$ and $\lim\limits_{a\to\infty}\ln R=\infty$, it results that there exists a_0 such that for $a<a_0$ the value $\ln R$ from (2.5) is less than γ for any value of c, hence for any value of b. Of course, a_0 from the last formula is chosen such that $\ln R=\gamma$.

We have $\lim\limits_{c\to 0}\ln R=-\infty$ and, because the last two terms in (2.5) can be written as $c\left(\frac{a\ln c}{c}-1\right)$, we have also $\lim\limits_{c\to\infty}\ln R=-\infty$, it results that for $a>a_0$ the domain of c for accepting the null hypothesis is outside an interval $[c_1,c_2]$ containing a_0. For $a=a_0$ the maximum of $\ln R$ is γ, hence for all values of c except a_0 we have $\ln R<\gamma$.

If $\overline{X}=0$, denoting by $c=\frac{\lambda_0}{b}$ as above, we obtain

(2.6) $$\ln R=\sum_{i=0}^{n-1}\ln(a+i)-n\ln c,$$

which is decreasing on c. To set for fixed a in the above formula $\ln R=\gamma$, we obtain

(2.6') $$c=\sqrt[n]{\frac{\prod\limits_{i=0}^{n-1}(a+i)}{\gamma_1}}.$$

Because $\lim\limits_{c\to 0}\ln R=\infty$ and $\lim\limits_{c\to\infty}\ln R=-\infty$, it results that the domain of c for fixed a such that we reject H_0 is $c<c_3(a)$. If we set $\overline{X}\to\infty$ in (2.4) it results that $\ln R\to\infty$, because the last term, $n\cdot\lambda_0\cdot\overline{X}$, tends to infinity with higher speed than $(n+a)\ln\left(n\cdot b\cdot\overline{X}+1\right)$. Therefore, setting $\overline{X}$ as common factor, we obtain the above infinite limit, and H_0 is rejected if $\overline{X}\to\infty$ for any values a and b.

The power of the test, after computing $\overline{X}_1$ and $\overline{X}_2$ is

(2.7) $$\pi(\lambda)=1+e^{-\lambda\overline{X}_2}\left(\sum_{i=0}^{n-1}\frac{\lambda^i\cdot\overline{X}_2^i}{i!}\right)-e^{-\lambda\overline{X}_1}\left(\sum_{i=0}^{n-1}\frac{\lambda^i\cdot\overline{X}_1^i}{i!}\right).$$

In the case of $Po(\lambda)$ we have

(2.8) $$\ln R_S=\sum_{i=0}^{S-1}\ln(a+i)+(S+a)\ln\left(\frac{b}{n\cdot b+1}\right)-a\ln b-S\ln\lambda_0+n\lambda_0.$$

If we compute the difference between two successive values (depending on R) of $\ln R$, we obtain

(2.9) $$\ln R_{S+1}-\ln R_S=\ln(a+S)+\ln\left(\frac{b}{n\cdot b+1}\right)-\ln\lambda_0=\ln\lambda_{expectation}-\ln\lambda_0.$$

Denote by $a_1=a+S$ and by $b_1=\frac{b}{n\cdot b+1}$. If we fix a and we differentiate on b, the condition $\frac{\partial\ln R_S}{\partial b}>0$ is equivalent with $a_1\cdot b_1<\frac{S}{n}$, which is finally equivalent with $a\cdot b<\frac{S}{n}$.

The condition such that S is the last value for which $\ln R$ decrease can be written as

(2.9') $$(a_1-n\cdot\lambda_0)\le\lambda_0<(a_1+1-n\cdot\lambda_0).$$

For $S \to \infty$, we have $\lim_{S\to\infty} \ln R_{S+1} - \ln R_S = \infty$, and from here $\lim_{S\to\infty} \ln R_S = \infty$. Therefore we reject the null hypothesis for large value of S.

If $S = 0$ we obtain

$$\ln R_0 = -a \ln (n \cdot b + 1) + n \cdot \lambda_0. \tag{2.8'}$$

We notice that $\ln R_0$ is decreasing on b for fixed a.

Consider now the pairs (a, b) on the top border of a region $(2.9')$, for a given $S = S_0$. We obtain the following values of $\ln R_0$ and its derivatives on a.

$$\begin{cases} \ln R_0 = a \ln\left(\frac{a+S-n\lambda_0}{a+S}\right) + n\lambda_0 \\ (\ln R_0)' = \frac{a}{a+S-n\lambda_0} - \frac{a}{a+S} + \ln(a+S-n\lambda_0) - \ln(a+S) \\ (\ln R_0)'' = \frac{n\lambda_0((2S-n\lambda_0)a+2S(S-n\lambda_0))}{(a+S-n\lambda_0)^2(a+S)^2} \end{cases}. \tag{2.10}$$

We have three cases depending on S. If $S > n\lambda_0$ we have $(\ln R_0)'' > 0$, hence $(\ln R_0)'$ increase from $\ln\left(\frac{S-n\lambda_0}{S}\right) < 0$ to zero. This is also the case $S = n\lambda_0$, but the increasement is from $-\infty$ to zero. It results that $\ln R_0' < 0$, and from here that $\ln R_0$ is decreasing. If $a = 0$ we obtain obviously $\ln R_0 = n\lambda_0$, and, using L'Hopital, $\lim_{a\to\infty} \ln R_0 = 0 > \gamma$. It results that in this case $\ln R_0 > \gamma$ for any $a > 0$.

If $S \leq \frac{n\lambda_0}{2}$, we have $a > n\lambda_0 - S$ to have all the logarithms in (2.10) defined. We have also $\ln R_0'' < 0$, hence $\ln R_0'$ is decreasing. The limit for $a \to \infty$ is also zero, and for $a \to n\lambda_0$ the first term must be considered (we know that in the cases $\frac{\infty}{\infty}$ and $\infty - \infty$ involving polynomial and logarithms, the terms involving polynomials have a higher "weight" than logarithms, and only them have to be considered). Therefore $\ln R_0$ increases from $-\infty$ to zero, and we use the Darboux property (in applications the bisection method).

In the last case, $\frac{n\lambda_0}{2} < S < n\lambda_0$, $\ln R_0''$ is first negative, and next positive. It results that $\ln R_0'$ decreases from ∞ to a negative value, and next increases to zero, hence it is also with one sign change, but now from plus to minus. Finally, it results that $\ln R_0$ increases from $-\infty$ to a positive value, and next decreases to zero. We use again the bisection method.

In the first case, on the top border of the region $(2.9')$ we have $a_1 \cdot b_1 = \lambda_0 < \frac{S}{n}$, hence $a \cdot b < \frac{S}{n}$ for the whole region. Therefore $\ln R_S$ and $\ln R_{S+2}$ are increasing on the whole region. Analogously, if $S + 1 < n\lambda_0$ (the other two cases, except the case of non-integer value of $n\lambda_0$ where we are in the first above case for S and in the third for $S + 1$) we have $\ln R_S$ and $\ln R_{S+2}$ decreasing on b for fixed a on the whole region.

It remains the above exception, when the hyperbolic line $a \cdot b = \lambda_0$ belongs to the region $(2.9')$. It results that $\ln R_S$ and $\ln R_{S+2}$ have both a maximum on this hyperbolic line. If $n\lambda_0 \in \mathbb{N}$, the top border for this value of S is in fact the hyperbolic line $a \cdot b = \lambda_0$.

We can prove analogously that on the hyperbolic lines $a_1 \cdot b_1 = \lambda_0$ we have $\ln R_S$ and $\ln R_{S+2}$ increasing. Of course, if $S < n\lambda_0$ we study the values only on $a > a_0$, where if $a = a_0$ we have $\ln R_0 = \gamma$. Using the expression of $\ln R_0$ we obtain

$$\ln R_S = \sum_{i=0}^{S-1} (\ln(a+i) - \ln(a+S)) + \ln R_0. \tag{2.11}$$

The derivated sum is the sum of $\frac{1}{a+i} - \frac{1}{a+S}$, and the limit of the sum as $a \to \infty$ is zero. Therefore we obtain $\ln R_S < \ln R_0$, as we have also obtained using the decreasement of $\ln R_S$ in terms of S from 0 to S. But we can not use this monotony to prove that $\ln R_{S+2} < \ln R_0$, because they are not in the same monotonic sequence: 0 is on decresing sequence, while $S + 2$ in the increasing one. But we can write analogously

(2.11')
$$\ln R_{S+2} = \sum_{i=0}^{S+1} (\ln (a+i) - \ln (a+S)) + \ln R_0.$$

To prove that $\ln R_{S+2} < \ln R_0$ we use in the last sum the terms up to $i = S-2$, which is analogously negative. The sum of the last three terms is $\ln\left(\frac{(a+S)^2-1}{(a+S)^2}\right)$, which is also negative (difference of squares).

Consider first the region $S > n\lambda_0$, where $\ln R_S$ and $\ln R_{S+2}$ are increasing on b for fixed a. We obtain $a_5 < a_4 < a_3 < a_2$ such that on the lower border $(a_1+1)\, b_1 = \lambda_0$ we have $\ln R_{S+2} = \gamma$ if $a = a_2$ and $\ln R_S = \gamma$ if $a = a_3$, and $\ln R_S = \gamma$ if $a = a_4$ and $\ln R_{S+2} = \gamma$ if $a = a_5$. The graphics of the region is presented in Fig. 2 *a*. Therefore, if $a \leq a_5$ for all pairs in the region we have $\ln R_S < \gamma$ and $\ln R_{S+2} \leq \gamma$, with equality in the last relation only for $a = a_5$ on the top border. If $a_5 < a \leq a_4$ we have for all pairs $\ln R_S \leq \gamma$, with equalyty only on the top border if $a = a_4$. $\ln R_{S+2} < \gamma$ on lower border, and $\ln R_{S+2} > \gamma$ on upper border, hence we use for $\ln R_{S+2}$ the bisection method. If $a_4 < a < a_3$ both function are less than γ on lower border, and greater than γ on upper border. Therefore we use the bisection method for $\max(\ln R_S, \ln R_{S+2})$. If $a > a_3$ we have no pair such that $\ln R_S \leq \gamma$, because it is already greater than γ on lower border. Therefore we have no additional pairs than that already mentioned.

Analogously, if $S+1 < n\lambda_0$ we have the same values of $a_2, ..., a_5$, but the order is $a_3 < a_2 < a_5 < a_4$, as in Fig. 2 *b*.

If $a \leq a_3$ we have $\ln R_S < \gamma$ and $\ln R_{S+2} < \gamma$ for all pairs. For $a = a_3$ we have on lower border $\ln R_S = \gamma < \ln R_0$. If $a_3 < a \leq a_2$ $\ln R_{S+2} <$
gamma for all pairs, and $\ln R_S$ is greater than γ on lower border and less than γ on upper border. We use again the bisection method for $\ln R_S$. On the last interval, which is now $(a_2, a_5]$ we use the bisection method for $\max(\ln R_S, \ln R_{S+2})$: both functions are greater than γ on lower border and less than γ on upper border. We have again no pair for $a > a_4$.

The value for which $\ln R_0 = \gamma$ is higher on top border, hence all the above accepting regions have to be intersected with the accepting region according $\ln R_0 > \gamma$. If the involved a is grater than the value on upper border, we have $\ln R_0 > \gamma$. If a is between these values, the region is obtain by bisection method: $\ln R_0 > \gamma$ on lower border, and $\ln R_0 < \gamma$ on upper border. In the other case, when a is lower than the value on lower border, the region for which we have $\ln R_S, \ln R_{S+2} < \ln R_0 < \gamma$ vanishes.

If we have $n\lambda_0 = S \notin \mathbb{N}$. We have a maximum for $\ln R_S$ and $\ln R_{S+2}$. Tacking non-integer S, we are in the first case for $\ln R_0 > \gamma$ on the whole hyperbolic line $a \cdot b = \lambda_0$. We obtain in the lower sub-region $a \cdot b \leq \lambda_0$ values for which $\ln R_S = \gamma$ or $\ln R_{S+2} = \gamma$ as in the first case, and on the upper sub-region $a \cdot b \geq \lambda_0$ the similar values as in the second above case. Therefore we divide the region where we accept H_0 if S is between the last value where $\ln R_S$ decrease and that value plus two and we reject it if $S = 0$ in two sub-regions obtained as in the first/ second case (see Fig. 2 *c*).

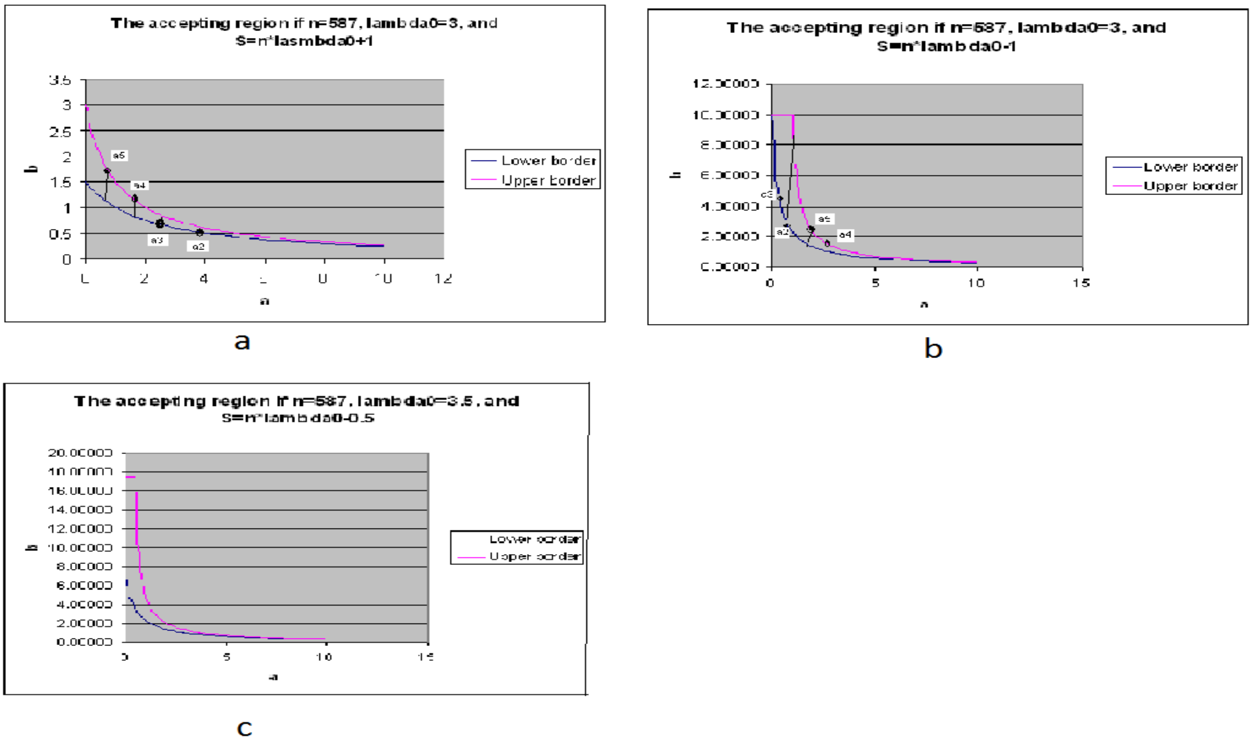

FIGURE 1. The accepting region if $a \cdot b > \frac{S}{n}$ in whole region (a); if $a \cdot b < \frac{S}{n}$ in whole region (b); if $a_1 \cdot b_1 < \frac{S}{n} < (a_1 + 1) \cdot b_1$ (c)

From the above bimodality, it results that for fixed a and b there exists $S_1 \le S_0 < S_0 + 1 < S_2$ such that we accept H_0 if and only if $S_1 \le S \le S_2$. The power of the test becomes

$$\pi(\lambda) = 1 - e^{-\lambda} \left(\sum_{i=S_1}^{S_2} \frac{\lambda^i}{i!} \right). \tag{2.12}$$

3. APPLICATIONS

Example 3.1. *Consider the monthly data on household savings (EP) between January 1997 and August 2003 (Jula and Jula, 2010). It results 80 values of this variable. Consider also the variable EP* $\sim \exp(\lambda)$. *We test first the null hypothesis* $H_0 : \lambda = \lambda_0$ *against the alternative hypothesis* $H_1 : \lambda \neq \lambda_0$, *with* $\lambda_0 = 0.025$ *or* $\lambda_0 = 0.04$.

The sample expectation is $\overline{X} = 46.0372$.

The formula (2.5) is written

$\ln R = \sum_{i=0}^{78} \ln\left(\frac{a+i}{79+a}\right) + 79 + a \cdot \ln a,$

with $a_0 = 0.14252$.

Next we fix a, and we obtain the following results.

TABLE 1. Bayesian signification two-sided test in the exponential case

θ_0	a	b_1, b_2, b_3	b	Accepting probability	Accepting interval	$\pi\left(\widehat{\lambda}\right)$
0.025	1	$0.00310, 5.275, 0.00079$	6	0.99769	$(29.30216, 54.23609)$	$\pi(0.02199) = 0.0472$
			0.002	0.99977	$(18.56583, 55.45965)$	$\pi(0.01936) = 0.2509$
	0.5	$0.00625, 9.05049, 0.00082$	10	0.96743	$(33.67465, 47.62736)$	$\pi(0.02186) = 0.3574$
			0.005	0.9707	$(28.69739, 48.40437)$	$\pi(0.02073) = 0.4722$
	0.1	$b_3 = 0.00085$	0.001	$> 1 - 5 * 10^{-6}$	$(4.34759, 65.95952)$	$\pi(0.02073) = 0.1289$
0.04	1	$0.005, 5.29, 0.00126$	6	0.00001	$(18.71088, 33.30456)$	$\pi(0.02199) = 0.9946$
			0.003	0.00001	$(10.9746, 35.0893)$	$\pi(0.02017) = 0.9982$
	0.5	$0.001, 8.58, 0.0013$	10	$< 5 * 10^{-6}$	$(21.44254, 29.27303)$	$\pi(0.02186) = 0.9999$
			0.005	$< 5 * 10^{-6}$	$(14.68424, 32.77903)$	$\pi(0.02073) = 0.9996$
	0.1	$b_3 = 0.00136$	0.005	$< 5 * 10^{-6}$	$(14.00813, 33.57903)$	$\pi(0.02063) = 0.9994$

In the above table we have consider one case in which, for the given prior $\Gamma(a, b)$ distribution of λ we accept the null hypothesis, and one in which we reject the null hypothesis. The signification of b_i with is as follows. For fixed a, we obtain c_1 and c_2 the roots of

$$\sum_{i=0}^{78} \ln\left(\frac{a+i}{a+79}\right) + 79 + a + a\ln c - c = \gamma.$$

If $\overline{X} = 0$ we have

$$c_3 = \sqrt[80]{\frac{\prod_{i=0}^{79}(a+i)}{\gamma_1}}.$$

The obtained domain for b is

$$\begin{cases} b \notin [b_1, b_2] \\ b > b_3 \end{cases},$$

where $b_i = \frac{1}{c_i}$.

Example 3.2. *Consider the exchange rate Euro versus RON in the period April 1st 1999— February 27 2015, according the Database of Romanian National Bank (The National Bank of Romania Database, 2015). We have daily data, and there are omitted holidays. We have finally* 4111 *data. We want to test in the Poisson case the null hypothesis $H_0 : \lambda = \lambda_0$, where $\lambda_0 \in \{3, 3.5, 4\}$, and in the other two cases $H_0 : \theta = \theta_0$, where $\theta_0 \in \{0.25, 0.5, 0.75\}$. For all tests consider $\varepsilon = 5\%$ (the standard treshold of Fisher), as in the previous example.*

We count the number of increasements in seven days for Poisson distribution.

Considering $p_0 = 0.5$ and $\varepsilon = 0.05$, we obtain by computation $\gamma = -2.94444$.

We have $S = 2021$ and $n = 587$. By the moments' method we obtain

$$\widehat{\lambda} = \frac{2021}{587} = 3.44293.$$

In the following table the signification of each matrix in the case of testing null hypothesis $H_0 : \lambda = \lambda_0$ for different values of λ_0 is as follows.

The first column contains several successive values of the statistics S. The second column contains the corresponding values for a such that $\ln R_S = \gamma$. The third column contains the corresponding values for a such that $\ln R_{S+2} = \gamma$. The fourth column contains the corresponding values for a such that $\ln R_{S-1} = \gamma$.

We obtain the following results.

Table 2. The values of a such that $\ln R_S = \gamma$, $\ln R_{S+2} = \gamma$ and $\ln R_{S-1} = \gamma$ for different values of S

Value of λ_0	Matrix				Value of λ_0	Matrix			
3	1756	20.9328	20.82417	20.82417	3.4	1990	25.93287	25.93287	25.93287
	1757	16.2312	16.22368	16.22368		1991	20.84258	20.83448	20.83448
	1758	12.13318	12.12661	12.12661		1992	16.21647	16.20936	16.20936
	1759	8.64502	8.63917	8.63915		1993	12.12694	12.12065	12.12065
	1760	6.05541	6.04984	6.04984		1994	8.71412	8.70839	8.70839
	1761	5.0554	5.04984	5.04984		1995	6.35294	6.34739	6.34739
	1762	5.87129	5.86559	5.86559		1996	5.7465	5.74095	5.74095
	1763	7.78108	7.77478	7.77478		1997	6.86399	6.85824	6.85824
	1764	10.3663	10.35913	10.35913		1998	9.00201	8.99568	8.99568
	1765	13.48098	13.47275	13.47275		1999	11.8037	11.79652	11.79652
	1766	17.06251	17.05313	17.05312		2000	15.13621	15.12806	15.12804
3.5	2049	24.62333	24.61457	24.61457	4	2343	23.25458	23.24665	23.24665
	2050	19.63077	19.62304	19.62303		2344	18.36413	18.35708	18.35708
	2051	15.12971	15.1299	15.1299		2345	14.018	14.0172	14.017
	2052	11.20031	11.19425	11.19425		2346	10.34891	10.34316	10.34316
	2053	8.03238	8.02675	8.02675		2347	7.68671	7.68114	7.68114
	2054	6.12245	6.1169	6.1169		2348	6.6867	6.68115	6.68114
	2055	6.09954	6.09396	6.09396		2349	7.53884	7.53318	7.53318
	2056	7.59452	7.58865	7.58865		2350	9.59601	9.58993	9.58993
	2057	9.9736	9.96706	9.96706		2351	12.40966	12.40289	12.40289
	2058	12.96936	12.96196	12.96196		2352	15.80064	15.79302	15.79302
	2059	16.47723	16.46883	16.46883		2353	19.68986	19.68128	19.68128

In the above matrices we notice that the values of a are bimodal in the three corresponding columns. From the row that a increase (for instance 1761 for $\lambda_0 = 3$) there is no value for a such that $\ln R_0 = \gamma$ for $\lambda_0 = 3.4$ or $\lambda_0 = 3.5$, or this property is true from the next row in the other two cases. There exists a such that $\ln R_0 = \gamma$ for the first six values of S: for the position $i = \overline{1,6}$ the corresponding value of a is $6 - i + 10^{-5}$ for $\lambda_0 = 3$ or $\lambda_0 = 4$, $6.8 - i + 10^{-5}$ for $\lambda_0 = 3.4$, respectively $6.5 - i + 10^{-5}$ for $\lambda_0 = 3.5$. We notice also that if weincrease i in the above formulae, we obtain negative values of a.

In the following we choose a value of S such that there exists not a such that $\ln R_0 = \gamma$. Next for choosen S we compute the limits of a for which b is in a closed interval $[b_1, b_2]$. We choose two values for a, one less than the lower limit and one in the corresponding interval. For each a we determine the closed interval $[b_1, b_2]$, and for the chosen a and b we compute the accepting probability and the power of the test. The results are presented in the following table. Finally we compute the interval for S such that we accept the null hypothesis for given λ_0, a and b.

Table 3. The accepting probability and the power of the test if $\lambda_0 = \lambda_{expectation}$

λ_0	S	(a_0, a_1)	a	$[b_1, b_2]$	b	$\begin{pmatrix} \lambda_{expectation} \\ p \\ \pi \end{pmatrix}$	$[S_1, S_2]$
3	1757	$(12.12661, 16.22368)$	10	$[0.42857, 0.5]$	0.45	$\begin{pmatrix} 3.44692 \\ 0 \\ 1 \end{pmatrix}$	$[1721, 1795]$
			15	$[0.25, 0.26136]$	0.26	$\begin{pmatrix} 3.44591 \\ 0 \\ 1 \end{pmatrix}$	$[1757, 1758]$

λ_0	S	(a_0, a_1)	a	$[b_1, b_2]$	b	$\begin{pmatrix} \lambda_{expectation} \\ p \\ \pi \end{pmatrix}$	$[S_1, S_2]$
3.4	1999	$(11.79652, 15.12804)$	10	$[0.23944, 0.25758]$	0.24	$\begin{pmatrix} 3.43558 \\ 0.96228 \\ 0.3946 \end{pmatrix}$	$[1960, 2041]$
			12.5	$[0.20359, 0.21008]$	0.205	$\begin{pmatrix} 3.43567 \\ 0.95224 \\ 0.5916 \end{pmatrix}$	$[1975, 2026]$
3.5	2055	$(6.09396, 7.58865)$	5	$[0.53846, 0.63636]$	0.57	$\begin{pmatrix} 3.44116 \\ 0.94475 \\ 0.634 \end{pmatrix}$	$[2027, 2085]$
			6.5	$[0.4375, 0.46875]$	0.45	$\begin{pmatrix} 3.44098 \\ 0.93705 \\ 0.7891 \end{pmatrix}$	$[2040, 2072]$
4	2350	$(9.58993, 12.40289)$	7	$[0.4, 0.44444]$	0.42	$\begin{pmatrix} 3.4409 \\ 0 \\ 1 \end{pmatrix}$	$[2310, 2392]$
			10.5	$[0.2963, 0.30815]$	0.2975	$\begin{pmatrix} 3.44111 \\ 0 \\ 1 \end{pmatrix}$	$[2327, 2376]$

4. Conclusions

We know that the exponential and Poisson distributions are connected: if the random time between two successive events is $\exp(\lambda)$ the distribution of the number of such events per time unit is $Po(\lambda)$.

Another connection is in Bayesian inference [1, 4, 5]: the used family of conjugated prior distributions is the Gamma family of dostributions. In both cases for the parameters of posterior distributions we add an integer number to the first parameter, a, and the posterior parameter b is b divided to a linear expression in b (formulae (1.3) and (1.4)).

When we compute $\ln R$ from (2.3) we have to compute for the constant in terms of λ $\ln\Gamma(a_1) - \ln\Gamma(a)$. From here we obtain, using the well known relation $\Gamma(a+1) = \Gamma(a) * a$ that becomes $\ln\Gamma(a+1) = \ln\Gamma(a) + \ln a$, the sums of logarithms from (2.4) and (2.8).

The other term for the constant is $a_1 \ln b_1 - a \ln b$, and the terms in λ are $(a_1 - a)\ln\lambda$ and $\lambda\left(\frac{1}{b_1} - \frac{1}{b}\right)$. All three expression are easy to derive on a and b, for fixed λ.

The above considerations were used in this paper to study the monotony of $\ln R$ in terms of a, b and λ.

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Department of Mathematics and Computer Science, Technical University of Civil Engineering Bucharest, Bucharest, Romania;, Macroeconomic Modeling Center, National Institute for Economic Research "Costin C. Kiriţescu", Romanian Academy
Email address: daniel.ciuiu@utcb.ro

ORDER RELATIONS IN INTERVAL-SPACES

Rodica-Mihaela Dăneţ
Technical University of Civil Engineering Bucharest, Romania
e-mail: rodica.danet@gmail.com
Marian-Valentin Popescu
Technical University of Civil Engineering Bucharest, Romania
e-mail: popescu.marianvalentin@gmail.com
Nicoleta Popescu
e-mail: popescu.snicoleta@gmail.com

Abstract. In 2013, the first author introduced the *interval-spaces* (abbreviated as *i-spaces*) and studied the problem of the extension of some *i-linear functionals*; this problem is difficult to solve, because the i-spaces are not vector spaces. (We mention that the considered intervals are closed intervals in an arbitrary real ordered vector space and, in particular, in ϒ). Later, we studied the extension of *i-linear operators* with values in a Dedekind complete Riesz space. The next step will be to study the extension of the i-linear operators between two i-spaces. Solving this problem is even more difficult, requiring first a deeper study of *order relations* over an i-space. In this paper we will begin this study for the space of closed intervals of real numbers, starting from what is known in the literature and looking for *new order relations*, to find the "*best*" relation for our purpose, and even a *total order relation*.
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Keywords: interval analysis, interval-linear operator, interval-sublinear operator.

1. Introduction

In 2013, at the seventh Positivity Conference (a Zannen Centennial Conference), Leiden University, the Netherlands, july 22-26, the first author of this paper introduced (see [6]) the *interval-spaces* (abbreviated as *i-spaces*) and studied ([6], [4] and [5]) the problem of the the extension of some *i-linear functionals*. In two previous papers, [7] and [8] we studied the extension of *i-linear operators* with values in a Dedekind complete Riesz space. For more details, see the Appendix at the end of this paper. We mention that the main difficulty to work in the i-space setting comes from the fact that these spaces, endowed with the natural algebraic operations, are not vector spaces. Our next goal will be to extend this study to the case where the i-linear operators act between two i-spaces. But, obviously, solving this new problem will be even more difficult, requiring as a first step, a deeper study of the order relations introduced in i-spaces. A second step will be to find the *best* such relation and to look for one that is *total*, which means that any two closed intervals can be compared. The order relations found in the literature (and which will be mentioned in this paper) do not have this property. In other words, they are only *partial* order relations.

2. Preliminaries

An *interval-space* (in short, *i-space*) is associated to an arbitrary real ordered vector space $(E,\leq)$. An *order interval* in E with the endpoints $\underline{a},\overline{a}\in E$ where $\underline{a}\leq\overline{a}$, denoted by $[a]=[\underline{a},\overline{a}]$, is defined as the set:

$$\left\{x\in E \middle| \underline{a}\leq x\leq\overline{a}\right\}.$$

The *interval-set* (in short, *i-set*) associated to E is the set

$$IE = \{[a] = [\underline{a}, \overline{a}] | \underline{a},\ \overline{a} \in E,\ \underline{a} \le \overline{a}\}$$

of all order intervals in E.

It is customary for the extremities $\underline{a}$ and $\overline{a}$ of the interval $[a] = [\underline{a}, \overline{a}]$ to be denoted by a_L and a_R, respectively. (Here "L" comes from "*Left*" and "R" from "*Right*".) Of course, with these notations, the interval $[a]$ is written $[a] = [a_L, a_R]$.

If $\underline{a} = \overline{a}$ is the element $a \in E$, then the interval $[a] = [\underline{a}, \overline{a}]$ consists only of the element a, and we say that $[a]$ is a *degenerate interval*. If $\underline{a} < \overline{a}$, then the order interval $[a] = [\underline{a}, \overline{a}]$ is *a non-degenerate interval*. We can identify the element $a \in E$ with the degenerate interval $[a] = [a, a]$.

We endow IE with the following algebraic operations:

1) The *addition*:

$$[a] \oplus [b] = \{a + b | a \in [a],\ b \in [b]\},$$

that is,

$$[a] \oplus [b] = [\underline{a} + \underline{b}, \overline{a} + \overline{b}],$$

if $[a] = [\underline{a}, \overline{a}]$ and $[b] = [\underline{b}, \overline{b}]$.

2) The *scalar multiplication* with reals:

$$\alpha[a] = \{\alpha a | a \in [a]\},\ \alpha \in \mathbb{R},$$

that is,

$$\alpha[a] = \begin{cases} [\alpha\underline{a}, \alpha\overline{a}], \text{ if } \alpha \in \mathbb{R},\ \alpha \ge 0 \\ [\alpha\overline{a}, \alpha\underline{a}], \text{ if } \alpha \in \mathbb{R},\ \alpha < 0 \end{cases},$$

where $[a] = [\underline{a}, \overline{a}]$.

We mention that, endowed with the previous algebraic operations, IE *is not* a real vector space; see for example, [6]. More precisely $(IE, \oplus)$ is a commutative semigroup (monoid) with a neutral element $\mathbf{0}$ $(\mathbf{0} = [0,0])$ but *is not* a group, because a non-degenerate interval $[a]$ has no inverse with respect to addition. Indeed, suppose that, for $[a] = [\underline{a}, \overline{a}]$ with $\underline{a} < \overline{a}$, there exists an inverse $[b] = [\underline{b}, \overline{b}]$. Then:

$[\underline{a}, \overline{a}] \oplus [\underline{b}, \overline{b}] = \mathbf{0} \Rightarrow [\underline{a} + \underline{b}, \overline{a} + \overline{b}] = [0,0] \Rightarrow \underline{a} + \underline{b} = 0$ and $\overline{a} + \overline{b} = 0 \Rightarrow \underline{b} = -\underline{a}$ and $\overline{b} = -\overline{a}$.

But $\underline{b} \le \overline{b}$ and therefore $-\underline{a} \le -\overline{a} \Leftrightarrow \overline{a} \le \underline{a}$, which is in contradiction with $\underline{a} < \overline{a}$.

If IE is an i-space, we say (see [6]) that a functional $f : IE \to \mathbb{R}$ is an *interval-linear functional* (in short, *i-linear functional*) if:

1. $f([a] \oplus [b]) = f([a]) + f([b])$, for all $[a], [b] \in IE$;
2. $f(\alpha[a]) = \alpha f([a])$, for all $[a] \in IE$ and $\alpha \in \mathbb{R}$.

If, in addition, F is an arbitrary Dedekind complete Riesz space, we say (see [7]) that an operator $L : IE \to F$ is an *interval-linear operator* (in short, *i-linear operator*) if:

1. $L([a] \oplus [b]) = L([a]) + L([b])$, for all $[a], [b] \in IE$;
2. $L(\alpha[a]) = \alpha L([a])$, for all $[a] \in IE$ and $\alpha \in \mathbb{R}$.

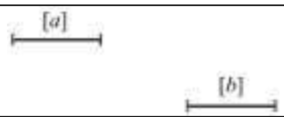
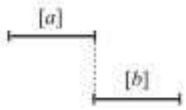
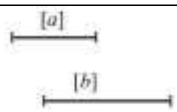
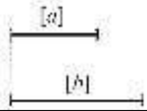
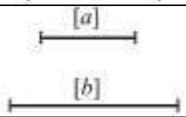
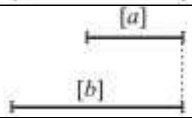
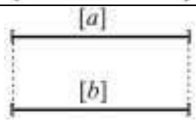
In this paper we will consider $E=\mathbb{R}$ and we will refer to the i-space $I\mathbb{R}$ as the *interval-space of reals* (in short, *i-space of reals*).

3. Order relations in interval-space of reals

Obviously, to make a decision choosing from several possibilities expressed quantitatively, we need to compare the values of respective amounts. In practice, most often, we know these values *with interval uncertainty*. Thus we come *to compare intervals*. All possible relations between intervals, for example on real line, which are generated by ordering of the end-points, are described in *Allen's interval algebra*. This algebra is a calculus for temporal reasoning, introduced by James F. Allen in 1983; see [1]. More precisely, to model *temporal knowledge* about the physical world, Allen introduced a *temporal logic based on intervals* and their qualitative relationships; see [11].

Informally speaking, *intervals* correspond to *events* (while *points* correspond to *instants*). Events, almost always has *duration*, and thus they can be represented by *intervals*. The end-points $\underline{a}$ and $\overline{a}$ of an event $[a]=[\underline{a},\overline{a}]$ are the *beginning* and the *ending* of the respective event. Obviously, *instantaneous* events are represented by *points* on the real line.

To understand what relationships Allen's interval algebra refers to, we mention that the following *13 basic relations* capture all possible relations between intervals; see [9].

Relation	Ilustration	Interpretations
1. $[a]<[b]$ 2. $[b]>[a]$		$[a]$ takes place before $[b]$
3. $[a]$ m $[b]$ 4. $[a]$ mi $[b]$		$[a]$ meets $[b]$ ("i" stands for "inverse") ($[b]$ met by $[a]$)
5. $[a]$ o $[b]$ 6. $[b]$ oi $[a]$		$[a]$ overlaps with $[b]$ ($[b]$ overlap by $[a]$)
7. $[a]$ s $[b]$ 8. $[b]$ si $[a]$		$[a]$ starts $[b]$ ($[b]$ started by $[a]$)
9. $[a]$ d $[b]$ 10. $[b]$ di $[a]$		$[a]$ during $[b]$ ($[b]$ includes $[a]$)
11. $[a]$ f $[b]$ 12. $[b]$ fi $[a]$		$[a]$ finishes $[b]$ ($[b]$ finished by $[a]$)
13. $[a]=[b]$		$[a]$ is equal to $[b]$

Allen's interval algebra defines many different relations between intervals. An important class of relations are the *order relations*; see [18].

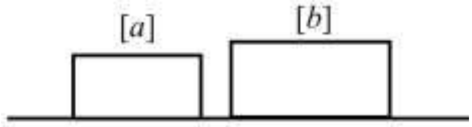
4. Order relations of intervals (Interval order relations). What is known?

Some informations about this subject can be found in [13]. Firstly, we remark that we *can summarize the relations of Allen's interval algebra*, considering that the following situations can occur, for two intervals (belonging, for example, to $I\mathbb{R}$):

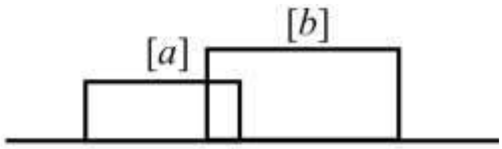
Type – I Non-overlapping intervals,
Type – II Partially overlapping intervals,
Type – III Completely overlapping intervals.

The following pictures are suggestive:

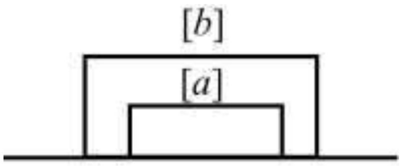
Type – I



Type – II



Type – III



Obviously, the first thing to compare two intervals would be to consider *an order relation in an interval-space*, that is, to define what it means that $[a]\le[b]$.

Many order relations on interval-spaces are considered in the literature; see [13]. For a brief review of some these order relations, remarking that some of them use *end-point bounds* of the intervals and/or *centre-radius bounds*, we change the notation $[a]=[\underline{a},\overline{a}]$ in $[a]=[a_L,a_R]$. (Notice again that " L " comes from "*Left*" and " R " from "*Right*".)

The interval $[a]=[a_L,a_R]$ is also denoted by $\langle a_C,a_W\rangle$, where $a_C=\dfrac{a_R+a_L}{2}$ and $a_W=\dfrac{a_R-a_L}{2}$ are the *centre* and, respectively, the *radius* of the interval. Here, "*W*" comes from the word "*Width*", since we remark that $2a_W$ is exactly the "width" of the interval $[a]$.

The *idea* to consider the notation $\langle a_C,a_W\rangle$ comes from the observation that every real number $a\in\mathbb{R}$ can be expressed as an interval-number $[a,a]$, with *zero width* and *centre* a, that is, $[a,a]=<a,0>$. The pioneer of the study of order relations on $I\mathbb{R}$ was Ramon E. Moore; see [17]. He first gave two *transitive* relations between two intervals $[a]=[a_L,a_R]$ and $[b]=[b_L,b_R]$, extending to $\overline{I\mathbb{R}}=\{[a_L,a_R] \mid a_L,a_R\in\overline{\mathbb{R}}=\mathbb{R}\cup\{\pm\infty\}\}$ the natural order of $\overline{\mathbb{R}}$:

(*) $[a]\prec[b]\Leftrightarrow a_R<b_L$ (Type – I intervals);

(**) $[a] \subseteq [b] \Leftrightarrow b_L \le a_L$ and $a_R \le b_R$ (Type – III intervals).

Also, Moore first defined the *equality* of two intervals:

(***) $[a] = [b] \Leftrightarrow a_L = b_L$ and $a_R = b_R$.

In the literature, the relation (*) is called the "left-to-right procedure by ranking intervals", $[a]$ being considered less than $[b]$, if $[a]$ is completely to the left of $[b]$. Concerning the relation (*), in 1985, Peter C. Fishburn [10] considered that "this is the *common* and by far the *most proeminent sense* of '*interval order*', because if $[a] \prec [b]$, then every value from $[a]$ is smaller than every value from $[b]$". In constrast to Fishburn's opinion, Roumen Anguelov [2] considered that "the use of the relation (*) is based on the view point that inequality between intervals should imply inequality between their interiors. This approach is rather limiting since (*) does not retain some eventual properties of the order on $\overline{\mathbb{R}}$. For example, a non-degenerate interval $[a]$ and the interval $[a]+\varepsilon$ are not comparable with respect to (*), when the positive real number ε is small enough". (Of course, here $[a]+\varepsilon$ is actually $[a] \oplus [\varepsilon, \varepsilon]$.) Obviously, the relation (*) is not an order relation unlike the relation (**), which is the set inclusion property of intervals.

Anguelov [2] considered that the inclusion relation on the set $\overline{I\mathbb{R}}$ is motivated by the *applications* of Interval Analysis to generating enclosures of solution set, for example for interval linear equations. However, the role of partial order relations extending the total order of $\overline{I\mathbb{R}}$ has also been recognized in computing; see [3].

Now I present *some order relations* "$\le$" on $I\mathbb{R}$ found in the literature, most of them being *Archimedean*, that is, for all $[a] \in \mathbb{R}$ whenever there exists some $[b] \in I\mathbb{R}$ such that $n[a] \le [b]$ for all positive integers n, then necessarily $[a] \le 0$. We analyze *what types* of intervals can be compared using these order relations, to see how far are they from being *total*.

Let $[a] = [a_L, a_R] = \langle a_C, a_W \rangle$ and $[b] = [b_L, b_R] = \langle b_C, b_W \rangle$ in *IE*.

1. The strong order

We remark that completing the relation (*) with the equality between two intervals, we obtain an order relation, called *the strong order*:

$$[a] \preceq [b] \Leftrightarrow \bar{a} < \underline{b} \text{ or } [a] = [b].$$

Of course, only the *Type–I* intervals may be compared using this relation. In other words, "$\preceq$" is not a total order relation.

2. The containment order

$$[a] \subseteq [b] \Leftrightarrow \underline{b} \le \underline{a} \text{ and } \bar{a} \le \bar{b}.$$

Only *Type–III* intervals can be compared by this relation. In other words, "$\subseteq$" is not a total order relation.

3. The weak order

$$[a] \le_{LR} [b] \Leftrightarrow a_L \le b_L \text{ and } a_R \le b_R.$$

According to [18] this order relation is a very natural sense of an interval order. For example, saying that an event *extended in time can be prior to another event* if it still underway when the subsequent event initiates. This means that intervals of *Type–I* and *Type–II can be compared* using this order relation but *not* the *Type–III.* This *partial order* was introduced and studied by Svetoslav Markov in [14] and [15] (see also [2] and [16]).

All three previous order relations have *end-points formulation*.

4. An order relation using the centre-radius formulation.

This order was introduced by H. Ishibuchi and H. Tanaka for *maximization problems of interval profits*, [12]. (Note that they also proposed to use the *weak order* for intervals in such problems.)

This order relation is denoted by "$\leq_{C,IW}$" and is defined as such:

$$\text{if } [a]=\langle a_C, a_W\rangle \text{ and } [b]=\langle b_C, b_W\rangle \text{ in } I\mathbb{R},$$

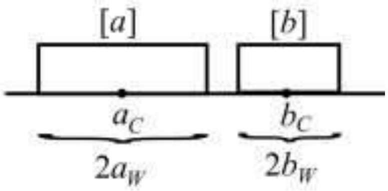
then

$$[a]\leq_{C,IW}[b] \Leftrightarrow a_C \leq b_C \text{ and } a_W \geq b_W.$$

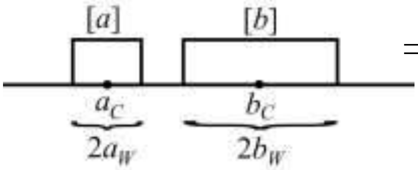
Here, the *centre* and the *radius* of the intervals are the expected value and the uncertainty of an imprecise parameter, respectively. Now we analyze which types of intervals can be compared by using this order relation.

We have:

Type–I



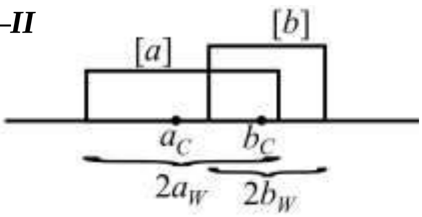
$\Rightarrow [a]\leq_{C,IW}[b]$;



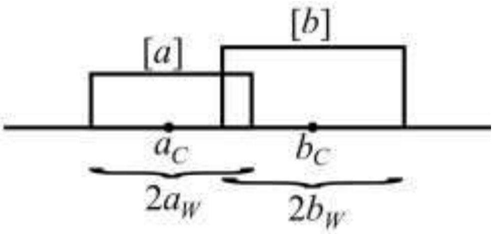
$\Rightarrow [a]\not\leq_{C,IW}[b]$ and $[b]\not\leq_{C,IW}[a]$.

So, if we use this order relation, *Type–I* intervals are partially comparable.

Type–II



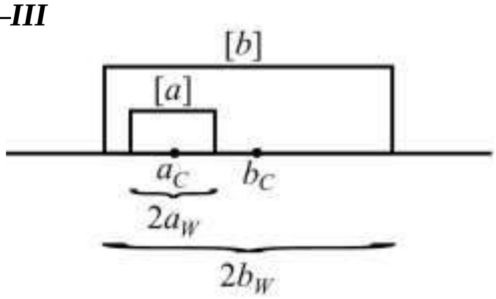
$\Rightarrow [a]\leq_{C,IW}[b]$,



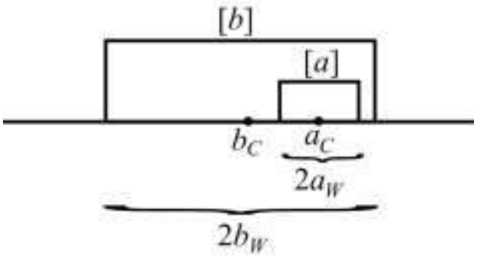
$\Rightarrow [a]\not\leq_{C,IW}[b]$ and $[b]\not\leq_{C,IW}[a]$.

Again, working with this order relation, *Type–II* intervals are partially comparable.

Type–III



$\Rightarrow [a]\not\leq_{C,IW}[b]$ and $[b]\not\leq_{C,IW}[a]$,



$\Rightarrow [b] \leq_{C,IW} [a]$ and $[a] \not\leq_{C,IW} [b]$.

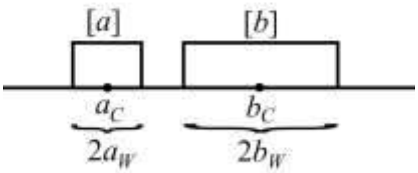
Hence, *Type–III* intervals are partially comparable by this order, too. In other words, "$\leq_{C,IW}$" is not a total order relation.

5. The following order relation (using another centre-radius formulation) also appeared in [12], being recommended for use it in *minimization problems of interval costs*.

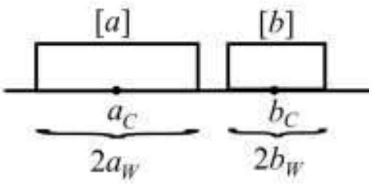
If $[a] = \langle a_C, a_W \rangle$ and $[b] = \langle b_C, b_W \rangle$ are in $I\mathbb{R}$, then:

$$[a] \leq_{CW} [b] \Leftrightarrow a_C \leq b_C \text{ and } a_W \leq b_W .$$

Type–I

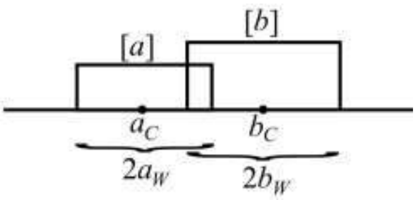


$\Rightarrow [a] \leq_{CW} [b]$,

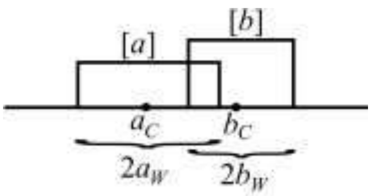


$\Rightarrow [a] \not\leq_{CW} [b]$, and $[b] \not\leq_{C,IW} [a]$.

Type–II

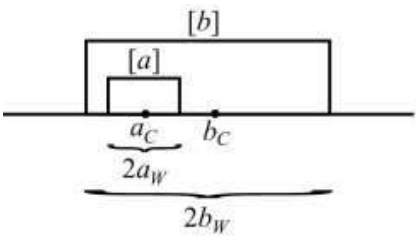


$\Rightarrow [a] \leq_{CW} [b]$,

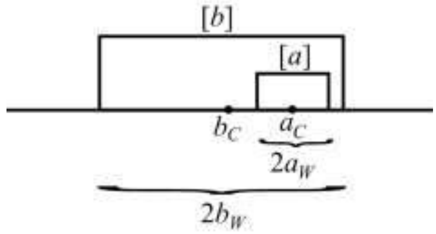


$\Rightarrow [a] \not\leq_{CW} [b]$ and $[b] \not\leq_{CW} [a]$.

Type–III



$\Rightarrow [a] \leq_{CW} [b]$,



$\Rightarrow [a] \not\leq_{CW} [b]$ and $[b] \not\leq_{CW} [a]$.

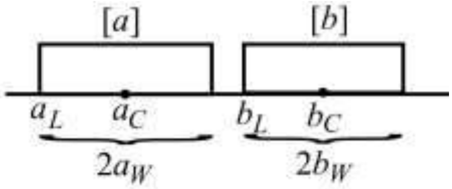
Hence, each type intervals contains partially comparable intervals. In other words "$\leq_{CW}$" is not a total order relation.

6. Let $[a]=[a_L, a_R]=\langle a_C, a_W \rangle$ and $[b]=[b_L, b_R]=\langle b_C, b_W \rangle$ in $I\mathbb{R}$.

The following order relation is mentioned in [12] as being useful for *maximization problems*:

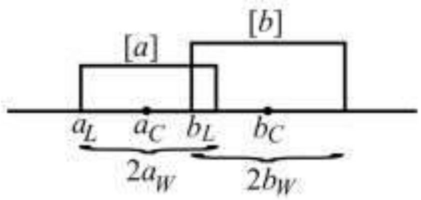
$$[a] \leq_{LC} [b] \Leftrightarrow a_L \leq b_L \text{ and } a_C \leq b_C .$$

Type–I



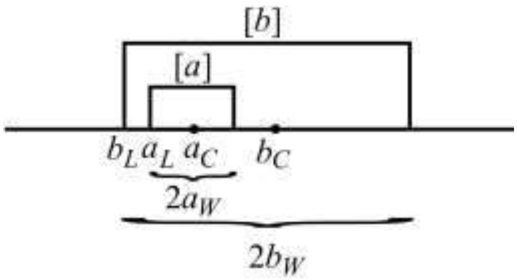
$\Rightarrow [a] \leq_{LC} [b]$.

Type–II

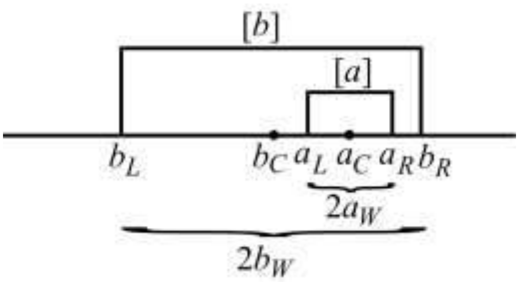


$\Rightarrow [a] \leq_{LC} [b]$,

Type–III



$\Rightarrow [a] \not\leq_{LC} [b]$ and $[b] \not\leq_{LC} [a]$,



$\Rightarrow [b] \leq_{LC} [a]$.

Hence two Type–III intervals are partially comparable. Consequently, "$\leq_{LC}$" is not a total order relation.

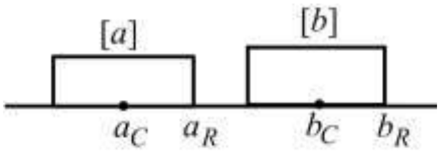
7. In [12], the following order relation is recommend to use for *minimization problems*. Let $[a]$ and $[b]$ be the following intervals:

$$[a]=[a_L, a_R]=\langle a_C, a_W \rangle \text{ and } [b]=[b_L, b_R]=\langle b_C, b_W \rangle .$$

The order relation "$\le_{RC}$" is defined as such:

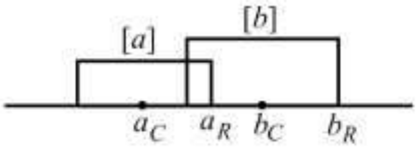
$$[a] \le_{RC} [b] \Leftrightarrow a_R \le b_R \text{ and } a_C \le b_C .$$

Type–I



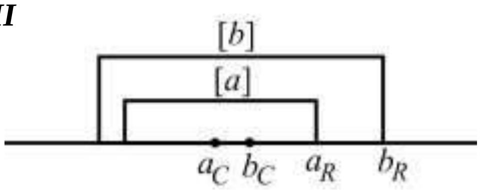
$\Rightarrow [a] \le_{RC} [b]$,

Type–II

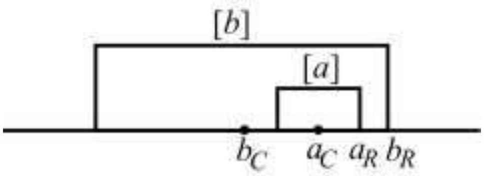


$\Rightarrow [a] \le_{RC} [b]$,

Type–III



$\Rightarrow [a] \le_{RC} [b]$,



$\Rightarrow [a] \not\le_{RC} [b]$ and $[b] \not\le_{RC} [a]$.

For this type, intervals are partially comparable. So, "$\le_{RC}$" is not a total order relation.

5. Looking for a total order relation in the interval-space $I\mathbb{R}$

It is desirable to define an order relation on the interval-space $I\mathbb{R}$ that is closer to a total order relation, if not is even a total order relation. Let $[a] = [\underline{a}, \overline{a}]$ and $[b] = [\underline{b}, \overline{b}]$ be in $I\mathbb{R}$. Obviously, if we put for example $[a] \le [b]$ if and only if $\overline{a} \le \overline{b}$, we will obtain a total reflexive and transitive (but not antisymmetric) relation on $I\mathbb{R}$. But this relation does not distinguish between intervals that have the same endpoint. Thus, this relation, for example, could not be used for *maximization problems of interval profits*.

Dual, if the relation is $[a] \le [b]$ if and only if $\underline{a} \le \underline{b}$, and we try to use this relation, for example, for *minimization problems of interval costs*.

6. New order relations in the interval-space $I\mathbb{R}$

In the previous works, we used the so-called *weak order* in the interval-space $I\mathbb{R}$ as in any arbitrary interval-space IE, too. For example, in $I\mathbb{R}$ this order relation is defined, as follows:

if $[a] = [\underline{a}, \overline{a}]$ and $[b] = [\underline{b}, \overline{b}]$ are two closed intervals, then

$$[a] \le [b] \text{ if } \underline{a} \le \underline{b} \text{ and } \overline{a} \le \overline{b}.$$

But this order relation is not a total order. Indeed, the Type–III intervals $[a] = [\underline{a}, \overline{a}]$ and $[b] = [\underline{b}, \overline{b}]$ with $\underline{a} < \underline{b} < \overline{b} < \overline{a}$ are not comparable.

In the following, we will propose four new similar relations and we will analyze if they are or not (total) order relations.

To define these relations we use only the endpoints of the intervals. Let us consider the following relations for $[a] = [\underline{a}, \overline{a}]$ and $[b] = [\underline{b}, \overline{b}]$ in $I\mathbb{R}$.

$$[a] \le_1 [b] \Leftrightarrow (\overline{a} < \overline{b}) \text{ or } (\overline{a} = \overline{b} \text{ and } \underline{a} = \underline{b}),$$
$$[a] \le_2 [b] \Leftrightarrow (\overline{a} < \overline{b}) \text{ or } (\overline{a} = \overline{b} \text{ and } \underline{a} \le \underline{b}),$$
$$[a] \le_3 [b] \Leftrightarrow (\underline{a} < \underline{b}) \text{ or } (\underline{a} = \underline{b} \text{ and } \overline{a} = \overline{b}),$$
$$[a] \le_4 [b] \Leftrightarrow (\underline{a} < \underline{b}) \text{ or } (\underline{a} = \underline{b} \text{ and } \overline{a} \le \overline{b}).$$

Proposition 1. *All these relations are order relations.*

Proof. We will prove, for example, that "$\le_2$" is an order relation, the checks for the other relations being similar. Hence we consider that

$$[a] \le_2 [b] \text{ if and only if } (\overline{a} < \overline{b}) \text{ or } (\overline{a} = \overline{b} \text{ and } \underline{a} \le \underline{b}).$$

We will prove that "$\le_2$" has the following properties:

(R) *Reflexivity*: $[a] \le_2 [a]$, for each interval $[a]$ in $I\mathbb{R}$;

(AS) *Antisymmetry*: $[a] \le_2 [b]$ and $[b] \le_2 [a]$ imply $[a] = [b]$, where $[a]$ and $[b]$ are arbitrary intervals in $I\mathbb{R}$;

(T) *Transitivity*: $[a] \le_2 [b]$ and $[b] \le_2 [c]$ imply $[a] \le_2 [c]$, where $[a]$, $[b]$ and $[c]$ are arbitrary intervals in $I\mathbb{R}$.

We suppose that $[a] = [\underline{a}, \overline{a}]$, $[b] = [\underline{b}, \overline{b}]$ and $[c] = [\underline{c}, \overline{c}]$ are arbitrary intervals from $I\mathbb{R}$.

Let us prove (R).

$[a] \le_2 [a] \Leftrightarrow (\overline{a} < \overline{a})$ or ($\overline{a} = \overline{a}$ and $\underline{a} \le \underline{a}$) which, obviously, is true.

Let us prove (AS).

$$[a] \le_2 [b] \text{ and } [b] \le_2 [a] \Leftrightarrow$$
$$\left[(\overline{a} < \overline{b}) \text{ or } (\overline{a} = \overline{b} \text{ and } \underline{a} \le \underline{b})\right] \text{and} \left[(\overline{b} < \overline{a}) \text{ or } (\overline{b} = \overline{a} \text{ and } \underline{b} \le \underline{a})\right] \Leftrightarrow$$

(AS.1): $\overline{a} < \overline{b}$ and $\overline{b} < \overline{a}$,

or (AS.2): $\overline{a} < \overline{b}$ and $\overline{b} = \overline{a}$ and $\underline{b} \le \underline{a}$,

or (AS.3): $\overline{a} = \overline{b}$ and $\underline{a} \le \underline{b}$ and $\overline{b} < \overline{a}$,

or (AS.4): $\overline{a} = \overline{b}$ and $\underline{a} \le \underline{b}$ and $\overline{b} = \overline{a}$ and $\underline{b} \le \underline{a}$.

But each of assertions (AS.1), (AS.2) and (AS.3) is contradictory. It remains to verify the assertion (AS.4) which is equivalent with $\underline{a} = \underline{b}$ and $\overline{a} = \overline{b}$, that is, $[a] = [b]$.

Let us prove (T).

Suppose that

$$[a] \le_2 [b] \text{ and } [b] \le_2 [c] \Leftrightarrow$$

$$\left[\left(\overline{a}<\overline{b}\right) \text{ or } \left(\overline{a}=\overline{b} \text{ and } \underline{a}\le\underline{b}\right)\right] \text{ and } \left[\left(\overline{b}<\overline{c}\right) \text{ or } \left(\overline{b}=\overline{c} \text{ and } \underline{b}\le\underline{c}\right)\right]\Leftrightarrow$$

(T.1): $(\overline{a}<\overline{b}$ and $\overline{b}<\overline{c}\,)\Rightarrow\overline{a}<\overline{c}$,

or (T.2): $\overline{a}<\overline{b}$ and $\overline{b}=\overline{c}$ and $\underline{b}\le\underline{c}\Rightarrow\overline{a}<\overline{c}$ and $\underline{b}\le\underline{c}$,

or (T.3): $\overline{a}=\overline{b}$ and $\underline{a}\le\underline{b}$ and $\overline{b}<\overline{c}\Rightarrow\overline{a}<\overline{c}$ and $\underline{a}\le\underline{b}$,

or (T.4): $\overline{a}=\overline{b}$ and $\underline{a}\le\underline{b}$ and $\overline{b}=\overline{c}$ and $\underline{b}\le\underline{c}\Rightarrow\overline{a}=\overline{c}$ and $\underline{a}\le\underline{c}$.

We remark that, for example, if we denote $p:\left(\overline{a}<\overline{c}\right)$, $q_1:\left(\underline{b}\le\underline{c}\right)$, $q_2:\left(\underline{a}\le\underline{b}\right)$ and $q_3:\left(\overline{a}=\overline{c} \text{ and } \underline{a}\le\underline{c}\right)$, then [(T.1) or (T.2) or (T.3) or (T.4)]$\Rightarrow$

$$p\vee\left(p\wedge q_1\right)\vee\left(p\wedge q_2\right)\vee q_3\Rightarrow p\vee q_3,$$

that is,

$$\overline{a}<\overline{c} \text{ or } q_3:\left(\overline{a}=\overline{c} \text{ and } \underline{a}\le\underline{c}\right)\Leftrightarrow[a]\le_2[c]. \qquad \Omega$$

Proposition 2.

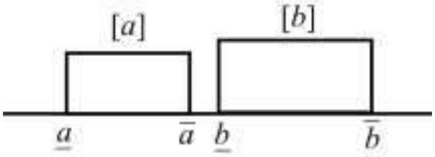
A) *The order relations "$\le_2$" and "$\le_4$" are total order relations.*

B) *The order relations "$\le_1$" and "$\le_3$" are not total order relations.*

Proof. ***A)***

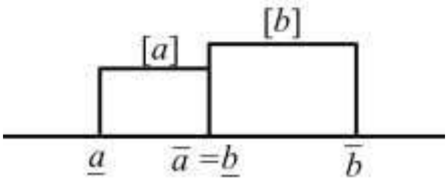
It suffices to prove that the order relation, "$\le_2$" $\left([a]\le_2[b]\Leftrightarrow\left(\overline{a}<\overline{b}\right) \text{ or } \left(\overline{a}=\overline{b} \text{ and } \underline{a}\le\underline{b}\right)\right)$ is a total order relation. That is, for all $[a],[b]\in I\mathbb{R}$ it follows $[a]\le_2[b]$ or $[b]\le_2[a]$.

Type–Ia)



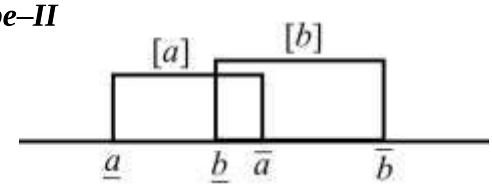
$\Rightarrow\overline{a}<\overline{b}\Rightarrow[a]\le_2[b]$.

Type–Ib)



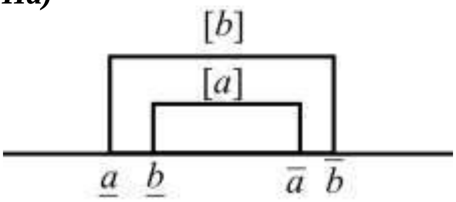
$\Rightarrow\overline{a}<\overline{b}\Rightarrow[a]\le_2[b]$.

Type–II



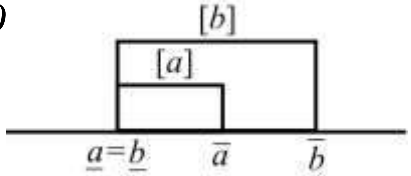
$\Rightarrow\overline{a}<\overline{b}\Rightarrow[a]\le_2[b]$.

Type–IIIa)



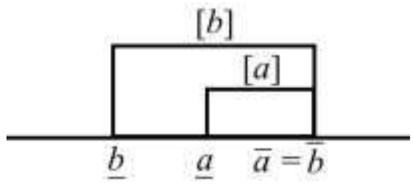
$\Rightarrow\overline{a}<\overline{b}\Rightarrow[a]\le_2[b]$.

Type–IIIb)



$\Rightarrow\overline{a}<\overline{b}\Rightarrow[a]\le_2[b]$.

Type–IIIc)



$\Rightarrow \overline{b} = \overline{a}$ and $\underline{b} < \underline{a} \Rightarrow [b] \leq_2 [a]$.

B)

It suffices to prove that the order relation "$\leq_1$" $\left([a] \leq_1 [b] \Leftrightarrow \left(\overline{a} < \overline{b}\right) \text{ or } \left(\overline{a} = \overline{b} \text{ and } \underline{a} = \underline{b}\right)\right)$ is not total. Analyzing the types of intervals considered in "*A)*", we notice that in the case "*Type–IIIc)*", it follows that $[a] \not\leq_1 [b]$ and $[b] \not\leq_1 [a]$ (since $\overline{a} = \overline{b}$ and $\underline{b} < \underline{a}$). Ω

Remark. Now let us compare the order relations "$\leq_1$" and "$\leq_2$". For this, we refer to the pictures that appear in the proof of Proposition 2, in the case "*A)*". It can be noted that the difference appears only in the situation of the "*Type–IIIc)*" intervals. Indeed, in this case we have $[b] \leq_2 [a]$ but $[a] \not\leq_1 [b]$, $[b] \not\leq_1 [a]$, in all other cases, we have $[a] \leq_2 [b]$ and $[a] \leq_1 [b]$ (since $\overline{a} < \overline{b}$).

This shows that the order relation "$\leq_2$" is "better" than "$\leq_1$".

Similar "$\leq_4$" is "better" than "$\leq_3$".

APPENDIX

Regarding the extension results for i-linear operators, mentioned in the Introduction of this paper

To begin with, we specify that in an arbitrary interval-space IE we can also introduce the *subtraction* operation:

$$[a]!\ [b] = [a] \oplus (-[b]),$$

where $[a], [b] \in IE$ and $-[b]$ means $(-1)[b]$.

Definition 1. (see, for example, [6]) An *interval-subspace* (in short, *i-subspace*) of IE is a nonempty subset IS of IE, closed under the algebraic operations (this means that for any $[a], [b] \in IS$ and $\alpha \in \square$, we have $[a] \oplus [b] \in IS$ and $\alpha [a] \in IS$).

Obviously $\mathbf{0} = [0,0] \in IS$ (because for any $[a] \in IS$, taking $\alpha = 0$, it follows that $\mathbf{0} = 0 \cdot [a] \in IS$). Also the *null set* O of IE, defined by $\text{O} = \{[-b, b] \mid b \geq 0,\, b \in E\}$, is an i-subspace of IE. We can define the *null set* O_{IS} of IS as the set $\text{O} \cap IS$. It follows that $\text{O}_{IS} = \{[a]!\ [a] \mid [a] \in IS\}$. (Indeed, note that for all $[b] \in IS$, we can write $[b] = \left[\frac{1}{2}b\right]!\ \left[\frac{1}{2}b\right]$ and $\left[\frac{1}{2}b\right] \in IS$, because IS is an i-subspace of IE.)

Definition 2. (see, for example, [6]) If IS is like in the previous definition, an *interval-sublinear functional* (in short an *i-sublinear functional*) on IS is a real-valued function $p : IS \to \square$ such that:

1) $p([a]\oplus[b]) \le p([a]) + p([b])$, for all $[a],[b]$ (that is, p is an *i-subadditive functional*);

2) $p(\alpha[a]) = \alpha p([a])$, for all $[a] \in IS$ and $\alpha > 0$ (that is, p is an *i-positively homogeneous functional*);

3) $p([a]\oplus[o]) = p([a])$, for all $[a] \in IS$ and $[o] \in \mathrm{O}_{IS}$.

The following results (see, for example, [6]) generalize in Interval Analysis the well-known Hahn-Banach existence Theorem, the Mazur-Orlicz Theorem and its consequence, the Hahn-Banach extension Theorem, all these for linear functionals.

Theorem 1. *Let IE be an arbitrary i-space and $IS \subseteq IE$ an i-subspace. Let also $s: IS \to \mathbb{R}$ be an i-sublinear functional. Then there exists an i-linear functional $l: IS \to \mathbb{R}$ such that $l([v]) \le s([v])$, for all $[v] \in IS$, that is, $l \le s$ on IS.*

Theorem 2. *Let IE, IS and $s: IS \to \mathbb{R}$ be as in the previous result. Let also A be a nonempty arbitrary set and $f: A \to \mathbb{R}$, $g: A \to IS$ two arbitrary maps. Then the following are equivalent:*

i) there exists an i-linear functional $l: IS \to \mathbb{R}$, such that:

1) $l \le s$ on IS;

2) $f(a) \le l([g(a)])$, for each $a \in A$;

ii) the inequality $\sum_{j=1}^{n} \lambda_j f(a_j) \le s\left(\bigoplus_{j=1}^{n} \lambda_j [g(a_j)]\right)$ holds for all finite subsets $\{a_1, a_2, ..., a_n\}$ in A and $\lambda_1 \ge 0, ..., \lambda_n \ge 0$ in $\mathbb{R}$.

Theorem 3. *Let IE and IS be as in Theorem 1, and $s: IE \to \mathbb{R}$ an i-sublinear functional. Let also $t: IS \to \mathbb{R}$ be an i-linear functional. Then the following are equivalent:*

i) there exists an i-linear functional $l: IE \to \mathbb{R}$, such that:

a) $l \le s$ on IE, and

b) $l = t$ on IS, that is, l is an i-linear extension of t;

ii) $t \le s$ on IS.

The following two results are the operatorial forms of the last two results mentioned above, were F is a Dedekind complete Riesz space.

Theorem 4.(the operatorial form of a Mazur-Orlicz type Theorem in *i-space* setting) *Let IE be an i-space and $S: IE \to F$ an i-sublinear operator. Let also A be a nonempty arbitrary set and $f: A \to F$, $g: A \to IE$ two arbitrary maps. Then the following are equivalent:*

i) there exists an i-linear operator $L: IE \to F$ such that:

a) $L \le S$ on IE;

b) $f(a) \le L([g(a)])$ for each $a \in A$;

ii) the inequality

$$\sum_{i=1}^{n} \lambda_i f(a_i) \le S\left(\bigoplus_{i=1}^{n} \lambda_i [g(a_i)]\right)$$

holds for all finite subsets $\{a_1,...,a_n\}$ *in* A *and* $\lambda_1 \geq 0,...,\lambda_n \geq 0$ *in* $\mathbb{R}$.

(Note that we denote $\left[g(a_i)\right]$ only to remind that actually, $g(a_i) \in IE$ is an interval in E.)

Remark. Note that in the previous result, since A is a nonempty arbitrary set, we can include the case when A is a set of intervals (in an arbitrary ordered vector space). If this is the case, we will write $[a_i]$ instead of a_i and $[a]$ instead of a. If moreover, $A \subseteq IE$, then g can be the inclusion and therefore we will write $[a_i]$, $f([a_i])$ and $f([a])$ instead of $\left[g(a_i)\right]$, $f(a_i)$ and $f(a)$, respectively.

Corollary 5. (Hahn-Banach extension type Theorem in the *i-space* setting)
Let IE *be an i-space,* $IG \subseteq IE$ *an i-subspace and* F *a Dedekind complete Riesz space. Let also* $S: IE \to F$ *an i-sublinear operator and* $T: IG \to F$ *an i-linear operator. Then the following are equivalent:*

i) there exists an i-linear operator $L: IE \to F$ *such that*:

a) $L \leq S$ *on* IE, *and*

b) $L = T$ *on* IG, *that is,* L *is an i-linear extension of T.*

ii) $T \leq S$ *on* IG.

The next step in our approach is to try to replace F with an interval-space IF.

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PROPERTIES OF SOME SERIES OF FUNCTIONS

GHIOCEL GROZA

Abstract. Newton interpolating series in one variable with real coefficients at two points are considered. Extending the case of purely periodic sequence, studied in a previous paper, we compare some properties of these series with those of classical power series.

Mathematics Subject Classification (2010): 40A30, 41A58, 41A10
Key words: polynomial approximation, interpolation, generalized divided differences, Newton interpolating series

1. Introduction

Consider an interval (a,b) and $S=\{\alpha_i\}_{i\geq 0}$ a fixed sequence of elements of (a,b). By using Newton interpolating polynomials we can construct formal series, called Newton interpolating series, such that

$$f=\sum_{i=0}^{\infty} a_i u_i,\ a_i\in\mathbb{R},$$

where

$$u_0=1,\ u_i=\prod_{j=0}^{i-1}(X-\alpha_j),\ i\geq 1.$$

Notice that particular convergent series of this form were used in number theory to prove the transcendence of some values of exponential series (see [8]). Then by means of these series it can be approximated solutions of a multipoint boundary value problem for a linear ordinary differential equation. For example is studied, in [4], the case of purely periodic interpolating sequence $\{\alpha_i\}_{i\geq 0}$, of period m (i.e. for every i, $\alpha_{i+m}=\alpha_i$) and applications in problems which arise in engineering in [3]. The approximation of solutions of variational problems or functional equations are other examples of applications of Newton interpolating series (see [2]).

2. On generalized divided differences

If $\{\alpha_i\}_{i\geq 0}$ is a sequence distinct of elements from an interval $[a,b]$ and $f:[a,b]\to\mathbb{R}$ is a continuous function, then we can define (see, for example, [1]) the divided difference of the order n of f with respect to $\alpha_0,\alpha_1,...,\alpha_n$, denoted by $f[\alpha_0,\alpha_1,...,\alpha_n]$. Thus $f[\alpha_0]:=f(\alpha_0)$,

$$f[\alpha_0,\alpha_1]:=\frac{f(\alpha_1)-f(\alpha_0)}{\alpha_1-\alpha_0},$$

and by recurrence, for $i\geq 1$,

$$f[\alpha_0,...,\alpha_i]:=\frac{f[\alpha_1,...,\alpha_i]-f[\alpha_0,...,\alpha_{i-1}]}{\alpha_i-\alpha_0}.$$

Since

$$f[\alpha_0, ..., \alpha_k] = \sum_{i=0}^{k} \frac{f(\alpha_i)}{\prod\limits_{j=0, j\neq i}^{k} (\alpha_i - \alpha_j)}, \tag{2.1}$$

the divided differences are symmetric functions of their arguments.

In the case of coincident elements by extending the usual notion it follows *generalized divided differences* (see [5], Ch.6, Sec. 1 or [6]. p. 14). Thus suppose that $\alpha_0, \alpha_1, ..., \alpha_m$ are arbitrary elements from $[a, b]$ such that there exist the distinct elements $\bar{\alpha}_0, \bar{\alpha}_1, ...\bar{\alpha}_r \in \{\alpha_0, ..., \alpha_m\}$ and k_i elements α_j are equal to $\bar{\alpha}_i$, for $i = 0, 1, ..., r$, $j = 0, 1, ..., m$, where $k_0 + k_1 + ... + k_r = m + 1$. Then, for $f \in C^m([a, b])$, we define

$$f[\alpha_0, \alpha_1, ..., \alpha_m] := \frac{1}{(k_0 - 1)!(k_1 - 1)!...(k_r - 1)!} \frac{\partial^{k_0+k_1+...+k_r-r-1} f[\bar{\alpha}_0, ..., \bar{\alpha}_r]}{\partial \bar{\alpha}_0^{k_0-1} \partial \bar{\alpha}_1^{k_1-1} ... \partial \bar{\alpha}_r^{k_r-1}}. \tag{2.2}$$

Here, for simplicity, we denoted $\frac{\partial^{k_0+k_1+...+k_r-r-1} f[\bar{\alpha}_0,...,\bar{\alpha}_r]}{\partial \bar{\alpha}_0^{k_0-1} \partial \bar{\alpha}_1^{k_1-1} ... \partial \bar{\alpha}_r^{k_r-1}}$, the derivative $\frac{\partial^{k_0+k_1+...+k_r-r-1} f[x_0,...,x_r]}{\partial x_0^{k_0-1} \partial x_1^{k_1-1} ... \partial x_r^{k_r-1}}$, computed for the distinct variables x_i at the point $(x_0, ..., x_r) = (\bar{\alpha}_0, \bar{\alpha}_1, ..., \bar{\alpha}_r)$.

Remark 2.1. By (2.1), for distinct elements $\bar{\alpha}_0, \bar{\alpha}_1, ...\bar{\alpha}_r$, divided differences $f[\bar{\alpha}_0, ..., \bar{\alpha}_r] = \sum\limits_{i=0}^{r} f(\bar{\alpha}_i)$ $R_i(\bar{\alpha}_0, ..., \bar{\alpha}_r)$, where R_i are rational functions of $\bar{\alpha}_0, \bar{\alpha}_1, ...\bar{\alpha}_r$. Hence we get that the partial derivatives of $f[\bar{\alpha}_0, ..., \bar{\alpha}_r]$ commute and the divided differences are well defined by (2.2). In this case we denote

$$f[\alpha_0, \alpha_1, ..., \alpha_m] = f[\bar{\alpha}_{0,k_0}, \bar{\alpha}_{1,k_1}, ..., \bar{\alpha}_{r,k_r}] := f[\underbrace{\bar{\alpha}_0, \bar{\alpha}_0, ..., \bar{\alpha}_0}_{k_0\ times}, \underbrace{\bar{\alpha}_1, ..., \bar{\alpha}_1}_{k_1\ times}, ..., \underbrace{\bar{\alpha}_r, ..., \bar{\alpha}_r}_{k_r\ times}].$$

3. Main results

Denote $(n_0, n_1, ..., n_{m-1}) \in \mathbb{N}^m$ by the multi-index $\mathbf{n}$ and $|\mathbf{n}| = n_0 + ... + n_{m-1}$. Let $S = \{\alpha_n\}_{n \geq 0}$ be a sequence of elements from an interval (a, b) such that only a finite number of them, denoted by $\bar{\alpha}_0, ..., \bar{\alpha}_{m-1}$, are distinct. Consider the set monomials, with respect to the variables $X - \bar{\alpha}_j$,

$$M_S = \left\{ w_{\mathbf{n}} = \prod_{j=0}^{m-1} (X - \bar{\alpha}_j)^{n_j}, \mathbf{n} \in \mathbb{N}^m \right\}$$

and the sequence of monomials

$$u_0 = 1,\ u_i = \prod_{j=0}^{i-1} (X - \alpha_j),\ i \geq 1.$$

Then $u_i = w_{\mathbf{i}}$, where $\mathbf{i} = (i_0, ..., i_{m-1})$, i_k, $k = 0, ..., m-1$ is the number of the terms $\alpha_0, ..., \alpha_{i-1}$ equals to $\bar{\alpha}_k$ and $i = |\mathbf{i}|$. Hence it follows that

$$u_i = \prod_{k=0}^{m-1} (X - \bar{\alpha}_k)^{i_k},\ i \geq 1.$$

Throughout this paper we'll consider $m = 2$.

Lemma 3.1. *If there exist a positive integer d such that, for every $j = 0, 1$, there are at most d consecutive terms of S equal to $\bar{\alpha}_j$, then for every $i \geq 1$,*

$$u_i' = \sum_{s=1}^{d+1} p_{i,s} u_{i-s}, \tag{3.1}$$

where $p_{i,s}$ are polynomials of degree less or equal to $s-1$ in $\bar{\alpha}_0, \bar{\alpha}_1$, with integer coefficients having absolute value less or equal to i. Here, in (3.1), if $j < 0$, we put $u_j = 0$.

Proof. Since $u_i = (X-\bar{\alpha}_0)^{i_0}(X-\bar{\alpha}_1)^{i_1} = w_{(i_0,i_1)}$, where $i = |\mathbf{i}| = i_0 + i_1$, we get

$$u_i' = (X-\bar{\alpha}_0)^{i_0-1}(X-\bar{\alpha}_1)^{i_1-1}(i_0(X-\bar{\alpha}_1)+i_1(X-\bar{\alpha}_0)) = w_{(i_0-1,i_1-1)}(iX - i_0\bar{\alpha}_1 - i_1\bar{\alpha}_0).$$

Hence we obtain

$$\|u_i\|_{[\bar{\alpha}_0,\bar{\alpha}_1]} = \left|u_i\left(x_i^{(M)}\right)\right|,$$

where $x_i^{(M)} = \frac{i_0\bar{\alpha}_1 + i_1\bar{\alpha}_0}{i}$ and $\|\cdot\|_{[a.b]}$ is the sup norm on $C([a,b])$.

There are two cases to consider. (A) $(i-1)_1 = i_1$. (B) $(i-1)_0 = i_0$. In case A denote $t = \max\{s : (i-r)_1 = i_1, r = 1,2,...,s\}$. Then $u_{i-1} = w_{(i_0-1,i_1)}$, $(i-s)_0 = i_0 - s, s = 1,2,...,t$ and $t \le d$. Since

$$w_{(i_0-s+1,i_1-1)} = w_{(i_0-s,i_1)} + (\bar{\alpha}_1 - \bar{\alpha}_0)w_{(i_0-s,i_1-1)}$$

and

$$u_{i-s} = w_{(i_0-s.i_1)},$$

we get

$$\begin{aligned} u_i' &= iu_{i-1} + c_1 w_{(i_0-1,i_1-1)} \\ w_{(i_0-1,i_1-1)} &= u_{i-2} + c_2 w_{(i_0-2,i_1-1)}, \\ w_{(i_0-2,i_1-1)} &= u_{i-3} + c_3 w_{(i_0-3,i_1-1)}, \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ w_{(i_0-t+1,i_1-1)} &= u_{i-t} + c_t w_{(i_0-t,i_1-1)}, \\ w_{(i_0-t,i_1-1)} &= u_{i-t-1}, \end{aligned}$$

where $c_1 = i_1(\bar{\alpha}_1 - \bar{\alpha}_0)$, $c_r = (\bar{\alpha}_1 - \bar{\alpha}_0), r = 2,...,t$. Hence it follows that u_i' has the form (3.1) in this case.

In case B the proof is the same. $\square$

Under the hypotheses of Lemma 3.1, consider a series of the form

(3.2) $$\sum_{i=0}^{\infty} a_i u_i,\ a_i \in \mathbb{R},$$

where

$$u_i = (X-\bar{\alpha}_0)^{i_0}(X-\bar{\alpha}_1)^{i_1},\ \bar{\alpha}_0 < \bar{\alpha}_1$$

(i.e. a *Newton interpolating series in one variable with real coefficients at* α_i, $i = 0,1,...$, with $\alpha_i \in \{\bar{\alpha}_0, \bar{\alpha}_1\}$).

Denote

(3.3) $$\rho := \limsup_{i\to\infty}\left|a_i u_i\left(x_i^{(M)}\right)\right|^{\frac{1}{i}}.$$

Lemma 3.2. *Under the hypotheses of Lemma 3.1, there exists* $\varepsilon > 0$ *such that, for every* $i > d$,

(3.4) $$\max\{|u_i(\bar{\alpha}_0 - \varepsilon)|, |u_i(\bar{\alpha}_1 + \varepsilon)|\} < \left|u_i\left(x_i^{(M)}\right)\right|.$$

Proof. Choose $\varepsilon = \frac{\bar{\alpha}_1 - \bar{\alpha}_0}{t}$, where t is a positive constant. Since

$$|u_i(\bar{\alpha}_0 - \varepsilon)| = \varepsilon^{i_0}(\bar{\alpha}_1 - \bar{\alpha}_0 + \varepsilon)^{i_1} = \left(\frac{\bar{\alpha}_1 - \bar{\alpha}_0}{t}\right)^i (t+1)^{i_1},$$

$$|u_i(\bar{\alpha}_1 + \varepsilon)| = \varepsilon^{i_1}(\bar{\alpha}_1 - \bar{\alpha}_0 + \varepsilon)^{i_0} = \left(\frac{\bar{\alpha}_1 - \bar{\alpha}_0}{t}\right)^i (t+1)^{i_0}$$

and

(3.5) $$\left|u_i\left(x_i^{(M)}\right)\right| = \left(\frac{\bar{\alpha}_1 - \bar{\alpha}_0}{i}\right)^i i_0^{i_0} i_1^{i_1},$$

we'll find $t \geq 1$ such that, for all $i > d$,

$$\max\left\{\frac{(t+1)^{i_1}}{t^i}, \frac{(t+1)^{i_0}}{t^i}\right\} < \frac{i_0^{i_0} i_1^{i_1}}{i^i}. \tag{3.6}$$

Since, for $j = 0, 1$, there at most d consecutive terms of S equal to $\bar{\alpha}_j$ and $i = i_0 + i_1 = |\mathbf{i}|$, it follows that

$$\frac{i_0}{d} - 1 \leq i_1 \leq d(i_0 + 1).$$

If $i_0 \leq i_1$, then (3.6) becomes

$$\frac{(t+1)^{i_1}}{t^i} < \frac{i_0^{i_0} i_1^{i_1}}{i^i}$$

or

$$\left(1 + \frac{1}{t}\right)^{\frac{i_1}{i_0}} < t \cdot \frac{1}{1 + \frac{i_1}{i_0}} \cdot \frac{1}{(1 + \frac{i_0}{i_1})^{\frac{i_1}{i_0}}}. \tag{3.7}$$

Since $i > d$ we get $i_0 i_1 \neq 0$ and $\frac{i_0}{i_1}, \frac{i_1}{i_0} \in (0, 2d]$. Then we obtain

$$\left(1 + \frac{1}{t}\right)^{\frac{i_1}{i_0}} \leq \left(1 + \frac{1}{t}\right)^d, \ t \cdot \frac{1}{1 + \frac{i_1}{i_0}} \cdot \frac{1}{(1 + \frac{i_0}{i_1})^{\frac{i_1}{i_0}}} \geq t \cdot \frac{1}{1 + 2d} \cdot \frac{1}{(2d)^{2d}}.$$

Hence (3.7) holds for t large enough and it follows (3.6) in this case.

In case $i_1 < i_0$ the proof is the same. □

Define the *(formal) derivative series* of (3.2) as the series obtained by its termwise differentiation and by reordering the terms in a suitable form to obtain a series of the same type. Thus we get

$$\sum_{i=0}^{\infty} a_i u_i' = \sum_{i=0}^{\infty} a_i^{(1)} u_i, \tag{3.8}$$

where $a_i^{(1)} = \sum_{k=1}^{d+1} a_{i+k} p_{i+k,k}$.

Theorem 3.3. *Under the hypotheses of Lemma 3.1 suppose that $\rho < 1$. Then, by choosing $\varepsilon > 0$ such that* (3.4) *holds, the following statements are true:*

(i) The series (3.2) *converges absolutely on $(\bar{\alpha}_0 - \varepsilon, \bar{\alpha}_1 + \varepsilon)$ and uniformly on every $[a, b] \subset (\bar{\alpha}_0 - \varepsilon, \bar{\alpha}_1 + \varepsilon)$.*

(ii) The derivative series (3.8) *converges absolutely on $(\bar{\alpha}_0 - \varepsilon, \bar{\alpha}_1 + \varepsilon)$ and uniformly on every $[a, b] \subset (\bar{\alpha}_0 - \varepsilon, \bar{\alpha}_1 + \varepsilon)$.*

(iii) If $f(x)$ is the sum of the series (3.2)*, for $x \in (\bar{\alpha}_0 - \varepsilon, \bar{\alpha}_1 + \varepsilon)$, then f is differentiable on $(\bar{\alpha}_0 - \varepsilon, \bar{\alpha}_1 + \varepsilon)$ and $f'(x)$ is the sum of the series* (3.8).

Proof. (i) Since, for every $x \in [a, b] \subset (\bar{\alpha}_0 - \varepsilon, \bar{\alpha}_1 + \varepsilon)$, by (3.3) and (3.4), there exists a positive constant $\rho_1 < 1$ such that, for all i large enough, $|a_i u_i(x)| \leq \rho_1^i$, (i) follows by Weierstrass M-test.

(ii) Consider $x \in (\bar{\alpha}_0 - \varepsilon, \bar{\alpha}_1 + \varepsilon)$. Then, for every fixed k, by (3.3) and (3.5), we get, for all i large enough,

$$|a_{i+k} u_i(x)| = |a_{i+k} u_{i+k}(x_{i+k}^{(M)})| \left|\frac{u_i(x)}{u_i(x_i^{(M)})}\right| \left|\frac{u_i(x_i^{(M)})}{u_{i+k}(x_{i+k}^{(M)})}\right| < \rho_1^{i+k} \left(\frac{\bar{\alpha}_1 - \bar{\alpha}_0}{i}\right)^i i_0^{i_0} i_1^{i_1}$$

$$\cdot \left(\frac{i+k}{\bar{\alpha}_1 - \bar{\alpha}_0}\right)^{i+k} \cdot \frac{1}{(i+k)_0^{(i+k)_0} (i+k)_1^{(i+k)_1}} = \rho_1^{i+k} \cdot \frac{1}{(\bar{\alpha}_1 - \bar{\alpha}_0)^k} \left(1 + \frac{k}{i}\right)^i (i+k)^k$$

$$\cdot \frac{i_0^{i_0}}{(i+k)_0^{(i+k)_0}} \cdot \frac{i_1^{i_1}}{(i+k)_1^{(i+k)_1}} < C i^k \rho_1^i,$$

where C is positive constants independent of i and $\rho_1 < 1$. Hence and by Lemma 3.1 it follows (ii).

(iii) By (i), (ii) and Theorem 7.17, from [7], it follows (iii). □

We say that a function $f : (c, d) \to \mathbb{R}$ *can be represented as Newton interpolating series at* $\bar{\alpha}_0, \bar{\alpha}_1 \in (c, d)$, if there exists a series of the form (3.2) which converges uniformly and absolutely to f on every $[a, b] \subset (c, d)$. By Theorem 3.3, if $\rho < 1$, f is an infinitely differentiable function. In this case the partial sums $S_i(x)$ of the series (3.2) define a sequence of polynomial functions which approximate uniformly f and $S_i^{(k_r)}(\bar{\alpha}_r) = f^{(k_r)}(\bar{\alpha}_r)$, $k_r = 0, 1, ..., i_r$, $r = 0, 1$. Moreover, the coefficients a_i of the series (3.2) are computed by means of generalized divided differences as

$$a_i = f[\bar{\alpha}_{0,i_0}, \bar{\alpha}_{1,i_1}].$$

Example 3.4. Notice that there are continuous functions $f : [a, b] \to \mathbb{R}$ which is the sum of a Newton interpolating series at $\{\alpha_i\}_{i \geq 0}$, but they are not differentiable at some points of $[a, b]$. For example, we consider $\bar{\alpha}_0 = -1$, $\bar{\alpha}_0 = 1$, $m = 2$, and $f(x) = |x|$. Then, for $x \in [-1, 1]$,

$$f(x) = \sqrt{1 - (1 - x^2)} = 1 + \sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{2^{2i-1} i} \binom{2i-2}{i-1} (x-1)^i (x+1)^i,$$

but f is not differentiable at $x = 0$. Consider the function $g : [-1, 1] \to \mathbb{R}$ defined by

$$g(x) = \begin{cases} 0, & \text{if } x \in [-1, 0] \\ x, & \text{if } x \in (0, 1]. \end{cases}$$

Then

$$g(x) = \frac{x + f(x)}{2} = \frac{1 + x}{2} + \sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{2^{2i} i} \binom{2i-2}{i-1} (x-1)^i (x+1)^i.$$

Hence it follows that $g(x)$ is a sum of a Newton interpolating series whose set of zeros is not a discrete set.

Motivated by Example 3.4, we call a function $f : (\bar{\alpha}_0 - \varepsilon, \bar{\alpha}_1 + \varepsilon) \to \mathbb{R}$, $\bar{\alpha}_0 < \bar{\alpha}_1$ a *Newton analytic function* if for every $\beta_0, \beta_1 \in (\bar{\alpha}_0 - \varepsilon, \bar{\alpha}_1 + \varepsilon)$, with $\beta_0 < \beta_1$, $f(x)$ is the sum of a Newton interpolating series (3.2), where

$$u_i = (X - \beta_0)^{i_0} (X - \beta_1)^{i_1},$$

on $[\beta_0, \beta_1]$ and its $\rho < 1$.

Proposition 3.5. *Let f be the sum of a Newton series* (3.2) *satisfying the hypotheses of Theorem 3.3. Then f is a Newton analytic function on $(\bar{\alpha}_0 - \varepsilon, \bar{\alpha}_1 + \varepsilon)$.*

Proof. Consider the linear function h defined by $h(y) = Ay + B$, $A, B \in \mathbb{R}$. If $\beta_0, \beta_1 \in (\bar{\alpha}_0 - \varepsilon, \bar{\alpha}_1 + \varepsilon)$, $\beta_0 < \beta_1$, we determine A, B such that $h(\beta_i) = \bar{\alpha}_i$, $i = 0, 1$. Hence $A = \frac{\bar{\alpha}_1 - \bar{\alpha}_0}{\beta_1 - \beta_0}$ and $B = \frac{\bar{\alpha}_0 \beta_1 - \bar{\alpha}_1 \beta_0}{\beta_1 - \beta_0}$. If $x = Ay + B$, then

$$f(x) = \sum_{i=0}^{\infty} a_i u_i (Ay + B) = \sum_{i=0}^{\infty} b_i v_i(y),$$

where $b_i = a_i A^i$ and $v_i = (Y - \beta_0)^{i_0} (Y - \beta_1)^{i_1}$. Then, by (3.3) and (3.5), we get

$$\tilde{\rho} = \limsup_{i \to \infty} \left| b_i v_i \left(y_i^{(M)} \right) \right|^{\frac{1}{i}} = \rho.$$

Now the proposition follows by Theorem 3.3. □

Theorem 3.6. *If $f : [\bar{\alpha}_0, \bar{\alpha}_1] \to \mathbb{R}$ is a Newton analytic function, then the set of its zeros is discrete.*

Proof. To prove the theorem, suppose the contrary, that there exist $\tilde{x}$ a zero of f and $\{x_j\}_{j\geq 0}$ distinct zeros of f such that $\tilde{x} = \lim_{j\to\infty} x_j$. If, for example $x_1 > \tilde{x}$, choose $\bar{\alpha}_0 = \tilde{x}$ and $\bar{\alpha}_1 = x_1$. Then there exists k such that $a_k \neq 0$ and

$$f(x) = \sum_{i=k}^{\infty} a_i u_i(x) = u_k(x)(Q_k(x) + a_{r+1}(x-\tilde{x})(x-x_1)^{r-k} + ...), \tag{3.9}$$

where $Q_k(x) = a_k + a_{k+1}(x-x_1) + ... + a_r(x-x_1)^{r-k}$.

There are two cases to consider. (A) $K = |Q_k(\tilde{x})| \neq 0$. (B) $K = |Q_k(\tilde{x})| = 0$. In case A, since $\tilde{x} = \lim_{j\to\infty} x_j$ we may choose x_j, $j > 1$, such that $|Q_k(x_j) + a_{r+1}(x_j - \tilde{x})(x_j - x_1)^{r-k} + ...| \geq K/2$. This would contradict (3.9) because $f(x_j) = 0$.

In case B, let s be the multiplicity of zero $\tilde{x}$ of f and $t = \min\{i : i_0 = s\}$. Then, if $i > t$ we denote

$$v_i = \begin{cases} u_0 & \text{if } i = 0 \\ (X-\tilde{x})^i & \text{if } i \in \{1, ..., s-1\} \\ (X-\tilde{x})^{s-1}(X-x_1)^{i-s+1} & \text{if } i \in \{s, ..., t-1\} \\ u_i & \text{if } i \geq t. \end{cases}$$

Hence

$$f(x) = \sum_{i=0}^{\infty} b_i v_i,$$

where $b_i = a_i$, for all $i \geq t$. Thus we reduced the problem to case A. Hence it follows the theorem. □

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Department of Mathematics and Computer Science, Technical University of Civil Engineering Bucharest, Bucharest, Romania

Email address: grozagc@yahoo.com

THE ROOTS OF RELIABILITY POLYNOMIALS FOR SERIES – PARALLEL NETWORKS

MARILENA JIANU AND LEONARD DĂUŞ

Abstract. The roots of the reliability polynomials became a topic of great interest during the last years, but the research concentrated mostly on roots of all-terminal networks, the roots of two-terminal networks receiving far less attention. This paper analyzes the properties of the roots of the reliability polynomials associated to a special type of two-terminal networks, namely the networks constructed by series and parallel compositions.

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Key words: Two-terminal reliability polynomial, series and parallel networks, polynomial roots.

1. Introduction. The reliability polynomial of a network

A network is a probabilistic graph, $\mathbf{N} = (V, E)$, where V is the set of nodes (vertices) and E is the set of (undirected) edges [9]. Each edge (representing a "device") is independently operational (closed) with probability $p \in [0, 1]$ and fails with probability $q = 1 - p$ (the vertices are supposed to be "perfect": they do not fail). The probability that every pair of nodes is connected by a path (a sequence of adjacent edges) made of operational edges is called *all-terminal reliability*. Given two special nodes S and T (called *terminals*), the *two-terminal reliability* is the probability that these two nodes are connected by a path of operational edges.

The reliability of a two-terminal network $\mathbf{N}$ of size n is given by the polynomial

$$\text{Rel}(\mathbf{N}; p) = \sum_{k=0}^{n} N_k(\mathbf{N})\, p^k (1-p)^{n-k}, \tag{1.1}$$

where the coefficients $N_k(\mathbf{N})$ are integers representing the number of sets of k edges that contain at least one path connecting S to T (k-pathsets) [9, 19].

Finding the exact expression of the reliability polynomial is a highly demanding computational task, belonging to the class of $\#P$ complete problems [9]. This is why the research in this area focused on two main directions: the first one is to find practical algorithms for bounding or approximating the reliability polynomials [1, 10, 11, 12, 17, 18], while the second direction is related to the analytical properties of reliability polynomials, such as convexity, shape properties and inflection points [3, 6, 13, 16], or the location of the roots in the complex plane [2, 4, 5, 7, 8]. The roots of all-terminal reliability polynomials were conjectured to belong to the unit disk centered at $z = 1$ in the complex plane [2]. This

"Brown-Colbourn conjecture" was proved to be false by Royle and Sokal [20], who found roots outside the disk $|z-1| \leq 1$ (by a slim margin only).

For two-terminal reliability polynomials, it is also an open question if the set of roots is bounded. The results reported by Tanguy for several types of two-terminal networks show that the roots are lying in a rectangular region of the complex plane [21]–[23]. Brown and DeGagné [8] prove that the closure of the roots of reliability polynomials of two-terminal networks contains the unit disks centered at the origin and at the point $z = 1$, respectively. In this paper we focus on the series-parallel networks formed by binary composition. An algorithm for computing the set of complex roots of this kind of networks is developed in [15]. We prove that the closure of these roots also contains the two closed disks

$$\mathcal{D}_0 = \{z \in \mathbb{C} : |z| \leq 1\} \text{ and } \mathcal{D}_1 = \{z \in \mathbb{C} : |z-1| \leq 1\}.$$

2. Networks constructed by series-parallel compositions

If $\mathbf{N}_1$ and $\mathbf{N}_2$ are two networks, then the composition of $\mathbf{N}_1$ and $\mathbf{N}_2$, denoted $\mathbf{N}_1 \bullet \mathbf{N}_2$, is obtained by replacing each device (edge) in $\mathbf{N}_1$ by a copy of $\mathbf{N}_2$, such that the terminals S, T of $\mathbf{N}_2$ replace the endpoints of the edge in $\mathbf{N}_1$. The reliability polynomial of the composed network $\mathbf{N}_1 \bullet \mathbf{N}_2$ is given by the formula

$$\operatorname{Rel}(\mathbf{N}_1 \bullet \mathbf{N}_2; p) = \operatorname{Rel}(\mathbf{N}_1; \operatorname{Rel}(\mathbf{N}_2; p)) = \operatorname{Rel}(\mathbf{N}_1; p) \circ \operatorname{Rel}(\mathbf{N}_2; p) \tag{2.1}$$

Let $\mathbf{C}^{(0)}$ be the network formed by two devices in series, $\mathbf{C}^{(1)}$ the network formed by two devices in parallel. We can generalize compositions of $\mathbf{C}^{(0)}$ and $\mathbf{C}^{(1)}$ to any m-length binary vector $\boldsymbol{u} = (u_1, \ldots, u_m) \in \{0,1\}^m$ as

$$\mathbf{C}^{\boldsymbol{u}} = \mathbf{C}^{(u_1)} \bullet \cdots \bullet \mathbf{C}^{(u_m)}. \tag{2.2}$$

For any positive integer m, we denote by $\mathcal{C}_m$ the set of all 2^m-sized networks formed by compositions of $\mathbf{C}^{(0)}$ and $\mathbf{C}^{(1)}$ and let $\mathcal{C} = \bigcup_{m\geq 1} \mathcal{C}_m$ be the set of all such networks.

The reliability polynomials of the elementary networks $\mathbf{C}^{(0)}$ and $\mathbf{C}^{(1)}$ are given by the formulas:

$$\operatorname{Rel}\left(\mathbf{C}^{(0)}; p\right) = p^2 = f_0(p), \tag{2.3}$$

$$\operatorname{Rel}\left(\mathbf{C}^{(1)}; p\right) = 1 - (1-p)^2 = f_1(p), \tag{2.4}$$

so the reliability polynomial of the network (2.2) is

$$\operatorname{Rel}(\mathbf{C}^{\boldsymbol{u}}; p) = f_{u_1} \circ f_{u_2} \circ \ldots \circ f_{u_m}(p). \tag{2.5}$$

For instance, the reliability polynomial of the network $\mathbf{C}^{(0,1,0)}$ presented in Fig. 1 is

$$\begin{aligned} \operatorname{Rel}\left(\mathbf{C}^{(0,1,0)}; p\right) &= f_0\left(f_1\left(f_0(p)\right)\right) \\ &= \left[1 - (1-p^2)^2\right]^2 \end{aligned}$$

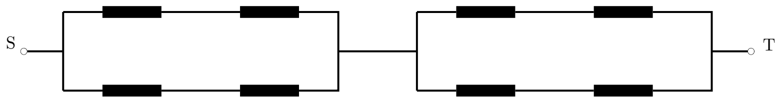

FIGURE 1. The network $\mathbf{C}^{(0,1,0)}$

It can be easily seen that

$$\underbrace{f_0 \circ \ldots \circ f_0}_{k}(p) = p^{2^k},$$

$$\underbrace{f_1 \circ \ldots \circ f_1}_{k}(p) = 1 - (1-p)^{2^k}.$$

It follows that, for each string $\boldsymbol{u}_{m,n} \in \{0,1\}^{m+n}$ of the form $\boldsymbol{u}_{m,n} = (\underbrace{1,\ldots,1}_{m},\underbrace{0,\ldots,0}_{n})$, the reliability polynomial of the network $\mathbf{C}^{\boldsymbol{u}_{m,n}}$ is

$$\operatorname{Rel}\left(\mathbf{C}^{\boldsymbol{u}_{m,n}};p\right) = 1 - \left(1 - p^{2^n}\right)^{2^m} =: P_{m,n}(p). \tag{2.6}$$

Lemma 2.1. *For any $m, n > 0$, the nonzero roots of the polynomial (2.6) are placed on $2^m - 1$ circles centered at the origin. Each circle has the radius less than or equal to $2^{2^{-n}}$ and contains exactly 2^n roots, equally spaced. The roots are given by the formula*

$$\alpha_{k,j}^{m,n} = \left(2\sin\frac{k\pi}{2^m}\right)^{2^{-n}} \exp\left[i\pi\left(\frac{k}{2^{m+n}} + \frac{4j-1}{2^{n+1}}\right)\right], \tag{2.7}$$

$k = 0, 1, \ldots, 2^m - 1$ *and* $j = 0, 1, \ldots, 2^n - 1$.

Proof. From (2.6), the roots of the polynomial $P_{m,n}(p)$ have to satisfy

$$\left(1 - p^{2^n}\right)^{2^m} = 1,$$

hence

$$1 - p^{2^n} = \cos\frac{2k\pi}{2^m} + i\sin\frac{2k\pi}{2^m}, \ k = 0, 1, \ldots, 2^m - 1.$$

It follows that

$$p^{2^n} = 2\sin^2\frac{k\pi}{2^m} - 2i\sin\frac{k\pi}{2^m}\cos\frac{k\pi}{2^m},$$

so

$$p^{2^n} = 2\sin\frac{k\pi}{2^m}\left[\cos\left(\frac{k\pi}{2^m} - \frac{\pi}{2}\right) + i\sin\left(\frac{k\pi}{2^m} - \frac{\pi}{2}\right)\right]. \tag{2.8}$$

For $k = 0$ we obtain that 0 is a multiple root of multiplicity 2^n. For each $k = 1, 2, \ldots, 2^m - 1$, the roots of the equation (2.8) are 2^n equidistant points on the circle of radius $r_k^{m,n}$,

$$r_k^{m,n} = \left(2\sin\frac{k\pi}{2^m}\right)^{2^{-n}} \leq 2^{2^{-n}} \tag{2.9}$$

and the formula (2.7) follows immediately. □

3. The main result

Theorem 3.1. *The closure of the set of roots of the reliability polynomials of all the networks in* $\mathcal{C}$ *contains the unit disk centered at the origin,* $\mathcal{D}_0 : |z| \leq 1$.

Proof. Let $z_0 = r(\cos\theta + i\sin\theta) \neq 0$ be a point inside the disk $|z| < 1$ and let $\varepsilon > 0$ be a small positive number such that $\varepsilon < r$ and $\varepsilon < 1 - r$. We shall prove that there exist $m, n \in \mathbb{N}$ such that at least one root $\alpha_{k,j}^{m,n}$ of the reliability polynomial of the network $\mathbf{C}^{\boldsymbol{u}_{m,n}}$ satisfies $\left|\alpha_{k,j}^{m,n} - z_0\right| < \varepsilon$.

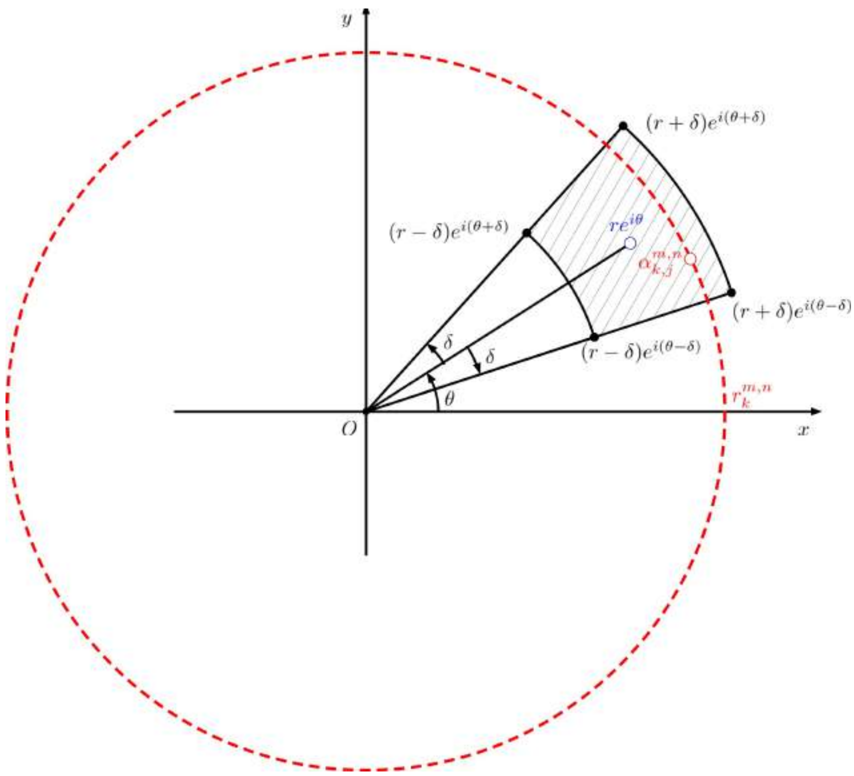

Figure 2

Consider the region $D_{z_0,\varepsilon}$ of the complex plane defined by

$$D_{z_0,\varepsilon} = \{\rho \exp(it) : r - \delta < \rho < r + \delta,\ \theta - \delta < t < \theta + \delta\},$$

where $\delta = \frac{\varepsilon}{2}$ (see Figure 2). Notice that the region $D_{z_0,\varepsilon}$ is inside the unit disk centered at the origin. We show that $|z - z_0| < \varepsilon$, for every $z \in D_{z_0,\varepsilon}$. It easy to see that the maximum distance to z_0 is attained for the points $(r + \delta)\exp(i(\theta \pm \delta))$, so we can write, for any $z \in D_{z_0,\varepsilon}$:

$$\begin{aligned}|z - z_0| &< |(r+\delta)\exp(i(\theta+\delta)) - r\exp(i\theta)| \\ &= |r\exp(i\theta)\,(\exp(i\delta)-1) + \delta\exp(i(\theta+\delta))| \\ &\le r\,|\exp(i\delta)-1| + \delta = r\sqrt{(\cos\delta-1)^2+\sin^2\delta} + \delta \\ &= 2r\sin\frac{\delta}{2} + \delta < \delta(r+1) < \varepsilon.\end{aligned}$$

Consider $n \in \mathbb{N}$ sufficiently large such that the following inequalities hold true:

$$\frac{\pi}{2^n} < \delta \tag{3.1}$$

and

$$\left(\frac{r-\delta}{r+\delta}\right)^{2^n} < \frac{2}{\pi}. \tag{3.2}$$

We prove that there exist $m \in \mathbb{N}$ and $k \in \{1, 2, \ldots, 2^{m-1}\}$ such that the circle (centered at the origin) of radius $r_k^{m,n}$ crosses the region $D_{z_0,\varepsilon}$, that is,

$$r - \delta < r_k^{m,n} < r + \delta. \tag{3.3}$$

Using the formula (2.9) the inequalities (3.3) can be written as:

$$(r-\delta)^{2^n} < 2\sin\frac{k\pi}{2^m} < (r+\delta)^{2^n}.$$

Since

$$\frac{2}{\pi}x \le \sin x \le x, \quad \text{for all } x \in \left[0, \frac{\pi}{2}\right], \tag{3.4}$$

we have

$$\frac{k}{2^{m-1}} \le \sin\frac{k\pi}{2^m} \le \frac{k\pi}{2^m},$$

for every $k = 0, 1, \ldots, 2^{m-1}$. Hence it is sufficient that the following inequalities are satisfied:

$$\frac{1}{2}(r-\delta)^{2^n} < \frac{k}{2^{m-1}} < \frac{1}{\pi}(r+\delta)^{2^n} \tag{3.5}$$

Note that $\frac{1}{2}(r-\delta)^{2^n} < \frac{1}{\pi}(r+\delta)^{2^n}$ by (3.2). By taking m sufficiently large such that

$$\frac{1}{2^{m-1}} < \frac{1}{\pi}(r+\delta)^{2^n} - \frac{1}{2}(r-\delta)^{2^n},$$

it follows that there exists at least one integer $k \in \{1, 2, \ldots, 2^{m-1}\}$ such that the inequalities (3.5) hold true.

We have shown that the circle of radius $r_k^{m,n}$ crosses the region $D_{z_0,\varepsilon}$ and we know that this circle contains 2^n equally spaced roots of the reliability polynomial $P_{m,n}$. Let $\overset{\frown}{AB}$ be the intersection between the circle of radius $r_k^{m,n}$ and the region $D_{z_0,\varepsilon}$. By the inequality (3.1) it follows that $\widehat{AOB} = 2\delta > \frac{2\pi}{2^n}$, so the arc $\overset{\frown}{AB}$ contains at least one root $\alpha_{k,j}^{m,n}$.

If $z_0 = \cos\theta + i\sin\theta$, then for any positive number $\varepsilon \in (0,1)$ we consider the region

$$D_{z_0,\varepsilon} = \{\rho\exp(it) : 1-\delta < \rho < 1,\ \theta-\delta < t < \theta+\delta\},$$

where $\delta = \frac{\varepsilon}{2}$. Similarly as above it can be proved that $|z - z_0| < \varepsilon$, for every $z \in D_{z_0,\varepsilon}$.

Consider $n \in \mathbb{N}$ sufficiently large such that the following inequalities hold true:

$$\frac{\pi}{2^n} < \delta \tag{3.6}$$

and

$$(1-\delta)^{2^n} < \frac{2}{\pi}. \tag{3.7}$$

We have to prove that there exist $m \in \mathbb{N}$ and $k \in \{1, 2, \ldots, 2^{m-1}\}$ such that

$$1 - \delta < r_k^{m,n} < 1,$$

or, equivalently,

$$(1-\delta)^{2^n} < 2\sin\frac{k\pi}{2^m} < 1.$$

Using again the inequalities (3.4) we find that it is sufficient that the following inequalities are satisfied:

$$\frac{1}{2}(1-\delta)^{2^n} < \frac{k}{2^{m-1}} < \frac{1}{\pi} \tag{3.8}$$

By taking m sufficiently large such that

$$\frac{1}{2^{m-1}} < \frac{1}{\pi} - \frac{1}{2}(r-\delta)^{2^n},$$

it follows that there exists at least one integer $k \in \{1, 2, \ldots, 2^{m-1}\}$ such that the inequalities (3.8) hold true. Since n was chosen such that $\frac{2\pi}{2^n} < 2\delta$ (see (3.6)), it follows that at least one root $\alpha_{k,j}^{m,n}$ is on the arc of intersection between the circle of radius $r_k^{m,n}$ and the region $D_{z_0,\varepsilon}$. $\square$

Corollary 3.2. *The closure of the set of roots of the reliability polynomials of all the networks in $\mathcal{C}$ contains the unit disk centered at 1, $\mathcal{D}_1 : |z-1| \le 1$.*

Proof. We consider the networks $\mathbf{C}^{\boldsymbol{u}_{m,n,s}}$, where $\boldsymbol{u}_{m,n,s} \in \{0,1\}^{m+n+s}$ is of the form $\boldsymbol{u}_{m,n,s} = (\underbrace{1,\ldots,1}_{m},\underbrace{0,\ldots,0}_{n},\underbrace{1,\ldots,1}_{s})$. The reliability polynomial of the network $\mathbf{C}^{\boldsymbol{u}_{m,n,s}}$ is

$$\mathrm{Rel}\left(\mathbf{C}^{\boldsymbol{u}_{m,n,s}};p\right) = \mathrm{Rel}\left(\mathbf{C}^{\boldsymbol{u}_{m,n}};\mathrm{Rel}\left(\mathbf{C}^{(1,\ldots,1)};p\right)\right) = P_{m,n}\left(\mathrm{Rel}\left(\mathbf{C}^{(1,\ldots,1)};p\right)\right), \tag{3.9}$$

where $P_{m,n}(p)$ is the reliability polynomial of the network $\mathbf{C}^{\boldsymbol{u}_{m,n}}$. By Lemma 2.1, this polynomial is written

$$P_{m,n}(p) = \prod_{k=0}^{2^m-1}\prod_{j=0}^{2^n-1}(p - \alpha_{k,j}^{m,n}),$$

where the roots $\alpha_{k,j}^{m,n}$ are given by the formula (2.7). Therefore, since

$$\mathrm{Rel}\left(\mathbf{C}^{(1,\dots,1)}; p\right) = 1 - (1-p)^{2^s},$$

the reliability polynomial $P_{m,n,s}(p)$ of the network $\mathbf{C}^{\boldsymbol{u}_{m,n,s}}$ is written

$$P_{m,n,s}(p) = \prod_{k=0}^{2^m-1} \prod_{j=0}^{2^n-1} \left(1 - \alpha_{k,j}^{m,n} - (1-p)^{2^s}\right). \tag{3.10}$$

We obtain that the roots of this reliability polynomial satisfy the equality

$$(p-1)^{2^s} = 1 - \alpha_{k,j}^{m,n}$$

which means that the roots of the polynomial $P_{m,n,s}(p)$ are placed on circles of radius $\left|1 - \alpha_{k,j}^{m,n}\right|^{2^{-s}}$ centered at 1 and each circle contains 2^s equally spaced roots.

Let $z_0 = 1 + r(\cos\theta + i\sin\theta)$, $0 < r \leq 1$ be an arbitrary point in the unit disk centered at 1 and let $\varepsilon > 0$ be a small positive number. In the same way as in the proof of Theorem 3.1, we take s large enough such that $\frac{2\pi}{2^s} < \varepsilon$.

By Theorem 3.1, $\left|1 - \alpha_{k,j}^{m,n}\right|$ takes values close enough to any number in $[0, 2]$, so we can find m, n, k, j such that

$$\left(r - \frac{\varepsilon}{2}\right)^{2^s} < \left|1 - \alpha_{k,j}^{m,n}\right| < \left(r + \frac{\varepsilon}{2}\right)^{2^s},$$

which means that the circle (centered at 1) of radius $\left|1 - \alpha_{k,j}^{m,n}\right|^{2^{-s}}$ is "close enough" to z_0, that is,

$$\left|1 - \alpha_{k,j}^{m,n}\right|^{2^{-s}} - r < \frac{\varepsilon}{2}.$$

Since the angle between any two consecutive roots on this circle is $\frac{2\pi}{2^s} < \varepsilon$, it follows that at least one root $\beta_{k,j,t}^{m,n,s}$ of $P_{m,n,s}(p)$ satisfies $\left|\beta_{k,j,t}^{m,n,s} - z_0\right| < \varepsilon$. □

In conclusion, Theorem 3.1 and Corollary 3.2 guarantee that $\mathcal{D}_0 \cup \mathcal{D}_1$ is contained into the closure of the roots of the reliability polynomials of networks constructed by series-parallel binary compositions.

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Department of Mathematics and Computer Science, Technical University of Civil Engineering Bucharest, Bucharest, Romania
Email address: marilena.jianu@utcb.ro

Department of Mathematics and Computer Science, Technical University of Civil Engineering Bucharest, Bucharest, Romania
Email address: leonard.daus@utcb.ro

A GEOMETRIC SOLUTION OF A MECHANICAL EQUILIBRIUM PROBLEM

ION MIERLUȘ – MAZILU, ȘTEFANIA CONSTANTINESCU, ALICE ANGHELESCU

Abstract. Most of the sciences problems' solutions imply mathematical analysis techniques. In this paper we present a plane geometric solution of a mechanical equilibrium problem. We reduce the determination of a mechanical equilibrium point to solving of some relations in a triangle and we give some conditions for the existence of the solution.

Mathematics Subject Classification (2010): 74G05, 74G25
Key words: mechanical equilibrium point

1. Preliminary Notions of Mechanics

Theorem 1.1 (Second Newton's Law of Motion)**.** *The change of motion of an object is proportional to the force impressed; and is made in the direction of the straight line in which the force is impressed.*

$$\overrightarrow{F} = m \cdot \overrightarrow{a} \tag{1.1}$$

Remark 1.2. *An equilibrium condition is given by* $\overrightarrow{a} = \overrightarrow{0}$, *hence* $\overrightarrow{F} = \overrightarrow{0}$.

Definition 1.3. *Being given a point O called* pole, *we define* the momentum of the force $\overrightarrow{F}$ with respect to O *the vector*

$$\overrightarrow{M_O} = \overrightarrow{OP} \times \overrightarrow{F}, \tag{1.2}$$

where P is the application point of the force $\overrightarrow{F}$.

Remark 1.4. *If* $\overrightarrow{M_O} = \overrightarrow{0}$ *then either the force acts in the direction of* $\overrightarrow{OP}$ *or* $\overrightarrow{F} = \overrightarrow{0}$.

Remark 1.5. *An equilibrium condition of a body is given by* $\overrightarrow{M_O}(\mathcal{S}) = \overrightarrow{0}$, *where $\mathcal{S}$ represents the resultant of the forces that act on the body.*

For much information about the mechanical notions above one can consult [1] and [2].

2. The Problem

Let a non-homogeneous bar AB, of length l. The bar is in equilibrium, having its ends tied by an inextensible thread, flexible, without weight, of length L, which passes, without friction through a fixed point O, located outside the bar.

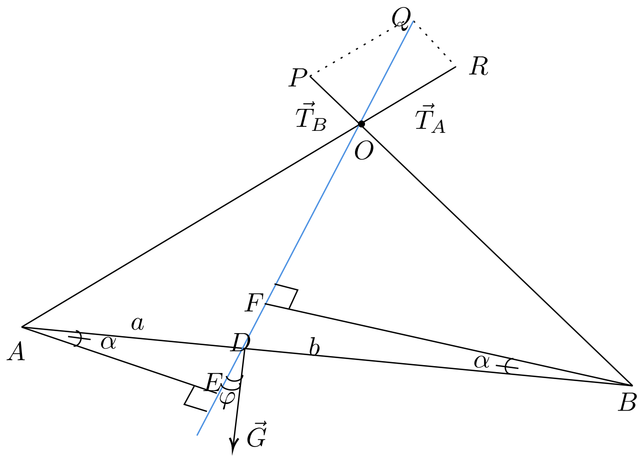
Find the equilibrium position of the bar and the magnitude of the binding forces (the tension forces), knowing that the tension forces are applied in the points A and B, having equal magnitudes, and the position of the center of gravity is given by point D, which divides the bar into the ratio $\frac{AD}{DB} = \frac{a}{b} = k$.

The Solution of the Problem.

Firstly, we construct $AE \perp OD$, $BF \perp OD$, hence $AE || BF$ și $\measuredangle EAD = \measuredangle DBF := \alpha$.

Let φ be the angle formed by the OD and the weight of the bar.

We translate the two tension forces such that their application point is O and then we construct the rhombus $OPQR$ with $|OP| = |\overrightarrow{T_B}|$ and $|OR| = |\overrightarrow{T_A}|$.



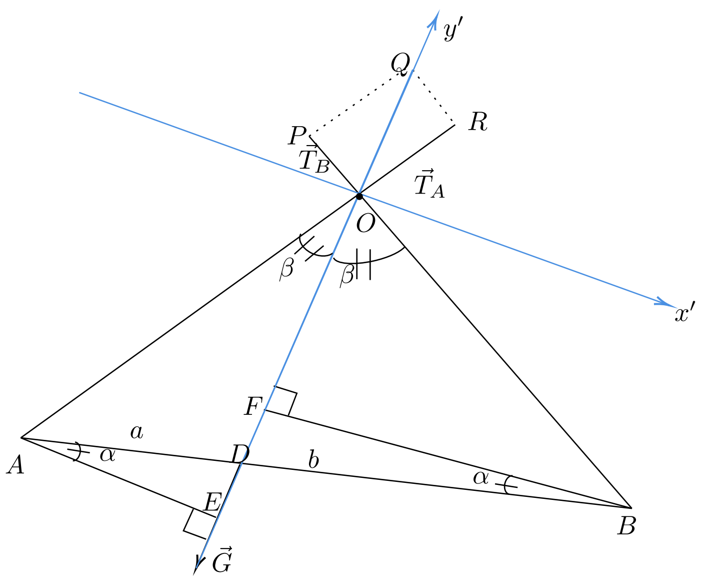
We denote by $\mathcal{S}$ the resultant of the forces that act on the bar and it is given by $\overrightarrow{T_A}, \overrightarrow{T_B}, \overrightarrow{G}$.

Because the bar is in equilibrium, then $\overrightarrow{T_A} + \overrightarrow{T_B} + \overrightarrow{G} = \overrightarrow{0}$ și $\overrightarrow{M_O}(\mathcal{S}) = \overrightarrow{0}$, i.e. $\overrightarrow{M_O}(\overrightarrow{T_A}) + \overrightarrow{M_O}(\overrightarrow{T_B}) + \overrightarrow{M_O}(\overrightarrow{G}) = \overrightarrow{0} \Longrightarrow \overrightarrow{M_O}(\overrightarrow{G}) = \overrightarrow{0}$, and we deduce that $\overrightarrow{G} \in (OD) \Longrightarrow \varphi = 0$, which implies

$$\overrightarrow{OQ} = -\overrightarrow{G}. \tag{2.1}$$

OQ is a bisectrix in rhombus, then OQ is bisectrix also for $\measuredangle AOB$, hence $\measuredangle AOD = \measuredangle BOD := \beta$.

To find the solution we must determine the coordinates $A(x_A, y_A), B(x_B, y_B)$ and we will choose the coordinates system $x'Oy'$ such that $OD := Oy$ and $Ox \perp Oy$.



As $AE \perp Oy$, respectively $BF \perp Oy$, then it remains to determine $\measuredangle\beta$ or $\measuredangle\alpha$. But finding the measure of one of these angles implies the determination of the other, because:

$$b\cos\alpha = OB\sin\beta \tag{2.2}$$

Applying the bisectrix theorem in $\triangle AOB$ we obtain

$$\frac{OB}{OA} = \frac{DB}{DA} = \frac{b}{a} \Longrightarrow \frac{OB}{OA+OB} = \frac{b}{a+b} \Longrightarrow \frac{OB}{L} = \frac{b}{l}. \tag{2.3}$$

From relation (2.2) we deduce that

$$\cos\alpha = \frac{L}{l}\sin\beta. \tag{2.4}$$

We will determine $\measuredangle\beta$.

Applying the law of cosines in $\triangle AOB$, from relation (2.3) we have:

$$\frac{a^2L^2}{l^2} + \frac{b^2L^2}{l^2} - 2\frac{abL^2}{l^2}\cos 2\beta = (a+b)^2,$$

wherefrom

$$\cos 2\beta = \frac{L^2(a+b)^2 - l^4}{2L^2ab}$$

and we obtain that

$$\cos^2\beta = \frac{l^2\left(1-\frac{l^2}{L^2}\right)}{4ab} = \left(\frac{a}{4b} + \frac{b}{4a} + \frac{1}{2}\right)\left(1 - \frac{l^2}{L^2}\right).$$

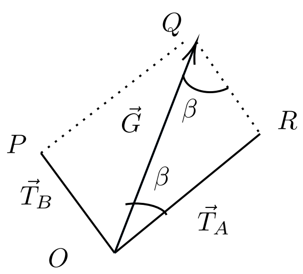
Denoting by $\frac{a}{b} := k$; $\frac{l}{L} := K$ we have

$$\cos^2\beta = \frac{1}{4}\cdot\frac{k^2+1}{k}\left(1-K^2\right).$$

Hence, knowing $\measuredangle\beta$ depends on k and K. Thus, we find x_A, y_A, x_B, y_B.

To find the magnitude of $|\overrightarrow{T_A}|$ we apply the cosines theorem and we obtain

$$|\overrightarrow{G}|^2 = 2|\overrightarrow{T_A}|^2 - 2\cos(\pi - 2\beta)|\overrightarrow{T_A}|^2 = 4|\overrightarrow{T_A}|^2\cos^2\beta \Longrightarrow |\overrightarrow{T_A}| = \frac{|\overrightarrow{G}|}{2\cos\beta}.$$



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Department of Mathematics and Computer Science, Technical University of Civil Engineering Bucharest, Bucharest, Romania

Email address: ion.mierlusmazilu@utcb.ro

Department of Mathematics and Computer Science, Technical University of Civil Engineering Bucharest, Bucharest, Romania

Email address: c_aurora32@yahoo.com

Faculty of Building Services Engineering, Technical University of Civil Engineering Bucharest, Bucharest, Romania

Email address: alicee_dag@yahoo.com

USING E-LEARNING IN STEM EDUCATION

Ion Mierluș Mazilu
Department of Mathematics and Computer Science
Technical University of Civil Engineering Bucharest
Bd. Lacul Tei 124, sector 2, 38RO-020396 Bucharest, Romania
E-mail: ion.mierlusmazilu@utcb.ro
Ștefania Constantinescu
Department of Mathematics and Computer Science
Technical University of Civil Engineering Bucharest
Bd. Lacul Tei, sector 2, 38RO-020396 Bucharest, Romania
E-mail: stefania.constantinescu@utcb.ro

Abstract: STEM is a broad term that groups together four academic disciplines; science, technology, engineering, and mathematics. STEM education is the teaching of science, technology, engineering, and mathematics in an academic context. You'll find STEM in all levels of education, from school curriculums, college subjects, and university degrees, right through to CPD courses and professional certifications. These four subjects are typically taught through hands-on learning and real-world projects – enabling students to prepare for a job in this growing field. STEM is an umbrella term that covers a range of subjects, and many academic disciplines that fall under this category. There are many skills you'll gain from studying STEM. In order to be minimized by the effects of pandemic, teaching staff of universities should be trained to use pedagogical and digital methods for active students and instruments that support the development of the student's personal knowledge. The primary context of the DigiSTEM project is STEM education. The objective is to promote innovative utilization of educational technology, learning analytics and use of open educational resources (OERs) in online, classroom and blended learning, especially in HEIs STEM subjects. The project aims to support professional development of HEI educators by increasing their technological and pedagogical skills and competence. The objective is to build HEI educators' competence of such instructional design that improves students' active learning, self-regulated learning and learning engagement with the help of educational technology and learning analytics to provide more effective and personalized support of learners.
Mathematics Subject Classification (2010): 97B10, 97M50
Key words: Mathematics education, STEM, pedagogical skills, competence

1. Introduction

Several national and international studies have shown that higher education institutions (HEIs) need support to recover from the Covid-19 pandemic consequences such as decrease of students' competence level (Aristovnik et al., 2020, Kinnari Korpela, 2021). During pandemic, universities have had to lower the competence requirements of the courses and to lighten their evaluations. In order to be minimized by the effects of pandemic, teaching staff of universities should be trained to use pedagogical and digital methods for active students and instruments that support the development of the student's personal knowledge. The primary context of the DigiSTEM project is STEM education. The objective is to promote innovative utilization of educational technology, learning analytics and use of open educational resources (OERs) in online, classroom and blended learning, especially in HEIs STEM subjects. The project aims to support professional development of HEI educators by increasing their technological and pedagogical skills and competence. The objective is to build HEI educators' competence of such instructional design that improves students' active learning, self-regulated learning and learning engagement with the help of educational technology and learning analytics to provide more effective and personalized support of learners. In Europe, there doesn't exist a framework or model for learning digital skills and pedagogy in HEI STEM education.

When teaching/learning STEM subjects, there are also special needs of digital tools/environments for example in presenting mathematical language, equations etc. The needs analysis of the project is based on literature review and educational research conducted in partner universities. Different educational research studies (e.g. Fernandez-Cruz & Fernandez-Diaz, 2016; Huang, Han & Wang, 2017) have highlighted that educators have an inadequate competence regarding effective use of digital tools and resources in education from both pedagogical and technological perspectives. The most common reasons for not to use digital technology in teaching/learning are a lack of confidence, a lack of competence and a lack of access to resources (e.g. Joy et al., 2014; Kinchin, 2012). The research has shown, that students would like to use digital learning possibilities and methods more widely in STEM courses (i.e. Rinneheimo et al., 2018; Kinnari-Korpela & Suhonen, 2017). However, utilizing educational technology in STEM subjects is very limited and it lags behind the expectations (e.g. Clark-Wilson, Oldknow, & Sutherland, 2011; De Witte & Rogge, 2014), even though using instructional design that utilizes educational technologies has a great and recognized potential to increase students' motivation, attractiveness of subject, promote active learning and improve learning outcomes (e.g. Dunn et al., 2015; Loch et al., 2014; Kinnari-Korpela, 2019; Kinnari-Korpela & Suhonen, 2017).

The needs analysis shows that there is a clear need for instructional design that increase pedagogical meaningful use of digital learning possibilities among STEM education and academics must be trained to utilize such instructional design. These are also needed to ensure that universities can better recover from the students' passivation and decreased learning outcomes caused by Covid-19 pandemic (Aristovnik et al., 2020, Kinnari-Korpela, 2021).

2. Objectives

The objective of the project is to increase digital and pedagogical competence of HEI educators and availability of digital resources in STEM subjects on a large scale to achieve long-lasting effects in everyday activity on the project partners and other European HEIs. By increasing such competences of educators, it gives them tools and knowledge to redesign their teaching and implement digital resources and activities (e.g. learning analytics, digital languaging, screencasts, visualizations and intelligent assessment) for different personalized learning scenarios.

The agenda for the modernization of Europe's higher education systems supports the project idea as it suggests the need to exploit new technologies and ICT to enrich teaching/learning experience and providing ubiquitous and personalized digital learning possibilities for students. Also, the Digital Education Action Plan set by the Commission has a priority to make better use of digital technology for teaching and learning. The project aims to enhance educators' technological and pedagogical competences by organizing different kind of pilot events and providing OERs and learning environment for competence development. By increasing such competencies of HEI educators (main target group), it gives them tools and knowledge to produce and implement digital resources for different personalized learning scenarios and resources that supports students' activation. Hence, the project aims to promote digital and pedagogical competence and skills of HEI educators nationally and transnationally in Europe. Simultaneously based on literature, this is expected to affect positively on HEI students' engagement and learning outcomes (secondary target group) but also support to recover from the Covid-19 pandemic consequences such as decrease of students' competence level.

3. Implementation

The main activities/results of the project are:

- developing DigiSTEM methodology that encapsulates innovative pedagogies, best practices and concrete examples for implementing digital learning/teaching of STEM and other similar subjects (PR1)
- building, maintaining and developing a digital platform for STEM subjects' digital teaching and learning to support educators' continuous professional development with high-quality resources (PR2)
- developing guidelines for European STEM educators to increase digital and pedagogical competence and implementing good practices and new methods smoothly into daily teaching activities and curricula (PR3)
- enhancing HEI educators' competence of such instructional design that improves students' active learning, self-regulated learning and learning engagement with the help of educational technology and learning analytics to provide more effective and personalized support of learners (PR1-PR3)
- to organize/participate 3 LTT (C1-C3) events and participate dissmination events (E1-E6) aimed at providing technological and pedagogical training/knowledge to STEM educators in the context of the project. Participants of training activities will be awarded with digital competence certificate.
- developing MOOC as a form of OER that will combine all the tangible results of the project to be able to promote and integrate developed good practices and innovative methods into daily activities of European HEIs. The MOOC is based on piloted PRs, which have been developed on the basis of feedback received during piloting. The project's pedagogical innovations and technological choices will be made as sustainable solutions that can be utilized after the project, for example 10 years after the end of the project.

4. Results

The objective is to build HEI educators' competence of such instructional design that improves students' active learning, self-regulated learning and learning engagement with the help of educational technology and learning analytics to provide more effective and personalised support of learners. In this way, it is possible to achieve the effectiveness of the project from the curriculum level to practice. The project is divided in three main Project Results (PR1-PR3). One of the partners is appointed as the coordinator of each output even though each output will be developed in close cooperation with all partners. The Project Results are described in previous section. As a consequence of the exploitation of Project's Results (PRs), it is expected to improve HEI students' motivation and to decrease dropouts (Kinnari-Korpela, 2019). Although the project focuses on HEI STEM subjects, the PRs can be applied to other disciplines with some adaptations. Hence, the project can contribute to improve educators' skills to apply modern methodologies, novel pedagogy and digital teaching/learning solutions on a large scale.

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APPLICATIONS OF THE YOUNG INTEGRAL IN DIFFERENTIAL EQUATIONS

LUCIAN NIȚĂ, ȘTEFANIA CONSTANTINESCU, ALICE ANGHELESCU

ABSTRACT. In this paper we present a concept of integral, namely the Young Integral and some of its applications in differential equations driven by irregular signals. We also give some results of existence, unicity and continuous dependence on data for this type of equations.

Mathematics Subject Classification (2010): 26A42, 60L20, 60L50
Key words: Young integral, rough paths, differential equations, finite p–variation functions

1. FUNCTIONS WITH FINITE p-VARIATION $(p \geq 1)$

Let $T > 0, J = [0, T], (E, |\cdot|)$ a Banach space and $X : J \to E$ a continuous function. Let, also, $p \geq 1$. We denote by $\mathcal{D}(J)$ the set of all subdivisions of J.

Definition 1.1. *We call* the p-variation of X on J, *the element, denoted* $\|X\|_{p,J}$ *defined by the formula*

$$\|X\|_{p,J} = \left(\sup_{D \in \mathcal{D}(J)} \sum_{i=0}^{r-1} |X_{t_i} - X_{t_{i+1}}|^p \right)^{\frac{1}{p}} \quad \textit{(where: } D = (t_0 < t_1 < \ldots < t_{r-1}), X(t) \overset{not}{=} X_t\textit{)}.$$

Remark 1.2. *If* $p = 1$, $\|X\|_{1,J}$ *is called* the variation of X on J.

Definition 1.3. *If* $\|X\|_{p,J} < \infty$, *we say that* X *has* finite *p-variation.*

Proposition 1.4. *If* $q > p \geq 1$ *then* $\|X\|_{q,J} \leq \|X\|_{p,J}$.

Notation. For $p \geq 1$, $\mathcal{V}^p(J, E) \overset{not}{=} \{X : J \to E | X \text{ is continuous function}, \|X\|_{p,J} < \infty\}$.

Remark 1.5. *From Proposition 1.4, we have:*

$$(p < q) \Longrightarrow \mathcal{V}^p(J, E) \subset \mathcal{V}^q(J, E).$$

Proposition 1.6. *a) The set* $\mathcal{V}^p(J, E)$ *is a vector subspace of the vector space* $C^0(J, E)$ *and the application* $X \mapsto \|X\|_{p,J}$ *is a seminorm;*
b) If the sequence $(X_n)_n \in \mathcal{V}^p(J, E)$ *is simple convergent to a function* $X : J \to E$, *then* $X \in \mathcal{V}^p(J, E)$ *and* $\|X\|_{p,J} \leq \underline{\lim}_{n \to \infty} \|X_n\|_{p,J}$.

For $X \in \mathcal{V}^p(J, E)$, we denote:

$$\|X\|_{\mathcal{V}^p(J,E)} := \|X\|_{p,J} + \|X\|_\infty, \quad (\|X\|_\infty = \sup_{t \in J} |X_t|)$$

.

Theorem 1.7. *a) The application* $X \mapsto \|X\|_{\mathcal{V}^p(J,E)}$ *is a norm;*
b) $(\mathcal{V}^p(J, E), \|\cdot\|_{\mathcal{V}^p(J,E)})$ *is a Banach space;*
c) If $1 \leq p < q$, *we have the continuous inclusions:*

$$\mathcal{V}^1(J, E) \subset \mathcal{V}^p(J, E) \subset \mathcal{V}^q(J, E) \subset C^0(J, E).$$

Definition 1.8. *Let $\alpha \in (0,1]$. The function $f : J \to E$ is called* α-Hölder, *if there exists $L \geq 0$ such that*

$$|f(x) - f(y)| \leq L|x-y|^{\alpha}, \forall x, y \in J.$$

Proposition 1.9. *If $X : J \to E$ is α-Hölder $(0 < \alpha \leq 1)$, then $X \in \mathcal{V}^{\frac{1}{\alpha}}(J, E)$.*

Example 1.10. *Let $f : [0,1] \to \mathbb{R}$,*

$$f(x) = \begin{cases} 1, & \text{if } x = 1, \\ -\frac{1}{2}, & \text{if } x = \frac{1}{2}, \\ \frac{1}{3}, & \text{if } x = \frac{1}{3}, \\ -\frac{1}{4}, & \text{if } x = \frac{1}{4}, \\ \vdots & \\ (-1)^{k+1}\frac{1}{k}, & \text{if } x = \frac{1}{k}, \\ \vdots & \\ \text{affine}, & \text{if } x \in \left(\frac{1}{k+1}, \frac{1}{k}\right). \end{cases}$$

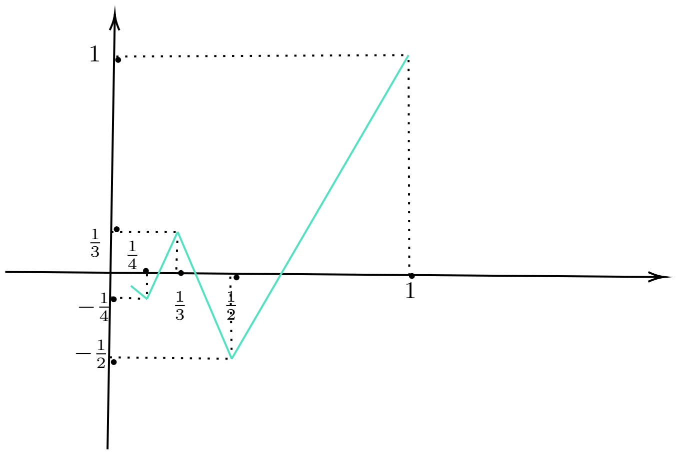

Fig. 1

i) One can prove that $\|f\|_{1,[0,1]} \geq \sum\limits_{k=1}^{\infty} \left(\frac{1}{k} + \frac{1}{k+1}\right) = \infty$, hence $f \notin \mathcal{V}^1([0,1], \mathbb{R})$.

ii) It can be proved that $\|f\|_{2,[0,1]}^2 \leq 2 \sum\limits_{k=1}^{\infty} \left(\frac{1}{k} + \frac{1}{k+1}\right)^2 < \infty$, hence $f \in \mathcal{V}^2([0,1], \mathbb{R})$.

Remark 1.11. *Example 1.10 shows that the inclusions from Theorem 1.7 c) are strictly.*

2. The Young integral

Let $J = [0,T]$, V, W Banach spaces, $p, q \geq 1$ such that $\frac{1}{p} + \frac{1}{q} > 1$. Let, also, $X \in \mathcal{V}^p(J, E)$, $Y \in \mathcal{V}^q(J, \mathcal{L}(V,W))$ ($\mathcal{L}(V,W) = \{f : V \to W | f$ is linear and continuous$\}$) and $D \in \mathcal{D}(J), D = (0 = t_0 < t_1 < \ldots < t_r = T)$. We denote

$$\int_D Y dX = \sum_{i=0}^{r-1} Y_{t_i} \left(X_{t_{i+1}} - X_{t_i}\right)$$

.

Definition 2.1. *We say that Y is* Young integrable with respect to X on J *if: $\exists I \in W$ such that $\forall \varepsilon > 0, \exists \delta_\varepsilon > 0$ with the property: $\forall D \in \mathcal{D}(J)$ with $\|D\| < \delta_\varepsilon$ we have $\left|\int_D Y dX - I\right| < \varepsilon$. [We denote:*

$\|D\| = \max\limits_{1\le i\le r} |t_i - t_{i-1}|$]. *In this case, I is called* the Young integral of Y with respect to X *on J and is denoted by* $\int_0^T Y dX$.

Remark 2.2. 1°) *It is easy to prove that $\int_0^T Y dX$ is linear both with respect to X and with respect to Y.*
2°) *One can see that Young integral is a generalization of the vector Riemann-Stieltjes integral.*

Example 2.3. *Let $a, b, c \in \mathbb{R}$ and the functions $X : [0,T] \to \mathbb{R}^3, X_t = (ta, b, 2tc)$, $Y : [0,T] \to \mathcal{L}\left(\mathbb{R}^3, \mathbb{R}^2\right)$, $Y_t : \mathbb{R}^3 \to \mathbb{R}^2, Y_t(x,y,z) = (x + ty - z, x + z)$. We consider the subdivision $D = (0 = t_0 < t_1 < \ldots < t_r = T)$ and compute:*

$$\int_D Y dX = \sum_{i=0}^{r-1} Y_{t_i}\left(X_{t_{i+1}} - X_{t_i}\right) = \sum_{i=0}^{r-1} Y_{t_i}\left((t_{i+1}a, b, 2t_{i+1}c) - (t_i a, b, 2t_i c)\right) =$$

$$= \sum_{i=0}^{r-1} Y_{t_i}\left((t_{i+1} - t_i)a, 0, 2c(t_{i+1} - t_i)\right)) = \sum_{i=0}^{r-1}\left((t_{i+1} - t_i)(a - 2c), (t_{i+1} - t_i)(a + 2c)\right) =$$

$$= \left((a - 2c)\sum_{i=0}^{r-1}(t_{i+1} - t_i), (a + 2c)\sum_{i=0}^{r-1}(t_{i+1} - t_i)\right) = (T(a - 2c), T(a + 2c)).$$

Theorem 2.4 (Young Theorem). *Let V and W Banach spaces and $p, q \ge 1$ such that $\frac{1}{p} + \frac{1}{q} > 1$, $X \in \mathcal{V}^p(J, V)$, $Y \in \mathcal{V}^q(J, \mathcal{L}(V, W))$. Then:*
1°) *$\forall t \in J$, there exists: $\int_0^t Y dX = \lim\limits_{\|D_n\|\to 0}\left(\int_{D_n} Y dX\right)$ (the limit doesn't depend on the sequence $(D_n)_n \subset \mathcal{D}(J)$ with $\|D_n\| \to 0$).*
2°) *As function of t,the limit from above belongs to $\mathcal{V}^p(J, W)$ and there exists $C_{p,q} > 0$ such that:*

$$\left\|\int_0^{\cdot} Y dX\right\|_{\mathcal{V}^p([0,T],W)} \le C_{p,q}\|Y\|_{\mathcal{V}^q([0,T],W)}\|X\|_{p,[0,T]}.$$

3. Differential equations driven by irregular signals

Let V, W, J like in the definition of Young integral, $X : J \to V, Y : J \to W, f : W \to \mathcal{L}(V, W)$ continuous functions.

Definition 3.1. *We say that X, Y and f satisfy* the differential equation (controlled by the signal X)*:*

$$dY_t = f\left(Y_t\right) dX_t \tag{3.1}$$

with the initial condition $Y_0 = \xi$, if we have $Y_t = \xi + \int_0^t f\left(Y\right) dX, \forall t \in J$.

Remark 3.2. *To find the solution of the equation (3.1) is equivalent to determine the fixed points of the functional $Y_\cdot \mapsto \xi + \int_0^{\cdot} f\left(Y\right) dX$.*

Theorem 3.3 (Picard-Lindelöf Theorem). *If X has finite variation and f is a Lipschitz function, then $\forall \xi \in W$ the equation (3.1) has an unique solution.*

[The idea of the proof: let $\mathcal{F}(Y_\cdot) = \xi + \int_0^{\cdot} f\left(Y\right) dX$; we divide $[0,T]$ in "smaller" intervals, such that on each interval $\mathcal{F}$ is a contraction.]

Theorem 3.4 (Peano Theorem). *Let $1 \le p < 2$ and $\gamma \in \mathbb{R}$ such that $p - 1 < \gamma \le 1$. Suppose that W finite dimensional, X and Y have finite p-variation and f is a γ-Hölder function. Then, the equation (3.1) has solutions.*

[The idea of the proof: let us consider the functional $\mathcal{F}$ as above, $B = \{Y \in \mathcal{V}^{p'}(J,W) | \|Y\|_{\mathcal{V}^{p'}(J,W)} \leq M\}$, for some $p' > p$ and $M > 0$. It can be proved that: $\mathcal{F}(B) \subset B$, $\mathcal{F}$ is continuous, $\mathcal{F}(B)$ is relative compact. Using the Schauder theorem we deduce that $\mathcal{F}$ has at least a fixed point.]

Now, we discuss the problem of continuous dependence on the initial data. First, we need some auxiliary results:

Lemma 3.5. *We suppose that $f : W \to \mathcal{L}(V,W)$ is a Lipschitz function and, more, there exists $C > 0$ such that $\|f \circ Y_1 - f \circ Y_2\|_{p'} \leq C\|Y_1 - Y_2\|_{p'}$, for any $Y_1, Y_2 \in \mathcal{V}_{p'}$. Then $\exists K > 0$ such that*

$$\|f \circ Y_1 - f \circ Y_2\|_{\mathcal{V}_{p'}} \leq K\|Y_1 - Y_2\|_{\mathcal{V}_{p'}}.$$

Proof. Let $D \in \mathcal{D}([0,T]), D = (0 = t_0 < t_1 < \ldots < t_r = T)$ and $L > 0$ the Lipschitz constant of f. Then

$$|f(Y_{1_t}) - f(Y_{2_t})| \leq L|Y_{1_t} - Y_{2_t}| \leq L\|Y_1 - Y_2\|_\infty \Longrightarrow \|f(Y_1) - f(Y_2)\|_\infty \leq L\|Y_1 - Y_2\|_\infty.$$

But $\|f \circ Y_1 - f \circ Y_2\|_{p',J} \leq C\|Y_1 - Y_2\|_{p',J}$. We obtain:

$$\|f \circ Y_1 - f \circ Y_2\|_{\mathcal{V}_{p'}} \leq L\|Y_1 - Y_2\|_\infty + C\|Y_1 - Y_2\|_{p',J} \leq K\|Y_1 - Y_2\|_{\mathcal{V}_{p'}},$$

where $K = \max(L, C)$. ☐

Remark 3.6. *In the particular case where f is linear, let $D = (0 < t_1 < t_2 < \ldots < t_r = T)$. We have:*

$$\sum_{i=0}^{r-1} \|f((Y_1 - Y_2)_{t_i} - (Y_1 - Y_2)_{t_{i+1}})\|^{p'} = \sum_{i=0}^{r-1} \|f(Y_1 - Y_2)_{t_i} - f(Y_1 - Y_2)_{t_{i+1}}\|^{p'} \leq$$

$$\leq L^{p'} \sum_{i=0}^{r-1} \|(Y_1 - Y_2)_{t_i} - (Y_1 - Y_2)_{t_{i+1}}\|^{p'}_{p',J}.$$

Using this relation, we get:

$$\|f \circ Y_1 - f \circ Y_2\|_{p',J} \leq L\|Y_1 - Y_2\|_{p',J}.$$

Lemma 3.7. *Let V a Banach space and $X : [0,T] \to V$ with finite p-variation. Then $\forall \varepsilon > 0, \exists \delta > 0$ such that $\forall s,t \in [0,T]$ with $0 \leq s < t \leq T$ and $|t - s| < \delta$ we have $\|X\|_{p,[s,t]} < \varepsilon$.*

Theorem 3.8. *Let f like in Lemma 3.5 and X a signal with finite p-variation which satisfy the differential equation $dY_t = f(Y_t)dX_t, t \in [0,T]$. We suppose that for each $\xi \in W$, the equation has an unique solution. Then Y depends continuously on the initial condition ξ on I.*

Proof. For each ξ, we denote by Y^ξ the unique solution on $[0,T]$ of the given equation with the property that $Y_0 = \xi$.

Let $(\xi_n)_n \subset W, \xi \in W$ such that $\xi_n \overset{|\cdot|_W}{\to} \xi$ and $p' > p$. According to Lemma 3.7, $\forall \varepsilon > 0, \exists \delta > 0$ such that $\forall s,t \in [0,T]$ with $|s - t| < \delta$ we have $\|X\|_{p,[s,t]} < \varepsilon$.

As Y^{ξ_n} are solutions of the equation, then they satisfy (3.1) and according to Young Theorem, Peano Theorem and to Lemma 3.5, we have succesively:

$$\left\|Y^{\xi_n} - Y^\xi\right\|_{\mathcal{V}^{p'}([0,t],W)} \leq |\xi_n - \xi| + \left\|\int_0^\cdot \left(f\left(Y_u^{\xi_n}\right) - f\left(Y_u^\xi\right)\right) dX_u\right\|_{\mathcal{V}^{p'}([0,t],W)} \leq |\xi_n - \xi| +$$

$$+ c_{p,p'}\left\|f\left(Y^{\xi_n}\right) - f\left(Y^\xi\right)\right\|_{\mathcal{V}^{p'}([0,t],W)}\|X\|_{p,[0,t]} \leq |\xi_n - \xi| + c_{p,p'}K\left\|Y^{\xi_n} - Y^\xi\right\|_{\mathcal{V}^{p'}([0,t],W)}\|X\|_{p,[0,t]}.$$

We suppose that t is small enough such that $\|X\|_{p,[0,t]} = M < \frac{1}{c_{p,p'}K}$. Then, it results that

$$\left\|Y^{\xi_n} - Y^\xi\right\|_{\mathcal{V}^{p'}([0,t],W)} \leq |\xi_n - \xi| + c_{p,p'}KM\left\|Y^{\xi_n} - Y^\xi\right\|_{\mathcal{V}^{p'}([0,t],W)} \Longrightarrow$$

$$\Longrightarrow \left\|Y^{\xi_n} - Y^\xi\right\|_{\mathcal{V}^{p'}([0,t],W)}(1 - c_{p,p'}KM) \leq |\xi_n - \xi|.$$

As $M < \frac{1}{c_{p,p'}K}$, then $1 - c_{p,q}KM > 0$ and as $\xi_n \to \xi$, we deduce that $\lim_{n\to\infty}\left\|Y^{\xi_n} - Y^\xi\right\|_{\mathcal{V}^{p'}([0,t],W)} = 0$, hence $Y^{\xi_n} \to Y^\xi$ in p'-variation norm.

Next, we divide the interval $[0,T]$ such that: $0<\frac{T}{m}<\frac{2T}{m}<\ldots<\frac{(m-1)T}{m}<T$, where $m\in\mathbb{N}^*$ is big enough such that $\|X\|_{p,[\frac{iT}{m},\frac{(i+1)T}{m}]}<\frac{1}{c_{p,p'}K}, \forall i=0,\ldots,m-1$. Similarly, we prove that $\lim\limits_{n\to\infty}\left\|Y^{\xi_n}-Y^{\xi}\right\|_{\mathcal{V}^{p'}([\frac{iT}{m},\frac{(i+1)T}{m}],W)}=0$.

But

$$\left\|\int_{\frac{iT}{m}}^{\cdot}\left(f\left(Y_u^{\xi_n}\right)-f\left(Y_u^{\xi}\right)\right)dX_u\right\|_{\mathcal{V}^{p'}([\frac{iT}{m},\frac{(i+1)T}{m}],W)}\le c_{p,p'}K\left\|Y^{\xi_n}-Y^{\xi}\right\|_{\mathcal{V}^{p'}([\frac{iT}{m},\frac{(i+1)T}{m}],W)}\|X\|_{p,[\frac{iT}{m},\frac{(i+1)T}{m}]}\le$$
$$\le\left\|Y^{\xi_n}-Y^{\xi}\right\|_{\mathcal{V}^{p'}([\frac{iT}{m},\frac{(i+1)T}{m}],W)}\to 0.$$

We can write

$$\left\|Y^{\xi_n}-Y^{\xi}\right\|_{\mathcal{V}^{p'}([0,T],W)}\le|\xi_n-\xi|+\sum_{i=0}^{m-1}\left\|\int_{\frac{iT}{m}}^{\cdot}\left(f\left(Y_u^{\xi_n}\right)-f\left(Y_u^{\xi}\right)\right)dX_u\right\|_{\mathcal{V}^{p'}([\frac{iT}{m},\frac{(i+1)T}{m}],W)}.$$

It results that $\lim\limits_{n\to\infty}\left\|Y^{\xi_n}-Y^{\xi}\right\|_{\mathcal{V}^{p'}([0,T],W)}=0$. □

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Department of Mathematics and Computer Science, Technical University of Civil Engineering Bucharest, Bucharest, Romania

Email address: lucian.nita@utcb.ro

Department of Mathematics and Computer Science, Technical University of Civil Engineering Bucharest, Bucharest, Romania

Email address: stefania.constantinescu@utcb.ro

Faculty of Building Services Engineering, Technical University of Civil Engineering Bucharest, Bucharest, Romania

Email address: alice.anghelescu@student.utcb.ro

IRRATIONALITY AND TRANSCENDENCE FOR SOME REAL NUMBERS

A TRIBUTE TO THE 80^{th} BIRTHDAY OF PROFESSOR GAVRIIL PĂLTINEANU

SEVER ANGEL POPESCU

Abstract. In 1766 J. H. Lambert used the continuous fractions expansion of the function $\tan x$ to prove the irrationality of π. In 1873, trying to prove the trancendency of e, C. Hermite [H] developed a new method to approximate the exponential function by rational functions. In 1947, starting from a basic idea of Hermite [H], I. Niven [N] gave a one page elementary proof for the irrationality of π. In an Algebra course delivered in 1970 at University of Warwick, Ian Stewart extended the ideas of Hermite and Niven to supply elegant proofs for the transcendence of e and π [M]. In this note we propose a general frame to unify all the above ideas and prove as particular cases the irrationality of π, π^2, e^q for q a nonzero rational number, and we apply the same principle to prove the transcendence of e. The proofs use only elementary Calculus. Professor Gavriil Păltineanu is a mathematician who appreciates a lot the Beauty of a mathematical proof. We hope that our proofs (inspired by [H], [N], [M] and [W]) will satisfy his high requirements.

Mathematics Subject Classification (2010): 11J72, 11J99, 01A55
Key words: irrationality of π, π^2 and e^q, transcendence of e, Hermite approximation, Hermite identity.

1. Introduction

The idea that π is not a rational number has appeared long time ago. Aristotle (384-322 BC) said that the diameter and circumference of a circle are not commensurable ([AZ], page 47). In 1737, L. Euler used the continuous fractions method to prove that e is irrational. In 1761, J. H. Lambert proved the irrationality of $\tan q$ for q a nonzero rational number. In particular, if π was a rational number, then $\tan(\pi/4) = 1$ would be an irrational number, a contradiction. So, we can conclude that π itself is an irrational number. Lambert used the approximation of $\tan x$ with rational functions given by the continuous fractions expansion of $\tan x$. Trying to prove the transcendency of e, the great mathematician C. Hermite [H] discovered in 1873 a new type of approximation for the function $f(x) = e^{bx}$ (where b is a fixed nonzero natural number), with rational functions which have integer coefficients, and used it to prove the transcendency of e. For this, Hermite fixed a natural number $n_1 \geq 1$ and constructed two polynomials $P_{n_0}(x)$, $Q_{n_1}(x)$ with integer coefficients, $\deg P_{n_0} \leq n_0$, $\deg Q_{n_1} \leq n_1$, such that

$$\lim_{n_0\to\infty} \frac{P_{n_0}(x)}{Q_{n_1}(x)} = e^{bx}$$

(Hermite-Padé approximation). Namely, for

$$f(x) = \frac{1}{n_0!} t^{n_0} (x-b)^{n_1}, \tag{1.1}$$

Hermite give the following formulas for these polynomials:

$$P_{n_0}(x) = \sum_{j=n_1}^{N=n_0+n_1} f^{(j)}(b) \cdot x^{N-j}, \; Q_{n_1}(x) = \sum_{j=n_1}^{N} f^{(j)}(0) \cdot x^{N-j}. \tag{1.2}$$

The reminder,

$$e^{bx} - \frac{P_{n_0}(x)}{Q_{n_1}(x)} = \frac{x^{N+1}}{Q_{n_1}(x)} I_{n_0}(x), \tag{1.3}$$

where

$$I_{n_0}(z) = \int_0^b f(x) e^{-zx} dx \tag{1.4}$$

goes uniformly to zero on $[0, b]$ when $n_0 \to \infty$.

This was the basic idea for many other proofs relative to the irrationality or transcendence of some real numbers. Starting from this deep idea of Hermite, replacing e^{-zx} with $\sin x$, I. Niven [N] succeeded to give in 1947 a one page elementary proof of the fact that π is an irrational number.

Taking some ideas from [M], in this note we succeeded to give a general frame which unifies many of the previous elementary methods to prove the irrationality or the transcendency of some real numbers. We apply this general frame in sections 3, 4 and 5 for proving the irrationality of π, π^2 and respectively of e^q, where q is a nonzero rational number, and for proving the transcendence of e in Section 6. All these proofs are particular applications of the general principle described in Section 2. Some alternative elementary proofs one can find in [AZ], [M] and [ZM].

2. The general frame

For any nonzero natural number n, let $f_n(x)$ be a polynomial of degree $d = d(n) \geq 1$ with real coefficients, let $r > 0$ be a fixed real number, and let $h(x)$, $x \in [0, r]$ be a fixed continuous function. We consider the following sequence of functions: $h_{-1} = h$, h_0, $h_1, ...$ such that $h'_{j+1} = h_j$ for any $j = -1, 0, 1, 2, ...$ We compute the following integral by integrating it $(d+1)$-times by parts:

$$I = I_n = I(f_n, h, r) = \int_0^r f_n(x) h(x)\, dx = \sum_{j=0}^{d} (-1)^j \left[f_n^{(j)}(r) h_j(r) - f_n^{(j)}(0) h_j(0) \right] \tag{2.1}$$

For the Hermite's function f defined in (1.1) (here $n = n_0$ and n_1 is a fixed nonzero natural number) and for $h(x) = e^{-zx}$, z a fixed real number, and $r = b$, we get the above Hermite-Padé approximation.

The main idea is the following. Making $r = a/b$, $a, b \in \mathbb{N} \setminus \{0\}$ in (2.1) and, if after multiplication both sides by $k_1 b^{k_2 n}$, with $k_1 \in \mathbb{R}$, $k_2 \in \mathbb{N} \setminus \{0\}$, the left side goes to zero when $n \to \infty$, and the right side remains a nonzero integer for any $n \geq n_0$, then we shall get a contradiction. Hence, r cannot be a rational number.

We shall use this general frame in the following sections to prove the irrationality of π, π^2, e^q, $q \in \mathbb{Q} \setminus \{0\}$ and the transcendency of e.

3. π is irrational

In 1837, the French mathematician P. Wantzel (1814-1848) made a connection between the ruler and compass problems and some algebraic number fields. In particular he pointed out that if one could prove the transcendency of π, then the ancient "quadrature problem" would be solved. We prove here in an elementary way only that π is irrational (see also [N]).

Assume that $\pi = a/b$, with $a, b \in \mathbb{N} \setminus \{0\}$ and take in Section 2 for any prime number p

$$f_p(x) = b^{p-1}x^{p-1}\frac{(a-bx)^p}{(p-1)!} \tag{3.1}$$

Now we take $r = \pi = a/b$ and $h(x) = \sin x$ in (2.1). Since $d = 2p-1$ and $h_{-1}(x) = \sin x$, $h_0(x) = -\cos x$, $h_1(x) = -\sin x$, $h_2(x) = \cos x$, ... so, in (2.1) one gets:

$$I = I_p = \int_0^\pi f_p(x)\sin x\, dx = \sum_{j=0}^{p-1}\left[\pm f_p^{(2j)}(\pi) \pm f_p^{(2j)}(0)\right]. \tag{3.2}$$

It is easy to see that

$$f_p^{(j)}(\pi) = \begin{cases} 0, & j < p \\ Mp, & j \geq p \end{cases}, \tag{3.3}$$

where Mp is a multiple of p in $\mathbb{Z}$, and

$$f_p^{(j)}(0) = \begin{cases} 0, & j < p-1 \\ a^p b^{p-1}, & j = p-1 \\ Mp, & j > p-1 \end{cases}. \tag{3.4}$$

Thus $I_p = Mp + a^p b^{p-1}$ for any prime number p. If p is large enough, then I_p is an integer number which is not divisible by p. In particular, it is not zero. But

$$I_p \leq a^2 b^{-1}\frac{(a^2)^{p-1}}{(p-1)!},$$

so $I_p \to 0$, when $p \to \infty$, a contradiction, because in $(-1, 1) \setminus \{0\}$ we have no integers.

Hence π is irrational.

4. π^2 is irrational

Suppose that $\pi^2 = a/b$, with $a, b \in \mathbb{N} \setminus \{0\}$ and for any prime number p let us take $f_p(x)$ like in (3.1), $r = \pi^2$ and $h(x) = h_{-1}(x) = \sin(\frac{1}{\pi}x)$. Since

$$h_{2j+1}(x) = \pm\pi^{2j+2}\sin\left(\frac{x}{\pi}\right),\ h_{2j}(x) = \pm\pi^{2j+1}\cos\left(\frac{x}{\pi}\right)$$

in this case, we get

$$I_p = \int_0^{\pi^2} f_p(x)\sin\left(\frac{1}{\pi}x\right)dx = \sum_{j=0}^{p-1}\pi^{2j+1}\left[\pm f_p^{(2j)}(\pi^2) \pm f_p^{(2j)}(0)\right].$$

Since $\pi^2 = a/b$, we get

$$\pi^{-1}b^{p-1}I_p = \sum_{j=0}^{p-1} a^j b^{p-1-j}\left[\pm f^{(2j)}(a/b) \pm f^{(2j)}(0)\right].$$

If in (3.3) we put π^2 instead of π, we finally get

$$\pi^{-1}b^{p-1}I_p = Mp + a^{\frac{3p-1}{2}}b^{\frac{3p-3}{2}}.$$

So, for p large enough, $\pi^{-1}b^{p-1}I_p$ is not divisible by p, i.e. it is not zero, and $\pi^{-1}b^{p-1}I_p \to 0$, when $p \to \infty$, a contradiction. Hence $\pi^2 \notin \mathbb{Q}$.

5. e^q, $q \in \mathbb{Q} \setminus \{0\}$, is irrational

It is sufficient to prove this statement for $q \in \mathbb{N} \setminus \{0\}$. This time we take

$$f_p(x) = x^{p-1}\frac{(1-x)^p}{(p-1)!}$$

$r = 1$, $d = 2p - 1$, and $h(x) = e^{qx}$. Assume that $e^q = a/b$, $a, b \in \mathbb{N} \setminus \{0\}$ and we consider the corresponding integral

$$I_p = \int_0^1 f_p(x)e^{qx}dx = e^q F(1) - F(0), \tag{5.1}$$

where

$$F(x) = \sum_{j=0}^{2p-1}(-1)^j \frac{1}{q^{j+1}} f_p^{(j)}(x).$$

Since $e^q = a/b$, we get

$$bq^{2p}I_p = a\sum_{j=0}^{2p-1}(-1)^j q^{2p-j-1} f^{(j)}(1) - b\sum_{j=0}^{2p-1}(-1)^j q^{2p-j-1} f^{(j)}(0). \tag{5.2}$$

As in (3.3), (3.4), we find that $f^{(j)}(1) = Mp$, a multiple of p in $\mathbb{Z}$, for any $j = 0, 1, ..., 2p-1$, while $f^{(j)}(0) = Mp$ for $j \neq p-1$ and $f^{(p-1)}(0) = 1$. Thus we see that $bq^{2p}I(1) = pM - bq^p$ and so, for p large enough, $bq^{2p}I_p$ is not equal to zero. But $bq^{2p}I_p \to 0$, when $p \to \infty$, a contradiction. Hence e^q is irrational for any nonzero rational number q.

Now we use here the remarkable idea of Hermite [H] with some improvements made by A. Hurwitz, P. Gordon and D. Hilbert (see [W] for more facts) in order to supply a very short and elementary proof for the transcendency of e.

Assume on contrary, namely that there exists a polynomial

$$H(x) = a_m x^m + a_{m-1}x^{m-1} + ... + a_0 \in \mathbb{Z}[x],$$

$m \geq 2$ (for $m = 1$ we just proved in previous section that e cannot be a rational number), $a_m, a_0 \neq 0$, such that

$$H(e) = a_m e^m + a_{m-1}e^{m-1} + ... + a_0 = 0. \tag{6.1}$$

Now we take (see also [M] or [W])

$$f_p(x) = x^{p-1}\frac{\prod_{i=1}^m (x-i)^p}{(p-1)!}$$

We take $h(x) = e^{-x}$, so $h_j(x) = (-1)^{j+1}e^{-x}$ for $j = -1, 0, 1, 2, ..., d$, where

$$d = \deg f_p = mp + p - 1.$$

For any $i = 1, 2, ..., m$ we consider the integral

$$I_p(i) = \int_0^i f_p(x)e^{-x}dx = -e^{-i}F(i) + F(0),$$

where

$$F(x) = \sum_{j=0}^{d} f_p^{(j)}(x).$$

Let us consider the following sum:

$$T_p = \sum_{i=1}^m a_i e^i I_p(i) = -\sum_{i=1}^m a_i \sum_{j=0}^d f_p^{(j)}(i) - F(0)a_0,$$

because of the equality (6.1). For any $i = 1, 2, ..., m$, let us see that $f_p^{(j)}(i) = Mp$ for any $j = 0, 1, .., d$, but $f_p^{(j)}(0) = Mp$ for $j \neq p-1$, while $f_p^{(p-1)}(0) = (-1)^m m!$. Thus, we see that $T_p = pM + (-1)^{m+1}a_0 m!$, where M is an integer. So, for p large enough, T_p is not divisible by p, in particular it is a nonzero integer, while $T_p \to 0$, when $p \to \infty$, a contradiction. Therefore e cannot be a root of a polynomial with rational coefficients, i.e. e is a transcendental number.

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Department of Mathematics and Computer Science, Technical University of Civil Engineering Bucharest, Bucharest, Romania

Email address: angel.popescu@gmail.com

A STUDY OF WATER QUALITY IN A WATER DISTRIBUTION NETWORK, BASED ON STATISTICAL ANALYSIS

Alina Elisabeta Sandu
Department of Mathematics and Computer Science
Technical University of Civil Engineering Bucharest
Bd. Lacul Tei 124, sector 2, 38RO-020396 Bucharest, Romania
E-mail: alina.sandu@utcb.ro

Abstract: The purpose of this study is to develop a mathematical model that provides a prediction on the quality of water in distribution networks, regarding compliance with national and international standards.
This prediction can be made using statistical analysis of water quality, with specific tools: correlation and logistic regression, which can be used to develop mathematical models. For an existing network, if we have data on the structure of the network, the materials of which it is composed and their age is known, a prediction can be made on the water quality in this network. Other types of data on network structure can be used in this type of analysis. The more data is used, the more viable study is obtained.
The developed mathematical model was implemented in a case study, using real data provided by ARA (Romanian Water Association, National Report 2012), on Romanian networks, using the calculation program SPSS (Statistical Package for the Social Sciences), trial version.
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1. Introduction

The pursuit of safe drinking water has long been and remains a major concern of public-health officials and water-treatment operators. Since the recognition of waterborne disease by the end of the 19th century, increased attention has been directed toward protecting the nation's health in addition to quenching its thirst. This requires a thorough understanding of the factors that cause the deterioration in drinking water.
For controlling the ever-increasing complexity of these fields, a proper understanding of their processes is important as is the ability to reason about them.
We need statistics because we want to draw more valid conclusions from limited amounts of data and significant differences are often masked by biological variability or experimental imprecision. On the other hand, the human mind excels at finding patterns and relationships and tends to generalize too much.
The purpose of many research projects is to assess relationships among a set of variables and regression techniques often used as statistical analysis tools in the study of such relationships. Research designs may be classified as experimental or observational. Regression analyses are applicable to both types; yet the confidence one has in the results of a study can vary with the research type. In most cases, one variable is usually taken to be the response or dependent variable, that is, a variable to be predicted from or explained by other variables. The other variables are called *predictors*, *explanatory variables*, or *independent variables*. Choosing an appropriate model and analytical technique depends on the type of variable under investigation.

The other variables are called predictors, explanatory variables, or independent variables. Choosing an appropriate model and analytical technique depends on the type of variable under investigation In this paper we considered the case with one independent variable (simple regression) and the case with several independent variables or covariates (multiple regression).
The algorithm of this statistical analysis can be synthesized by the following steps:
Step 1: It will be analyzed all the data we have about a given water distribution network for identified the data to use in the statistical analysis to be made.
Step 2: There are highlighted the variables which have a significant correlation between each other. Only these variables will be effectively used in the statistical analysis to be made.
Step 3: It will be analyzed the normal distribution of the used variables. If these variables are normally distributed it will be chouse for calculus the linear regression or the logistic regression if the variables are not normally distributed. In real life, because the data are so various, there are not normally distributed, so the logistic regression it will be chouse for calculus.
Step 4: With the help of SPSS program, will be calculated the correlation coefficients of the independent variables choused in step 2. The values of these coefficients will tell us which data are more influence in the water quality prediction.
Step 5: The equation of the logistic regression it will be written, substituting the coefficients with the ones calculated in step 4. From this equation it will be calculated the ratio of the odd chances and the probability to have a water quality in accordance with legal norms.
Step 6: Tests are made for validate the model, so it will be verified if the values generated by calculus are in accordance with the real data. In this study, 3 tests were made, all indicated the relationship between the predictors and the variables.

2. Theoretical considerations.

2.1. The correlation

In many study cases, we are interested in the way in which the variability of a data set it is reflected in the variability of another data set (if the data are correlated). For determinate this influence, we can perform a X^2 test for one sample. This completes a contingency table containing the data observed and the expecting data. The test result will indicate the types of existent correlations between variables. Then we will determine the correlation coefficients between the two variables.
The correlation coefficients represent the association grade between two variables. These coefficients indicate the way how the value of one variable it is modified by the value of the other variable (if these two variables are associated). The correlation coefficients shows if there is a correlation between these two variables and how tide it is the relation between them.
Between two variables there are many correlation types:

- perfect positive;
- strong positive;
- weak positive;
- perfect negative;
- strong negative;
- weak negative;
- no correlation.

There are many correlation coefficients, and the most used are:

- *Pearson correlation coefficient* (detonated with r_{xy} or r), which is a parametric coefficient.
- *Spearman coefficient* (detonated with r_s), which is a non-parametric coefficient.

2.2. The Pearson correlation coefficient

The correlation coefficients , which describes the link between two variables, are different.
One of the most used correlation coefficients is the Pearson correlation coefficient (be noted r_{xy} or r), which is a parametric coefficient; the complete name of this coefficient is "Pearson product moment correlation coefficient " . The values of the Pearson correlation coefficient may vary between *+1* and *-1*. A value close to *+1* indicates a powerful positive correlation, a value close to *-1* indicates a powerful negative correlation, and a value close to *0* indicates that between these variables there is no correlation.
The Pearson correlation coefficient can be calculated with next formula:

$$r_{xy} = \frac{\frac{1}{n} \cdot \sum_i (x_i - \bar{x}) \cdot (y_i - \bar{y})}{s_x \cdot s_y} \quad (1)$$

where:

- n – the sample size formed by pair measurements *(x,y)*;
- x_i – the individual measures of the variable x;
- y_i - the individual measures of the variable y;
- $\bar{x}$ - the arithmetic media of the x variables ;
- $\bar{y}$ - the arithmetic media of the y variables;
- s_x - standard deviation for the x values;
- s_y - standard deviation for the y values .

The numerator from the equation (1) it is called *covariance* (denoted by s_{xy}) or variability couple

$$s_{xy} = \frac{1}{n} \cdot \sum_i (x_i - \bar{x}) \cdot (y_i - \bar{y}) \quad (2)$$

The covariance is a measure of the degree in which variation of the variable suits the variation of the other variable. The correlation coefficient is the ratio between covariance and the total variability (the product of the two-standard deviation).
If covariance is equal to total variability, then the correlation coefficient is equal to unit ($r = 1$). If the covariance is much smaller than the variability, then r approaches to zero.
Testing the significance of coefficients r_{xy} it is very important especially when working with relatively small samples and it is possible to obtain a coefficient different from zero by random choosing of a total uncorrelated data. The signification of these coefficients can be tested on a sample of any dimension (but over 10 pairs of values) by using t distribution (Student):

$$t = r \cdot \sqrt{\frac{n-2}{1-r^2}} \quad (3)$$

where:

n - the sample size formed by pair measurements *(x,y)*;
r – the correlation coefficient.

When is testing the signification of the correlation coefficient, can be used a one-tailed distribution *t* or two-tailed distribution. So we will have the next assumptions:

	two – tailed	one – tailed
H_0	ρ = 0	ρ = 0
H_1	ρ ≠ 0	ρ > 0 or ρ < 0

In both cases it is testing the null hypothesis, H_0, this means to verify if the population correlation it is zero (as noted, assumptions stated above using parameter ρ, representing population correlation).
When we expecting that the relationship between the two variables to be going in one particular direction, we will use one-tailed distribution for finding out if ρ is positive or negative. In these cases, the signification level from the table *t*, must be half of that used in two-tailed distribution. If we are not sure of the direction of relationship between the two variables, it will be using two-tailed test.

2.3. Simple Logistic Regression [5]

The usual regression analysis goal is to describe the mean of a dependent variable Y as a function of a set of predictor variables. The logistic regression, however, deals with the case where the basic random variable Y of interest is a dichotomous variable taking the value 1 with probability π and the value 0 with probability $(1-\pi)$. Such a random variable is called a *point-binomial* or *Bernouilli variable*, and it has the simple discrete probability distribution

$$\Pr(Y = y) = \pi^y (1-\pi)^{1-y} \qquad y = 0,1 \tag{4}$$

Suppose that for the *i* th individual of a sample $(i = 1,2,..., n)$, Y_i is a Bernoulli variable with

$$\Pr(Y_i = y_i) = \pi_i^{y_i} (1-\pi_i)^{1-y_i} \qquad y_i = 0,1 \tag{5}$$

The logistic regression analysis assumes that the relationship between π_i and the covariate value x_i of the same person is described by the logistic function

$$\pi_i = \frac{1}{1+\exp[-(\beta_0 + \beta_1 x_i)]} \qquad i = 1,2,..., n, \tag{6}$$

The basic logistic function is given by

$$f(z) = \frac{1}{1+e^{-z}} \tag{7}$$

were, as in this simple regression model,

$$z_i = \beta_0 + \beta_1 x_i \tag{8}$$

or, in the multiple regression model,

$$z_i = \beta_0 + \sum_{j=1}^{k} \beta_1 x_{ji} \tag{9}$$

representing an index of combined risk factors.
Under the simple logistic regression model, the likelihood function is given by

$$L = \prod_{i=1}^{n} \Pr(Y_i = y_i) = \prod_{i=1}^{n} \frac{[\exp(\beta_0 + \beta_1 x_i)]^{y_i}}{1+\exp(\beta_0 + \beta_1 x_i)} \qquad y_i = 0,1 \tag{10}$$

from which we can obtain maximum likelihood estimates of the parameters β_0 and β_1. As mentioned previously, the logistic model has been used both extensively and successfully to describe the probability of developing $(Y=1)$ some disease over a specified time period as a function of a risk factor X.

2.4. Multiple Regression Analysis [5]

The effect of some factor on a dependent or response variable may be influenced by the presence of other factors through effect modifications (i.e., interactions). Therefore, to provide a more comprehensive analysis, it is very desirable to consider a large number of factors and sort out which ones are most closely related to the dependent variable. In this section we discuss a multivariate method for risk determination. This method, which is multiple logistic regression analysis, involves a linear combination of the explanatory or independent variables; the variables must be quantitative with particular numerical values for each patient. A covariate or independent variable, such as a patient characteristic, may be dichotomous, polytomous, or continuous (categorical factors will be represented by dummy variables). Examples of dichotomous covariates are gender and presence/absence of certain co morbidity. Polytomous covariates include race and different grades of symptoms; these can be covered by the use of *dummy variables*. Continuous covariates include patient age and blood pressure. In many cases, data transformations (e.g., taking the logarithm) may be desirable to satisfy the linearity assumption.

Suppose that we want to consider *k* covariates simultaneously; the simple logistic model of previous section can easily be generalized and expressed as

$$\pi_i = \frac{1}{1+\exp\left[-\left(\beta_0 + \sum_{j=1}^{k}\beta_j x_{ji}\right)\right]} \qquad i = 1,2,\ldots,n \tag{11}$$

or, equivalently,

$$\ln\frac{\pi_i}{1-\pi_i} = \beta_0 + \sum_{j=1}^{k}\beta_j x_{ji} \tag{12}$$

This leads to the likelihood function

$$L = \prod_{i=1}^{n}\frac{\left[\exp\left(\beta_0 + \sum_{i=1}^{k}\beta_j x_{ji}\right)\right]^{y_i}}{1+\exp\left(\beta_0 + \sum_{j=1}^{k}\beta_j x_{ji}\right)} \qquad y_i = 0,1 . \tag{13}$$

Also similar to the univariate case, $\exp(\beta_i)$ represents one of the following:

1. The odds ratio associated with an exposure if X_i is binary;
2. The odds ratio due to a 1-unit increase if X_i is continuous.

After $\hat{\beta}_i$ and its standard error have been obtained, a 95% confidence interval for the odds ratio above is given by

$$\exp\left[\hat{\beta}_i \pm 1.96\,\mathrm{SE}\left(\hat{\beta}_i\right)\right] \tag{14}$$

These results are necessary in the effort to identify important risk factors for the binary outcome. Of course, before such analyses are done, the problem and the data have to be examined carefully. If some of the variables are highly correlated, one or fewer of the correlated factors are likely to be as good predictors as all of them; information from similar studies also has to be incorporated so as to drop some of these correlated explanatory variables. The use of products such as X_1X_2 and higher power terms such as X_1^2 may be necessary and can improve the goodness of fit. It is important to note that we are assuming a *(log) linear* regression model, in which, for example, the odds ratio due to a 1-unit increase in the value of a continuous X_i is independent of x. Therefore, if this *linearity* seems to be violated, the incorporation of powers of Xi should be seriously considered. The use of products will help in the investigation of possible effect modifications. Finally, there is the messy problem of missing data; most packaged programs would delete a subject if one or more covariate values are missing.

3. A study case

This study is about elaborate of a mathematical model that can predict the water quality in network distribution systems, and confirming the required national and international standards.

This prediction can be made by using statistical analysis of water quality, with specific instruments: correlation and logistic regression, these can be utilized in elaborate mathematical models.

For an existing network, if there are known data about the network structure, the materials of the equipment, the age of the network, it can be made a prediction about water quality in this network. In this kind of analysis can be use another type of data referring to network structure. The more data it is used, the more viable it is the study.

The developed mathematical model it was implemented in a study case that uses real data from ARA (Romanian Water Association, National Report 2012), about the Romanian networks, using SPSS program (Statistical Package for the Social Sciences), trail version.

These are data from 13 districts of Romania, exactly 23 towns, summing over 6.400 kilometres of water distribution network.

The first step of the algorithm is the analysis of all the data we have about the water distribution network and we have to select the ones to be used in the next statistical analysis. Once we decided which data will be used, *the second step of the algorithm* is to determinate the correlations between them. In our case, the poor structure, the age of the network and the nonconforming sample of water quality have the most significant correlation between them.

In the next table, table 1, it is presented the real data from ARA National Report[4]. These data were used in this study case.

Table1. A synthesized data from water distribution networks of Romania [4]

No.	District	Town	Km of network	Poor structure [%]	Age of network > 30 years [%]	Nonconforming sample of water quality [%]
1.	Buzău	Buzău	178,05	79,57	13,96	2
2.	Brăila	Brăila	400	44,4	50	16
3.	Caraș Severin	Caransebeș	390	65	53	6
4.	Constanța	Constanța	1351,9	70,43	71,4	2,2
5.	Dolj	Craiova	427	68	73	7,9
6.	Timiș	Deta	22,6	45,6	46	15
		Jimbolia	67,8	67,6	63,3	14,3
		Timișoara	617	27,7	63,4	2
7.	Bihor	Oradea	608	28,5	27,5	0
		Salonta	57,31	45,5	47,33	
8.	Mureș	Tg. Mureș	298,23	74,22	20	4,59
		Sighișoara	87	65	34	11
9.	Hunedoara	Hunedoara	187	95,3	17,5	0
		Hațeg	27,62	14	2	0
10.	Bistrița Năsăud	Bistrița	554,3	46	5	3,04
11.	Cluj	Cluj Napoca	626	30	2	5
		Dej	100,7	36,43	13,9	8
12.	Sălaj	Zalău	102,6	1,4	0	0,19
		Jibou	29,91	24,9	0	5,2
		Simleul Silvanei	37,85	41,6	0	3,3
		Cehu Silvanei	21,052	18,3	0	2,9
13.	Vaslui	Bârlad	220,9	62,2	15	0,57
		Negrești	18	100	40	0

From the data mentioned above we will use the following parameters:

- poor structure (PS [%]),
- age of network (AN [%]);
- nonconforming sample of water quality (NS [%]).

In the third step of the algorithm is been established what mathematical instruments can be used for statistical analysis to be made. We studied the repartition of these parameters, if the variables are normally distributed. In our case, nonconforming sample of water quality is a dependent variable and so it will be used logistic regression, which can be binomial or multinomial. The binomial logistic regression it will be used if there are only two categories of variables. The multinomial logistic regression will be used if there are three or mare categories of variables involved. The predictors are independent variables which can be score variables, nominal variables or a combination of these two. The best predictors are those who have the more significant *B-coefficients* (the regression coefficients). The *B* value from logistic regression it will be applied at the natural logarithm of a value named *chance ratio,*

which represents the rate of the frequencies of two alternative results. This logarithm is known by the name of *logit*, there for the term of logistic regression. The *chance ratio* is actually the probability of membership of one category of another.
With SPSS program we can generate classification table, and these tables will indicate predicting belonging to a certain category, based on predictor variable. In these tables it is precisely indicated the number of correct classifications, so they are a good indicator of the quality of prediction.
The fourth step of the algorithm is to calculate the correlation coefficient. With the help of logistic regression analysis, it can be indicated an improved correspondence between predicted category appurtenance and the real appurtenance in a category.
In this study, were considered the independent variables PS (poor structure) and AN (age of network) and the dependent variable NS (nonconforming sample of water quality).
As a result, it will be obtaining the correlation matrix between selected variables for the analysis. One cell of the table contains the value of correlation coefficient, the critical probability of the signification test and the number of values which were withheld for the calculus (after treated the missing samples).
It will be presented below the calculation of Pearson correlation coefficient for the specific data of this study. The study has as a base a real data referring to poor structure and the age of the networks from water distribution networks of Romania, and also the water quality, precisely the nonconforming sample of water quality, because from the many data referring to the actually state of a network, this in particular has the biggest impact to water quality.
In the next table, table 2, it is presented the result from the SPSS program, with the obtained correlations between the variables:

***Table 2.* The Pearson correlation coefficient**

Pearson Correlation		PS	AN	NS
PS		1	0,416*	0,064
AN		0,416*	1	0,385
NS		0,064	0,385	1

From the calculus presented in the table 2, it appears that the correlation (0,064) between PS (poor structure) and NS (nonconforming sample of water quality) it is lower than the correlation (0,385) between AN (age of network) and NS (nonconforming sample of water quality), which means that the age of network influence more the quality of water than poor structure of the distribution system.
If these corelations are calculated, *the five step of the algorithm* is to write the equation of the logistic regression. This equation it will be obtained from equation (12), by replacing the coefficients with those being calculated with SPSS program and represented in the following table 3.

***Table 3.* The variables from the logistic regression equation**

		B	S.E.	Wald	df	Sig.	Exp(B)
Pas 1[a]	Intercept	5.605	0.144	0.958	1	0.032	
	PS	0.0208	0.177	0.464	1	0.049	1.0210
	AN	-0.105	0.121	1.264	1	0.026	0.9003

Using the coefficients from table 3, the column of B coefficients, will be obtain the following logistic regression equation:

$$\ln(chanceR) = 5.605 + 0.0208 \cdot (PS) - 0.105 \cdot (AN) \tag{15}$$

where $chanceR$ it is notated the chance ratio from conform water quality and non-conform water quality and it is been given by the expression:

$$chanceR = e^{5.605+0.0208\cdot(PS)-0.105\cdot(AN)} \tag{16}$$

Using equations (12) and (15) results:

$$chanceR = \frac{p}{1-p} \tag{17}$$

Where p is the probability [%] of having a conform water quality in accord with law standards and it will be calculated from the next equation obtained from equation (17):

$$p = \frac{chanceR}{chanceR + 1} \tag{18}$$

So, if in a water distribution network, we know the precents about poor structure, respectively the precents of the age of network, it can be calculated, using the described model, the probability of the water quality to be in accord with the law standards.

In the next table, table 4, are presented the calculus made with SPSS program for determinate the probability to have a water quality in accord with law standards, when are known the age of the network and the poor structure of the materials.

***Table 4*. The chance ratio calculated with logistic regression model**

Nr. crt.	Town	PS (Poor structure) [%]	AN (Age of network) > 30 years [%]	Chance ratio calculated with the logistic regression model	The probability of the water quality to be in accord with law standards [%]
0	1	2	3	4	5
1.	Constanța	70,43	71,4	0,652427	39
2.	Dej	36,43	13,9	134,72	99,2
3.	Deta	45,6	46	5,6	84,8
4.	Salonta	45,5	47,33	4,863	82,9
5.	Brăila	44,4	50	3,591	78,2
6.	Caransebeș	65	53	4,023	80
7.	Cluj Napoca	30	2	411,2	99,76
8.	Tg. Mureș	74,2	20	155,8	99,3
9.	Timișoara	67,6	63,3	0,621	59
10.	Hațeg	14	2	294,8	99,66
11.	Craiova	68	73	0,524	34,4
12.	Oradea	28,5	27,5	27,39	96,48

In this table, table 4, on the fourth column, it was calculated the chance ratio, $chanceR$, using the equation (16) by substituting the values PS and AN with the correspondent ones from the table 4, columns (2) and (3) and in the five column it was been calculated the probability to have a water quality in accord with the law standards, using the equation (18) and the results from the fourth column of the same table.

From the results in the table 4 there are some notable:

- The probability of having water quality in accord with the law standards in the case of PS 70,4% in the network and AN 71,4% is only 39%.
- The probability in rising to 99,2% in case of PS 36,4% and AN 13,9%.

The last step of the algorithm is to made tests to see if the values generated by the model are in accord with the real data.

Using the elimination regressive conditional model, were analyzed differentiating characteristics of the quality of non-conformable water and the conformable water. PS was a significant predictor (Sig.=0.04). This model was almost the same exact, regarding conform water quality(correct in proportion of 90,3%), like in non-conformable water quality case (87,4%).

The Cox and Snell coefficient pseudo R^2 was 0,975 (table 5) indicating a good correlation between the model and the actual data.

***Table 5*. Pseudo R-Square**

Cox and Snell	0,975
Nagelkerke	0,979

As can be seen, there are two methods used for the calculation. All indicates the combined relation between predictors and categorical variables. The value 0 means that there is no multiple correlation, value 1 means there is a perfect multiple correlation. The values obtained indicate o pretty good prediction.

4. Conclusions

In order to evaluate the water quality in the water distribution network, we have applied the logistic regression method, from which we can conclude that the model which contains the variables PS, AN, NS explains its evolution. The correlations between variables are presented in table 2 and it can be observed that AN (age of network) has a big importance in water quality evaluation.

Mathematical models which have the capacity to accurately describe the correlations between poor structure (PS), age of network (AN) and nonconforming sample of water quality (NS) are excellent instruments for analysis and interpretation of experimental data as well.

Using mathematical models, it can be accurately describing the correlation between different characteristics of water distribution network, such as poor structure or age of network, and the quality of water. Statistical analysis, using logistic regression and correlations, is an excellent instrument for interpretation of experimental data and for prediction in water quality for distribution networks.

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LAPLACE TRANSFORM METHOD IN THERMODYNAMICS

Narcisa Teodorescu
Department of Mathematics and Computer Science
Technical University of Civil Engineering Bucharest
Bd. Lacul Tei 124, sector 2, 38RO-020396 Bucharest, Romania
E-mail: narcisa.teodorescu@utcb.ro
Vlad-Daniel Lupea
Faculty of Building Services Engineering
Technical University of Civil Engineering Bucharest
Bd. Pache Protopopescu 66, Bucharest, Romania
E-mail: vlad-daiel.lupea@student.utcb.ro

Abstract: Laplace transform method is very important tools in engineering but also in mechanics, biology, economy. In this article, we will present how to use this method to obtain the temperature distribution and the heat flow along a rod connected between two thermal reservoirs.
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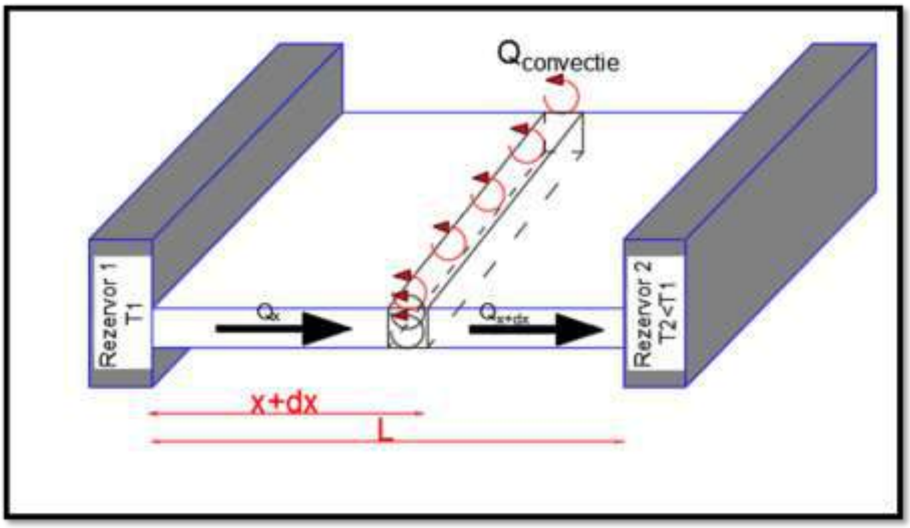
Key words: Laplace transforms, heat flow

1. Introduction

The Laplace transform expresses the conflict between pure and applied mathematics splendidly. Laplace transforms only come alive when they are used to solve real problems.
Heat is one of the forms of energy which flows by virtue of temperature gradient from a point at a higher temperature to another point at a lower temperature. In many situations of practical importance, it is generated at a uniform rate itself within the conducting medium. The rate of generation of heat has to be controlled, otherwise, the growth in temperature resulting from it leads to the failure of the conducting medium. The distribution of temperature within the uniform conducting rod and the rate of loss of heat by the uniform conducting rod to its surroundings assume significant importance in the construction of thermal systems. There are three modes which transfer heat from one point to another namely: conduction, convection and radiation.

2. The physical problem

We consider a conducting rod having length L, uniform area of cross-section ‘ a ’ and perimeter ‘P’ connected between two thermal reservoirs R_1 and R_2 at points $x = 0$ and $x = L$. The thermal reservoirs are maintained at fixed temperatures T_1 and T_2 respectively. If the temperature of the surroundings of the uniform conducting rod is denoted by T_s and is kept constant, then the convective heat will flow from the rod to the surroundings which lead to a loss of heat from the uniform conducting rod to the surroundings.
Conduction of heat through a uniform rod with heat loss by convection is shown in the following figure:



Let us consider an infinite small section of thickness dx of the conducting rod, located at a distance of from the reservoir R_1.

Let $T(x)$ the temperature of the uniform conducting rod and is a function of variable x. It is assumed to be constant for the infinite small section of the uniform conducting rod.

Heat conducted into the element at plane x is given by

$$\left(\dot{Q}_x\right)_{in} = -\lambda a \frac{\partial T(x)}{\partial x},$$

Heat conducted out of the element at the plane $(x+dx)$ is given by

$$\left(\dot{Q}_{x+dx}\right)_{out} = \left(\dot{Q}_x\right)_{in} + dx\frac{\partial}{\partial x}\left(\dot{Q}_x\right)_{in}$$

Expanding using Taylor's series and retaining only first two terms, we get

$$\left(\dot{Q}_{x+dx}\right)_{out} = -\lambda a\left(\frac{\partial T(x)}{\partial x} + \left(\frac{\partial^2 T(x)}{\partial x^2}\right)dx\right)$$

Heat convected of the element of length dx between the planes at x and $(x+dx)$ is given by

$$\dot{Q}_{convected} = \alpha(Pdx)(T(x)-T_s)$$

Where λ represents the thermal conductivity of the material of uniform conducting rod and α represents the coefficient of heat transfer by convection.

Making use steady-state heat balance, we can write

$$\left(\dot{Q}_x\right)_{in} = \left(\dot{Q}_{x+dx}\right)_{out} + \dot{Q}_{convected}.$$

Using the above equations we get

$$\frac{\partial^2 T(x)}{\partial x^2} - \frac{\alpha P}{\lambda a}(T(x)-T_s) = 0$$

Let us substitute

$$\theta(x) = T(x) - T_s \quad \text{and} \quad \frac{\alpha P}{\lambda a} = \beta^2$$

Thus we will obtain the equation

$$\frac{\partial^2 \theta(x)}{\partial x^2} - \beta^2\theta(x) = 0 \qquad (1)$$

3. Presentation of the mathematical problem

To solve equation (1) by Laplace transform, we need to write the boundary conditions as given below:

At $x = 0$, $T = T_1$ and at $x = L$, $T = T_2$.

In terms of temperature $\theta(x)$, we can write $\theta(0) = \theta_1$ and $\theta(L) = \theta_2$.

Taking Laplace transform of equation (1), we get

$$L\left[\frac{\partial^2 \theta(x)}{\partial x^2}\right] - \beta^2 L[\theta(x)] = 0$$

This equation gives

$$q^2 \bar{\theta}(q) - q\theta(0) - \frac{\partial \theta(0)}{\partial x} - \beta^2 \bar{\theta}(q) = 0 \tag{2}$$

Applying boundary condition $\theta(0) = \theta_1$, equation (2) becomes

$$q^2 \bar{\theta}(q) - q\theta_1 - \frac{\partial \theta(0)}{\partial x} - \beta^2 \bar{\theta}(q) = 0 \tag{3}$$

Let us substitute $\frac{\partial \theta(0)}{\partial x} = \varepsilon$ (it is a constant).

Equation (3) becomes

$$q^2 \bar{\theta}(q) - \beta^2 \bar{\theta}(q) = \varepsilon + q\theta_1$$

$$\text{Or } \bar{\theta}(q) = \frac{\varepsilon}{(q^2 - \beta^2)} + \frac{q\theta_1}{(q^2 - \beta^2)} \tag{4}$$

Taking inverse Laplace transform of equation (4), we get

$$\theta(x) = \frac{\varepsilon}{\beta} \sinh \beta x + \theta_1 \cosh \beta x \tag{5}$$

To find the value of constant ε, applying boundary condition $\theta(L) = \theta_2$, equation (5) provides

$$\theta_2 = \frac{\varepsilon}{\beta} \sinh \beta L + \theta_1 \cosh \beta L$$

Upon rearranging and simplification of above equation, we get

$$\theta(x) = \frac{\theta_1 \sinh \beta(L - x) + \theta_2 \sinh \beta x}{\sinh \beta L}$$

Using $\theta(x) = T(x) - T_s$ and $\beta = \sqrt{\frac{\alpha P}{\lambda a}}$, we can write

$$T(x) = T_s + \frac{(T_1 - T_s)\sinh\sqrt{\frac{\alpha P}{\lambda a}}(L - x) + (T_2 - T_s)\sinh\sqrt{\frac{\alpha P}{\lambda a}}x}{\sinh\sqrt{\frac{\alpha P}{\lambda a}}L} \tag{6}$$

Equation (6) provides the distribution of temperature along the length of the rod.

The most important parameter is the total amount of heat that can be removed by the uniform conducting rod from the reservoir at a higher temperature T_1 and lost it to the surroundings.

The total heat (Q) emitted from the surface of the uniform conducting rod to its surroundings can be calculated by integrating the expression for heat convected from the surface of an infinite small section of the uniform conducting rod to its surroundings:

$$Q = \int_0^L \alpha P (T(x) - T_s) dx$$

Or $Q = \int_0^L \alpha P \theta(x) dx$

Or $Q = \int_0^L \alpha P \frac{\theta_1 \, sinh \beta(L - x) + \theta_2 \, sinh \beta x}{sinh \beta L} dx$

On solving the integration and applying the limits, we get

$$Q = \alpha P \frac{(\theta_1 + \theta_2)(cosh \beta L - 1)}{\beta \, sinh \beta L} \tag{7}$$

Using $\beta = \sqrt{\frac{\alpha P}{\lambda a}}$ in equation (7), we get

$$Q = \alpha P \frac{(\theta_1 + \theta_2)\left(cosh \sqrt{\frac{\alpha P}{\lambda a}} L - 1\right)}{\sqrt{\frac{\alpha P}{\lambda a}} \, sinh \sqrt{\frac{\alpha P}{\lambda a}} L}$$

Or $Q = \sqrt{\alpha P \lambda a} \frac{(\theta_1 + \theta_2)\left(cosh \sqrt{\frac{\alpha P}{\lambda a}} L - 1\right)}{sinh \sqrt{\frac{\alpha P}{\lambda a}} L}$

Or $Q = \sqrt{\alpha P \lambda a} \frac{(T_1 + T_2 - 2T_s)\left(cosh \sqrt{\frac{\alpha P}{\lambda a}} L - 1\right)}{sinh \sqrt{\frac{\alpha P}{\lambda a}} L}$ (8)

Equation (8) provides the total heat emitted from the surface of the uniform conducting rod to its surroundings and reveal that the flow of heat can be increased by increasing the surface of the uniform conducting rod across which the convection of heat occurs.

4. Conclusion

The Laplace transform has a very useful characteristic, namely that many relations and operations currently performed on the original correspond to simpler relations and operations performed on the image. The temperature distribution in the conducting rod and the rate of its heat loss are important characteristics in the construction of thermal systems.

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ON INTEGRAL INEQUALITIES

DANIEL TUDOR, DAN CARAGHEORGHEOPOL AND MARIANA ZAMFIR

ABSTRACT. In this paper, we aim to solve some problems of real analysis involving integral inequalities.

Mathematics Subject Classification (2010):26D15
Key words: integral, inequality

1. Introduction

We begin this paper by presenting the integral variant of some well-known inequalities, namely: Cauchy-Bunyakowsky-Schwarz inequality, Holder's inequality and Young's inequality. In the last section, we will solve some examples using these tools.

2. Cauchy-Bunyakowsky-Schwarz inequality for integrals

For two integrable functions f, g on a closed interval $I = [a, b]$, it follows that

$$\left(\int_a^b f(t)g(t)dt\right)^2 \leq \int_a^b f^2(t)dt \int_a^b g^2(t)dt$$

3. Holder's inequality

Let S be a measurable subset of $\mathbf{R_n}$ and f, g are measurable real- or complex-valued functions on S, then Hölder's inequality is

$$\int_S |f(x)g(x)|dx \leq \left(\int_S |f(x)|^p dx\right)^{\frac{1}{p}} \left(\int_S |g(x)|^p dx\right)^{\frac{1}{p}}$$

for all $p, q \in (0, 1)$ with $\frac{1}{p} + \frac{1}{q} = 1$.

4. Young's inequality

Let f be a real-valued, continuous, and strictly increasing function on $[0, c]$ with $c > 0$. If $f(0) = 0$, a in $[0, c]$, and b in $[0, f(c)]$, then

$$\int_0^a f(x)dx + \int_0^b f^{-1}(x)dx \geq ab,$$

where f^{-1} is the inverse function of f. Equality holds iff $b = f(a)$.

5. Examples[1]

Problem 1. Let

$$A = \left\{ f \mid f : [0,1] - > \mathbf{R} \; continuous \; with \int_0^1 f^2(x)dx \leq 1 \right\}.$$

Determine

$$M = max \left\{ \int_0^1 xf(x) \mid f \in A \right\}.$$

Solution. Using the Cauchy-Bunyakowsky-Schwarz inequality, we have

$$\left(\int_0^1 xf(x)dx \right)^2 \leq \int_0^1 f^2(x)dx \int_0^1 x^2 dx \leq 1 \cdot \frac{1}{3} = \frac{1}{3},$$

for any $f \in A$. Therefore, $M \leq \frac{1}{\sqrt{3}}$. For the function $f(x) = \sqrt{3}x$, we get $\int_0^1 xf(x)dx = \frac{1}{\sqrt{3}}$, therefore $M = \frac{1}{\sqrt{3}}$.

Problem 2. Find all continuous functions $f : [0,1] \to [0, \infty)$ such that

$$\int_0^1 f(x)dx \cdot \int_0^1 f^2(x)dx \cdot \ldots \cdot \int_0^1 f^{2020}(x)dx = \left(\int_0^1 f^{2021}(x)dx \right)^{1010}$$

Solution. For any $k \in \mathbf{N}$, we have, using Holder's inequality,

$$\left(\int_0^1 (f^k(x))^{\frac{k+1}{k}} dx \right)^{\frac{k}{k+1}} \cdot \left(\int_0^1 1dx \right)^{\frac{1}{k+1}} \geq \int_0^1 f^k(x)dx$$

which can be rewritten as

$$\left(\int_0^1 (f^{k+1}(x))dx \right)^k \geq \left(\int_0^1 (f^k(x))dx \right)^{k+1} \tag{5.1}$$

Next, let $a_k = \left(\int_0^1 (f^{k+1}(x))dx \right)^k$. Then, we can rewrite the above inequality as

$$a_k \geq a_{k-1} \cdot \left(\int_0^1 (f^k(x))dx \right)^2.$$

Repeatedly iterating thisinequality until we get to the initial term yields

$$\begin{aligned} a_k &\geq a_{k-1} \cdot \left(\int_0^1 (f^k(x))dx \right)^2 geqa_{k-2} \cdot \left(\int_0^1 (f^{k-1}(x))dx \right)^2 \cdot \left(\int_0^1 (f^k(x))dx \right)^2 \geq \ldots \\ &\geq a_1 \cdot \left(\int_0^1 (f^2(x))dx \right)^2 \cdot \ldots \cdot \left(\int_0^1 (f^k(x))dx \right)^2 \\ &\geq \left(\int_0^1 (f(x))dx \right)^2 \cdot \left(\int_0^1 (f^2(x))dx \right)^2 \cdot \ldots \cdot \left(\int_0^1 (f^k(x))dx \right)^2. \end{aligned}$$

It follows that

$$\left(\int_0^1 (f^{k+1}(x))dx\right)^{\frac{k}{2}} \geq \int_0^1 f(x)dx \cdot ... \cdot \int_0^1 f^k(x)dx.$$

The above inequalities become equalities iff f is a constant function. Consequently, for $k = 2020$, the functions that satisfy the given equality are the constant functions.

Problem 3. Let $f : [0,1]ß[0,1]$ be a continuous and bijective function with $f(0) = 0$. Prove that, for all $\alpha \geq 0$,

$$(\alpha+2)\int_0^1 x^\alpha(f(x)+f^{-1}(x))dx \leq 2$$

Solution. f is a continuous and bijective function, therefore it is strictly monotonous. Since $f(0) = 0$, it must follow that $f(1) = 1$ and f is strictly increasing. Using Young's inequality, we have

$$\int_0^x f(t)dt + \int_0^x f^{-1}(t)dt \geq x^2,$$

for all $x \in [0,1]$. This can be rewritten as

$$\int_0^x \left(f(t)+f^{-1}(t)-2t\right) dt \geq 0.$$

Define $G : [0,1] \to \mathbf{R}$ by $G(x) = \int_0^x \left(f(t)+f^{-1}(t)-2t\right) dt$. Then $G(1) = G(0) = 0$ and $G(x) \geq 0$ for all $x \in [0,1]$. Furthermore, G is differentiable and for all $\alpha \geq 0$, we have

$$\int_0^1 x^\alpha(f(x)+f^{-1})dx = 2\int_0^1 x^{\alpha+1}dx + \int_0^1 x^\alpha(f(x)+f^{-1}(x)-2x)dx =$$

$$\frac{2}{\alpha+2} + \int_0^1 x^\alpha G'(x)dx = \frac{2}{\alpha+2} + \lim_{\alpha\to 0}\left(G(1) - G(\alpha)\cdot\alpha^\alpha - \int_\alpha^1 \alpha^{\alpha-1}G(x)dx\right) \leq \frac{2}{\alpha+2}.$$

In conclusion, multiplying both sides by $\alpha+2$, we reach the desired conclusion,

$$(\alpha+2)\int_0^1 x^\alpha(f(x)+f^{-1}(x))dx \leq 2.$$

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Department of Mathematics and Computer Science, Technical University of Civil Engineering Bucharest, Bucharest, Romania
Email address: daniel.tudor@utcb.ro

Department of Mathematics and Computer Science, Technical University of Civil Engineering Bucharest, Bucharest, Romania and "Titu Margulescu" Institute of Physical Chemistry of the Romanian Academy, Bucharest, Romania
Email address: dancaraghe@gmail.com

Department of Mathematics and Computer Science, Technical University of Civil Engineering Bucharest, Bucharest, Romania
Email address: zamfirvmariana@yahoo.com

ABOUT GENERALIZATION OF DOWSON RESULT FOR RESTRICTIONS AND QUOTIENTS OF SPECTRAL OPERATORS

Mariana Zamfir
Department of Mathematics and Computer Science
Technical University of Civil Engineering of Bucharest
Bd. Lacul Tei 122-124, Sector 2, 38RO-020396 Bucharest, Romania
E-mail: zamfirvmariana@yahoo.com

Abstract: In [3], [4], H.R. Dowson has shown that, for a spectral operator T and for a closed subspace Y invariant to T, the restriction operator $T\,|\,Y$ and the quotient operator $\dot{T}$ are spectral operators if the spectrum $\sigma(T\,|\,Y)$ is totally disconnected (i.e. $\dim\sigma(T\,|\,Y)=0$). In this paper, we prove that these assertions occur if the intersection of the spectra $\sigma(T\,|\,Y)$ and $\sigma(\dot{T})$ is totally disconnected, i.e. $\dim\left(\sigma(T\,|\,Y)\cap\sigma(\dot{T})\right)=0$.

Mathematics Subject Classification (2010): 47B47, 47B40.
Key words: spectral ($S-$spectral) operator; spectral measure; restriction and quotient of an operator; totally disconnected set.

1. Introduction

The purpose of the present paper is to investigate some results of the theory of spectral operators in arbitrary complex Banach spaces in a systematic way. The restrictions and the quotients of spectral operators with respect to closed invariant subspaces are presented here, and it is shown that they are also spectral (respectively, S-spectral) operators.

To make our work self-contained, the first section may be thought of as preliminaries. Let us briefly remind several notions and definitions which will be further needed.

Throughout our paper, we shall denote by **C** the complex plane, by $\mathbf{B}(X)$ the Banach algebra of all linear bounded operators on a given complex Banach space X, and by $\mathcal{P}_X$ the set of all projectors on X.

If Y is a linear closed subspace of X invariant to an operator $T\in\mathbf{B}(X)$ (i.e. $TY\subseteq Y$), then $T\,|\,Y$ is the restriction operator of T to Y and $\dot{T}$ means the quotient operator induced by T on the quotient space $\dot{X}$.

Furthermore, if $T\in\mathbf{B}(X)$ and $Y\subseteq X$ is a closed subspace invariant to T, then, for any $x\in X$, we have $\dot{x}=x+Y=\{z\in X\ ;\ z-x\in Y\}$, $\dot{X}=X/Y=\{\dot{x}\ ;\ x\in X\}$, and the linear application $\varphi: X\to\dot{X}$, $\varphi(x)=\dot{x}=x+Y$, is the canonical map of the quotient space $\dot{X}$.

As usual, the *spectrum* of an operator $T\in\mathbf{B}(X)$ is denoted by $\sigma(T)=\mathbf{C}\setminus\rho(T)$ and it is defined as the set of all complex numbers $\lambda\in\mathbf{C}$ for which the operator $\lambda I-T$ is no inversable in $\mathbf{B}(X)$, where the *resolvent set* $\rho(T)$ consists of all complex numbers $\lambda\in\mathbf{C}$ for which the operator $\lambda I-T$ is inversable in $\mathbf{B}(X)$.

Definition 1.1. ([2], [5]) An operator $T \in \mathbf{B}(X)$ is said to have the *single-valued extension property* if for any analytic function $f : D \to X$ (where $D \subset \mathbf{C}$ is open), with the condition $(\lambda I - T)\, f(\lambda) \equiv 0$, it follows that $f(\lambda) \equiv 0$.

For the operator $T \in \mathbf{B}(X)$ having the single-valued extension property and for $x \in X$, we denote by $\rho_T(x)$ the set of all elements $\xi \in \mathbf{C}$, with the property that there is an X-valued analytic function $\lambda \to x_T(\lambda)$, defined on a neighborhood of ξ, which verifies the condition $(\lambda I - T)\, x_T(\lambda) \equiv x$.

The open set $\rho_T(x)$ is called the *local resolvent set of* x *with respect to* T and the closed set $\sigma_T(x) = \mathbf{C} \setminus \rho_T(x)$ is called the *local spectrum of* x *with respect to* T.

We also have the inclusions $\rho(T) \subset \rho_T(x)$, $\sigma_T(x) \subset \sigma(T)$, and also we denote by

$$X_T(F) = \{x \in X \ ;\ \sigma_T(x) \subset F,\ F \subset \mathbf{C}\}.$$

Definition 1.2. ([3]) A subset of the complex plane $\mathbf{C}$ is called *totally disconnected* (*of dimension* 0) if and only if the connected component of each point is the set consisting of the point itself.

Dunford's work on restrictions and quotients of spectral operators with respect to a closed invariant subspace ([3], [4]) is summarized in Section 2. More over, Dowson has shown that the restriction $T\,|\,Y$ and the quotient $\dot{T}$ of a spectral operator $T \in \mathbf{B}(X)$ are spectral operators if the spectrum $\sigma(T\,|\,Y)$ is totally disconnected.

In Section 3, we show that the restriction $T\,|\,Y$ and the quotient $\dot{T}$ of a spectral operator $T \in \mathbf{B}(X)$ are spectral operators if the intersection $\sigma(T\,|\,Y) \cap \sigma(\dot{T})$ is totally disconnected.

2. Dowson Result for restrictions and quotients of spectral operators

Definition 2.1. ([2]) For the complex plane $\mathbf{C}$, we denote by $\mathcal{B}(\mathbf{C})$ the family of all Borelian sets of $\mathbf{C}$.

We call an application $E : \mathcal{B}(\mathbf{C}) \to \mathcal{P}_X$ *spectral measure* if the following conditions are fulfilled:

1) $E(\mathbf{C}) = I_X$ and $E(\varnothing) = 0_X$;
2) $E(B_1 \cap B_2) = E(B_1)\, E(B_2)$, $B_1, B_2 \in \mathcal{B}(\mathbf{C})$;
3) $E\left(\bigcup_{m=1}^{\infty} B_m\right) x = \sum_{m=1}^{\infty} E(B_m)\, x$, $B_m \in \mathcal{B}(\mathbf{C})$, $B_m \cap B_p = \varnothing$, $m \neq p$, $x \in X$;
4) $\sup_{B \in \mathcal{B}(\mathbf{C})} \| E(B) \| < \infty$.

An operator $T \in \mathbf{B}(X)$ is called a *spectral operator* if there exists a spectral measure $E : \mathcal{B}(\mathbf{C}) \to \mathcal{P}_X$ such that the following conditions are satisfied:

5) $T\,E(B) = E(B)\,T$, $B \in \mathcal{B}(\mathbf{C})$;
6) $\sigma(T\,|\,E(B)X) \subset \bar{B}$, $B \in \mathcal{B}(\mathbf{C})$.

The spectral measure E verifying 5) and 6) is uniquely determined by T and it is called *the spectral measure* of T ([2]).

Proposition 2.1. ([3]) Let $T \in \mathbf{B}(X)$ be a spectral operator with its spectral measure E and let Y be a closed linear subspace of X invariant to T.
Then the following statements are established:

1. If the restriction operator $T \mid Y$ is a spectral operator, then Y is also invariant to the spectral measure E (i.e. $T E(B) X \subset E(B) X$, $B \in \mathcal{B}(\mathbf{C})$), $E \mid Y$ is the spectral measure of $T \mid Y$, and $\sigma(T \mid Y) \subset \sigma(T)$.
2. If Y is invariant to both T and E, then the restriction operator $T \mid Y$ is a spectral operator, with $E \mid Y$ its spectral measure.

Proposition 2.2. ([4]) Let $T \in \mathbf{B}(X)$ be a spectral operator, with the spectral measure E and let Y be a closed linear subspace of X invariant to T.
Then the following assertions are verified:

1. If the quotient operator $\dot{T}$ is a spectral operator having the spectral measure $\dot{E}$, then Y is also invariant to E, $\dot{E}$ is the spectral measure induced by E on the quotient space $\dot{X} = X / Y$, and $\sigma(\dot{T}) \subset \sigma(T)$.
2. If Y is invariant to both T and E, then the quotient operator $\dot{T}$ is a spectral operator on $\dot{X}$, having the spectral measure $\dot{E}$.

Theorem 2.1. ([4]) Let $T \in \mathbf{B}(X)$ be a spectral operator, with its spectral measure E and let $Y \subseteq X$ be a closed linear subspace invariant to T.
Then the following statements are equivalent:

1. Y is also invariant to the spectral measure E.
2. the restriction operator $T \mid Y$ is a spectral operator, with the spectral measure $E \mid Y$.
3. the quotient operator $\dot{T}$ induced by T on the quotient space $\dot{X} = X / Y$ is a spectral operator, with the spectral measure $\dot{E}$ induced by E on $\dot{X}$.

Theorem 2.2. ([3], [4]) Let $T \in \mathbf{B}(X)$ be a spectral operator with totally disconnected spectrum (i.e. $\dim \sigma(T) = 0$). Then the restriction $T \mid Y$ and the quotient $\dot{T}$ are spectral operators, for any closed linear subspace Y of X invariant to T.

Theorem 2.3. ([3], [4]) Let $T \in \mathbf{B}(X)$ be a spectral operator and let Y be a closed linear subspace of X invariant to T such that the spectrum $\sigma(T \mid Y)$ is totally disconnected (i.e. $\dim \sigma(T \mid Y) = 0$). Then the restriction $T \mid Y$ and the quotient $\dot{T}$ are spectral operators.

Theorem 2.4. ([4]) Let X be a reflexive Banach space, let $T \in \mathbf{B}(X)$ be a spectral operator, and let Y be a closed linear subspace of X invariant to T such that the spectrum $\sigma(\dot{T})$ is totally disconnected (i.e. $\dim \sigma(\dot{T}) = 0$). Then the restriction $T \mid Y$ and the quotient $\dot{T}$ are spectral operators.

3. Generalization of Dowson Result for restrictions and quotients of spectral operators

Proposition 3.1. ([8]) Let $T \in \mathbf{B}(X)$ be a spectral operator having the spectral measure E, and let Y be a closed linear subspace of X invariant to T, with $X_T\left(\sigma(T\,|\,Y) \setminus \sigma(\dot{T})\right) \subset Y$. Let also denote by T_Y the restriction operator $T\,|\,Y$ and by S the set $\sigma(T\,|\,Y) \cap \sigma(\dot{T})$. Then the following assertion holds

$$T\,|\,Y = T_Y = T_1 \oplus T_2$$

where T_1 and T_2 are two linear bounded operators satisfying the properties:

1. $T_1 = T_Y\,|\,E\left(\sigma(T\,|\,Y) \setminus \sigma(\dot{T})\right) Y$ is a spectral operator.
2. $\sigma(T_2) = \sigma\left(T_Y\,|\,E(S)\,Y\right) \subset \tilde{S} \cap \sigma(T_Y)$

($\tilde{S} = \mathbf{C} \setminus D^{\infty}$, where D^{∞} is the unbounded component of $\mathbf{C} \setminus S$).

Proposition 3.2. ([8]) Let $T \in \mathbf{B}(X)$ be a spectral operator with its spectral measure E, let Y be a closed linear subspace of X invariant to T, and let S be the set $\sigma(T\,|\,Y) \cap \sigma(\dot{T})$. Then, it follows that

$$\dot{T} = \dot{T}_1 \oplus \dot{T}_2$$

where the quotient operators $\dot{T}_1$ and $\dot{T}_2$ have the properties:

1. $\dot{T}_1 = \dot{T}\,|\,\varphi\left(E\left(\sigma(\dot{T}) \setminus \sigma(T\,|\,Y)\right) X\right)$ is a spectral operator.
2. $\sigma(\dot{T}_2) = \sigma\left(\dot{T}\,|\,\varphi\left(E\left(\sigma(T\,|\,Y)\right) X\right)\right) \subset S$.

Theorem 3.1. ([8]) Let $T \in \mathbf{B}(X)$ be a spectral operator having the spectral measure E, let Y be a closed linear subspace of X invariant to T, with $X_T\left(\sigma(T\,|\,Y) \setminus \sigma(\dot{T})\right) \subset Y$, and let S be the set $\sigma(T\,|\,Y) \cap \sigma(\dot{T})$ such that $\tilde{S} = S$. Then the restriction $T\,|\,Y$ and the quotient $\dot{T}$ are $S-$spectral operators.

Theorem 3.2. ([8]) Let $T \in \mathbf{B}(X)$ be a spectral operator and let Y be a closed linear subspace of X invariant to T such the set $S = \sigma(T\,|\,Y) \cap \sigma(\dot{T})$ is totally disconnected (i.e. $\dim S = \dim \sigma(T\,|\,Y) \cap \sigma(\dot{T}) = 0$). Then the restriction $T\,|\,Y$ and the quotient $\dot{T}$ are spectral operators.

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ON A PROPERTY OF THE NORMAL POLYNOMIALS IN Q [X]

SEVER ACHIMESCU, VICTOR ALEXANDRU

ABSTRACT. A polynomial in **Q** [**X**] is said to be normal if it is irreducible and its splitting field is generated by one of its roots.

First we give a new proof of a well known arithmetic characterization of the fact that a polynomial $f \in \mathbf{Q}[\mathbf{X}]$ is normal:

Theorem Let $f \in \mathbf{Q}[\mathbf{X}]$ be irreducible. The following statements are equivalent:

(1) f is normal

(2) There is a finite set $A \subset \mathbf{N}$ such that for any prime $p \in \mathbf{N} - A$ satisfying: p does not divide $disc(f)$ and the image $\bar{f}$ of f in $\mathbf{F_p}[\mathbf{X}]$ has a root in $\mathbf{F_p}$ it follows that $\bar{f}$ can be factorized as a product of linear factors in $\mathbf{F_p}[\mathbf{X}]$.

Second we discuss various examples including polynomials with applications in engineering, such as $T_n(X) = cos(n \cdot arccos(X))$.

Mathematics Subject Classification (2010):
Key words:

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DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, TECHNICAL UNIVERSITY OF CIVIL ENGINEERING BUCHAREST, BUCHAREST, ROMANIA

Email address: sever.achimescu@utcb.ro

SOME REMARKS ON DE BRANGES LEMMA

GHEORGHE BUCUR

Abstract. After a short history on the subject we formulate a similar assertion in a very large context.
Applications of this assertion are also considered.

Mathematics Subject Classification (2010): 41A10, 46J10
Key words: Nachbin family, weighted space, radon measure, dual space, polar set, extreme point, convex cone, ordered ideal, antisymmetric set with respect to an algebra, antisymmetric set with respect to a pair $(\mathcal{M}, \mathcal{W})$ or $(\mathcal{C}, \mathcal{W})$, antialgebraic set with respect to a pair $(\mathcal{M}, \mathcal{W})$.

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Faculty of Mathematics and Computer Science, University of Bucharest, Bucharest, Romania/ Institute of Mathematics of the Romanian Academy
Email address: gheorghebucur42@gmail.com

CHEN INEQUALITIES FOR SPACELIKE SUBMANIFOLDS IN STATISTICAL MANIFOLDS OF TYPE PARA-KÄHLER SPACE FORMS

SIMONA DECU (MARINESCU)

Abstract. We prove some inequalities between intrinsic and extrinsic curvature invariants, namely involving the Chen first invariant and the mean curvature of totally real and holomorphic spacelike submanifolds in statistical manifolds of type para-Kähler space forms. Furthermore, we investigate the equality cases of these inequalities. We consider also an illustrative example. This talk is based on a joint work with S. Haesen [2].
Key words: Chen inequality; statistical manifold of type para-Kähler space form

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Department of Applied Mathematics, Bucharest University of Economic Studies, Bucharest, Romania
Email address: simona.decu@gmail.com, simona.marinescu@csie.ase.ro

PYTHAGOREAN SUBMANIFOLDS AS APPLICATIONS OF MATRIX PYTHAGOREAN TRIPLES

MUHITTIN EVREN AYDIN

Abstract. This talk is based on jointly works with Adela Mihai (Technical University of Civil Engineering Bucharest and Transilvania University of Brasov) and Cihan Ozgur (Izmir Democrasy University) [1, 2, 3]. A Pythagorean surface in 3-dimensional Euclidean space is a complete surface where the three fundamental forms are a matrix Pythagorean triple. We know that such a surface becomes an ordinary sphere whose Gaussian curvature is the Golden Ratio. In this talk, we introduce and investigate the so-called Pythagorean submanifolds, a particular class of isometric immersions into Euclidean spaces.

Mathematics Subject Classification (2010): see http://www.ams.org/msc/
Key words: Matrix Pythagorean triple, Pythagorean immersion, principal curvature.

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Department of Mathematics, Faculty of Science, Firat University, Elazig, Turkey
Email address: meaydin@firat.edu.tr

SLANT CURVES ON SOME ALMOST CONTACT METRIC MANIFOLDS

ŞABAN GÜVENÇ, CIHAN ÖZGÜR

Abstract. We consider curves with C-parallel and C-proper mean curvature vector fields in trans-Sasakian manifolds and non-Sasakian contact metric manifolds. In particular, we obtain the curvature characterizations of slant curves in trans-Sasakian manifolds and Legendre curves in non-Sasakian contact metric manifolds. We give some examples of these kinds of curves.

Mathematics Subject Classification (2010): 53C25, 53C40, 53A05
Key words: Trans-Sasakian manifold, contact metric manifold, Legendre curve, slant curve, C-parallel mean curvature vector field, C-proper mean curvature vector field.

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Balikesir University, Faculty of Arts and Sciences, Department of Mathematics, 10145, Balikesir, Turkey
Email address: sguvenc@balikesir.edu.tr

İzmir Democracy University, Department of Mathematics, Faculty of Arts and Sciences, Karabağlar, İzmir, Turkey
Email address: cihan.ozgur@idu.edu.tr

GEOMETRIC INEQUALITIES ON ISOTROPIC SPACELIKE SUBMANIFOLDS IN PSEUDO-RIEMANNIAN SPACE FORMS

MARIUS MIREA AND ALEXANDRU CIOBANU

Abstract. Both spacelike and isotropic submanifolds of pseudo-Riemannian spaces have interesting properties, studied in Mathematics and Physics as well. We will present new inequalities for isotropic spacelike submanifolds of pseudo-Riemannian space forms, respectively a corresponding inequality of a generalized Euler inequality and a Ricci inequality.

Mathematics Subject Classification (2010): 53C40, 53C25 (Geometry)
Key words: Spacelike submanifolds, isotropic submanifolds, pseudo-Riemannian space forms.

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Doctoral School of Mathematics, University of Bucharest, Romania
Email address: marius-ion.mirea@my.fmi.unibuc.ro

Interdisciplinary Doctoral School, Transilvania University of Braşov, Romania
Email address: alexandru.ciobanu@unitbv.ro

SEMI-INVARIANT $\xi^{\perp}-$RIEMANNIAN SUBMERSIONS ADMITTING RICCI SOLITON

RAMAZAN SARI AND İNAN ÜNAL

Abstract. In this paper, we work Ricci solitons on semi invariant Riemannian submersions from a contact manifold. We investigate the foliations of the submersions as a Ricci soliton. Moreover, we examine Einstein conditions.

Mathematics Subject Classification (2010): 53C05, 53C10, 53C25
Key words: Riemannian submersion; semi-invariant Riemannian submersion; Ricci soliton. space forms.

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Gümüshaciköy Hasan Duman Vocational School, Amasya University, Amasya, Turkey
Email address: ramazan.sari@amasya.edu.tr,

Department of Computer Engineering, University of Munzur, Tunceli, Turkey
Email address: inanunal@munzur.edu.tr

A STUDY OF CONFORMAL RIEMANNIAN MAPS

RAKESH KUMAR

Abstract. A major flaw in Riemannian geometry is the shortage of suitable kind of smooth maps from one manifold to another that compare their geometric properties. The study of such smooth maps between Riemannian manifolds plays a role of central importance for comparing structure between two manifolds. Due to wider applications in modern physics and geometry, conformal Riemannian maps have been introduced as the generalization of conformal immersions, horizontally conformal submersions and Riemannian maps. We study conformal Riemannian maps between the Riemannian manifolds. We derive a Bochner type identity and conditions for such maps to be harmonic. Later, we study conformal Riemannian maps whose total manifold admits a Ricci soliton and present a non-trivial example of such conformal Riemannian maps. We also obtain conditions for fiber and range space of such maps to be Ricci soliton and Einstein. We derive conditions for conformal Riemannian maps whose total manifold admits a Ricci soliton to be harmonic and biharmonic.

Mathematics Subject Classification (2010):
Key words:

Department of Mathematics, Punjabi University, Patiala, India
Email address: rakesh_bas@pbi.ac.in