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AN APPLICATION OF THE $p-$ ADIC NEWTON METHOD TO APPROXIMATE A ZERO OF A CONTINUOUS FUNCTION ON $\mathbf{Z_p}$

SEVER ACHIMESCU

ABSTRACT. Let $f : \mathbf{Z_p} \to \mathbf{C_p}$ be a continuous function such that there exists a special sequence of polynomials f_n uniformly convergent to f. Our goal is to approximate a zero of f by applying the Newton "tangent" method to each f_n for n large enough.

Mathematics Subject Classification (2010): 12
Key words: Newton method, p-adic approximation

1. Introduction

See [1] for definitions of $\mathbf{Z_p}$, $\mathbf{C_p}$ and their topology as (ultra)metric spaces, the p-adic absolute value $|\cdot|_p$ and its corresponding valuation $v = v_p$.

In this paper we find a class of continuous non-differentiable functions $f : \mathbf{Z_p} \to \mathbf{C_p}$ of the form $f(x) = \Sigma_{n=0}^{\infty} a_n P_n(x)$ with $|a_n|_p \to 0$ and $P_n(x) = \frac{x(x-1)...(x-n+1)}{n!}$ such that one can get a zero of f as a limit of a sequence of certain zeros of f_n obtained by applying the Newton method to f_n.

2. Background

Everything in this section is quoted from [1].

Let $f : \mathbf{Z_p} \to \mathbf{C_p}$ be a continuous non-differentiable function. By the Mahler Theorem, the following is true: if we denote $a_n = (\nabla^n f)(0)$ and $P_n(x) = \frac{x(x-1)...(x-n+1)}{n!}$, then $f(x) = \Sigma_{n=0}^{\infty} a_n P_n(x)$ and the sequence of polynomials $f_n(x) = \Sigma_{i=0}^{n} a_i P_i(x)$ converges uniformly to f. Conversely, if $|b_n|_p \to 0$ then $\Sigma_{n=0}^{\infty} b_n P_n(x)$ converges uniformly and defines a continuous function of $x \in \mathbf{Z_p}$.

Definition: $f : \mathbf{Z_p} \to \mathbf{C_p}$ is said to be differentiable at x_0 if there exists the limit $lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0}$ in $\mathbf{C_P}$. denoted $f'(x_0)$.

Theorem: Let $f : \mathbf{Z_p} \to \mathbf{C_p}$ be a continuous function and for a fixed $y \in \mathbf{Z_p}$ let $f(x+y) = \Sigma_{n=0}^{\infty} a_n(y) P_n(x)$ be its Mahler expansion. Then f is differentiable at y iff $|a_n(y)/n|_p \to 0$.

Proposition: Let $P \in \mathbf{Z_p}[X]$ and $x_0 \in \mathbf{Z_p}$ such that $P(x_0) = 0\, mod\, p^k$. If $v(P'(x_0)) < \frac{k}{2}$ then $\hat{x_0} = x_0 - \frac{P(x_0)}{P'(x_0)}$ satisfies

(1) $P(\hat{x_0}) = 0\, mod\, p^{k+1}$

(2) $\hat{x_0} = x_0\, mod\, p^{k-v(P'(x_0))}$

(3) $v(P'(\hat{x_0})) = v(P'(x_0))$.

Moreover, the sequence $x_0, x_{m+1} = \hat{x_m}$ converges to a zero of f.

The above proposition describes the Newton method. We will apply it in the next section for $k = 1$ and $x_0 = 1$.

3. Main result

Theorem:

Let $f(x) = \Sigma_{n=0}^{\infty} a_n P_n(x) = p - 1 + x + \Sigma_{n=2}^{\infty} a_n P_n(x)$ with $|a_n|_p \to 0$ and $|\frac{a_n}{n}|_p < 1$, $\forall n \geq 2$. Let $f_n(x) = \Sigma_{i=0}^{n} a_i P_i(x)$. Then we can find a zero of f following the algorithm described in the following three propositions.

Proposition1 Each f_n , $n \geq 1$satisfies the hypothesis of the Proposition of the Newton method stated in the previous section (starting with $x_0 = 1$ for each f_n):

$$f_n(1) = p\,,\,\forall n \geq 1$$

and

$$|f_n^{,}(1)|_p = 1(i.e.v(f_n^{'}(1)) = 0 < \frac{1}{2}\,,\,\forall n \geq 1$$

it proof.

The first statement follows from definitions of f_n. We prove the second statement by induction on n.

Let us assume that $|f_n^{,}(1)|_p = 1$ and let us prove that $|f_{n+1}^{,}(1)|_p = 1$. We have that

$$f_{n+1}(x) = f_n(x) + a_{n+1}P_{n+1}(x)$$

thus

$$f_{n+1}^{'}(x) = f_n^{'}(x) + a_{n+1}P_{n+1}^{'}(x)$$

From the definition of P_n we have that $P_{n+1}(x) = P_n(x)\frac{x-n}{n+1}$. Taking the derivatives and plugging in $x = 1$ we obtain $P_{n+1}^{'}(1) = P_n^{'}(1)\frac{1-n}{n+1}$. Since $|P_n^{'}(1)(1-n)|_p \leq 1$ and , by hypothesis, $|\frac{a_{n+1}}{n+1}|_p < 1$ it follws that

$$|f_{n+1}^{,}(1)|_p = max(|f_n^{,}(1)|_p, |\frac{a_{n+1}}{n+1}|_p |P_n^{'}(1)(1-n)|_p) = |f_n^{,}(1)|_p = 1\,,\,QED$$

Proposition2 Let $(x_{nm})_m$ be the sequence obtained by applying the Newton method to f_n starting with $x_0 = 1$. The diagonal sequence $(x_{mm})_m$ is a Cauchy sequence.

proof

Recall that $\mathbf{C_p}$ is complete and $\mathbf{Z_p}$ is a closed subset of $\mathbf{C_p}$ therefore a sequence in $\mathbf{Z_p}$ is Cauchy iff it is convergent in $\mathbf{Z_p}$. First we note that it is easy to prove by induction on $n \geq 1$ that $(x_{nm})_n$ is Cauchy. Then, since we already know that $(x_{nm})_m$ is Cauchy, it follows by a standard argument that the diagonal sequence $(x_{mm})_m$ is Cauchy.

Proposition3 Let ξ be the limit of $(x_{mm})_m$. Then $f(\xi) = 0$

This follows by a standard argument, using the fact that the sequence of polynomials f_n converges uniformly to f.

References

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SETS OF FUNCTIONS WHICH HAS PROPERTY (V) THE VON NEUMANN DENSITY THEOREM

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Abstract. A set of functions $F \subset C(X;[0,1])$ has the property (V), if $1-f \in F$ and $f \cdot g \in F$, $\forall f, g \in F$. Von Neumann is the one who drew attention to these collections of functions in [4].
Moreover, he claims, without proof, a density theorem for such families of functions. A careful analysis of these sets and their properties was made by R.I. Jewett in [3].
In this paper we present new, more accessible proofs for the majority of the results from [3].
We especially mention our proof of Lemma 1 which plays an essential role in the whole paper of [3].
Also we mention some new results as Propositions 3.1, 3.2 and 3.3 that make the connection between the Uryson sets and the sets which has the property (V) and the Corollary 4.1 from which the Theorem 4.18 of [5] immediately follows.
Mathematical Subject Classification: Primary 41A10; Secondary 46J10.
Keywords: Sets of functions which has the property (V) or which has the property (VN), point-separating property, lattice.

1. Introduction

If X is a Hausdorff compact space and $I = [0,1]$, then we will denote by:

$$C(X;\, I) = \{ f : X \to I,\ \text{continuous} \}.$$

Obviously, this set is **not a vector space**, but it has some interesting properties:

a) Contains the constant functions.
b) $1 - f \in C(X;I),\ \forall f \in C(X;I)$.
c) $f \cdot g \in C(X;I),\ \forall f, g \in C(X;I)$.
d) $\varphi \cdot f + (1-\varphi) \cdot g \in C(X;I),\ \forall \varphi, f, g \in C(X;I)$.
e) $C(X;\, I)$ is a lattice i.e., $f \vee g,\ f \wedge g \in C(X,[0,1]),\ \forall f, g \in C(X,[0,1])$.

Definition 1.1. *We say that a subset* $F \subset C(X;I)$ *has the property* (V) *if:*

$$(i)\quad 1 - f \in F,\ \forall f \in F.$$
$$(ii)\quad f \cdot g \in F,\ \forall f, g \in F.$$

The result stated by Von Neumann in [4], without proof, says that any subset $F \subset C\left(I^n;I\right)$ which has the property (V), contains the projections and at least a constant $c \in (0,1)$ is dense in $C\left(I^n;I\right)$. This result was proved by R.I. Jewett in [3].

The main result of [3] is Theorem 3, which generalizes Von Neumann's theorem. The proof of this theorem is based on several preliminary results i.e., Lemmas 1-5 and Theorems 1-2.

In this paper we present new, more accessible proofs for the majority of the results from [3].

In particular we mention our proof of Lemma 1 which plays an essential role in the whole paper of [3].

2. Preliminaries

Definition 2.1. *We say that a subset* $F \subset C\left(X;[0,1]\right)$ *has the property* (VN) *if:*

$$\varphi \cdot f + (1-\varphi)\cdot g \in F,\ \forall \varphi, f, g \in F .$$

Remark 2.1. *If the subset* F *has the property* (VN) *and contains the constant functions* 0 and 1*, then* F *has the property* (V).

Indeed, $1-\varphi \in F,\ \forall \varphi \in F$ because $1-\varphi = \varphi\cdot 0 + \left(1-\varphi\right)\cdot 1$ and $\varphi \cdot f \in F,\ \forall \varphi, f \in F$ because $\varphi \cdot f = \varphi \cdot f + \left(1-\varphi\right)\cdot 0$.

The definition of a family of functions that has the (VN) property and some density results for such families was studies by J.B. Prolla in [6].

Also, these families of functions were also studied in [2] and [5], where the connection between the property (VN)) and the notion of Uryson families was shown.

It is clear that the closure of a set which has the property (V) is also a set which has the property (V) and any intersection of such sets has the property (V) also. Thus, any subset of $C\left(X;\ [0,1]\right)$ is contained in a smallest subset which has the property (V), respectively in a smallest closed subset with this property.

In the particular case $X = I^n$ where $I = [0.1]$, we will denote by $\mathcal{P}_n$ the smallest subset of the set $C\left(I^n;[0,1]\right)$ that has the property (V) and contains the n projections, i.e., the functions:

$$(x_1,\dots,x_i,\dots,x_n) \to x_i : I^n \to I,\ \forall i = \overline{1,n} .$$

Remark 2.2. *Let* $F \subset C\left(X;[0,1]\right)$ *be a subset which has the property* (V) *and* $p \in \mathcal{P}_2$. *If we denote by* $h : X \to [0,1]$ *the function:*

$$h(x) = p[f(x), g(x)],\ \forall x \in X ,$$

then $h \in F,\ \forall f, g \in F$.

Indeed, if we denote by $\mathcal{Q}$ the set of all functions $q : I^2 \to [0,1]$ with the property that $q(f,g) \in F,\ \forall f, g \in F$, then the set $\mathcal{Q}$ has the property (V) and contains the projection functions, so $\mathcal{Q} \supset \mathcal{P}_2$.

Lemma 2.1. *Let* $a, b \in \mathbb{R}$ *be two real numbers such that* $0 \le a < b \le 1$. *Then, for any* $\varepsilon > 0$ *there are* $m, n \in \mathbb{N}^*$ *such that the polynomial function:*

$$p(x) = \left(1 - x^m\right)^n,\ x \in [0,1]$$

has the following properties:

$$(i)\ p(x) > 1-\varepsilon,\ \forall x \in [0,a]$$
$$(ii)\ p(x) < \varepsilon,\ \forall x \in [b,1]\ .$$

Proof. It is obvious that we can assume that $0 < a < b < 1$. From Bernoulli's inequality we have:

$$\left(1-x^m\right)^n \geq \left(1-a^m\right)^n \geq 1-n\cdot a^m,\ \forall x \in [0,a].$$
$$\left(1-x^m\right)^n \leq \left(1-b^m\right)^n \leq \frac{1}{\left(1+b^m\right)^n} \leq \frac{1}{1+n\cdot b^m} \leq \frac{1}{n\cdot b^m},\ \forall x \in [b,1]..$$

We notice that we can choose $m,n \in \mathbb{N}^*$ such that:

$$1-n\cdot a^m \geq 1-\varepsilon \text{ and } \frac{1}{n\cdot b^m} < \varepsilon,$$

inequalities equivalent to:

$$\frac{1}{\varepsilon\cdot b^m} < n < \frac{\varepsilon}{a^m}.$$

Indeed, this is possible if we choose $m \in \mathbb{N}$ such that:

$$\frac{2}{\varepsilon\cdot b^m} < \frac{\varepsilon}{a^m} \quad \Leftrightarrow \quad \frac{2}{\varepsilon^2} < \left(\frac{b}{a}\right)^m \text{ and } n = 2\cdot\left[\frac{1}{\varepsilon\cdot b^m}\right] > \frac{1}{\varepsilon\cdot b^m}$$

because we can suppose that $\frac{1}{\varepsilon\cdot b^m} > 1$.

Corollary 2.1. (See [5], Theorem 2.1) *Let* $k \in \mathbb{N}^*, k \geq 2$ *and let* $a,b \in \mathbb{R}$ *be two real numbers such that* $0 \leq a < \frac{1}{k} < b \leq 1$. *Then, for any* $\varepsilon > 0$ *there is* $m \in \mathbb{N}^*$ *such that the polynomial function:*

$$p(x) = \left(1-x^m\right)^{k^m},\ x \in [0,1]$$

has the following properties:

$$(i)\ p(x) > 1-\varepsilon,\ \forall x \in [0,a]$$
$$(ii)\ p(x) < \varepsilon,\ \forall x \in [b,1]\ .$$

The statement follows from Lemma 1, if we choose $n = k^m$ and take into account that $k\cdot b > 1$.

Corollary 2.2. *For any point* $a \in I$ *and any* $\varepsilon > 0,\ \delta > 0$ *there exists a polynomial function* $p \in \mathcal{P}_1$ *with the properties:*

$$(1)\ p(x) > (1-\varepsilon)^2,\ \forall x \in [a-\delta, a+\delta]$$
$$(2)\ p(x) < \varepsilon,\ \forall x \in [0, a-2\cdot\delta] \cup [a+2\cdot\delta, 1]\ .$$

Proof. From Lemma 2.1 it follows that there are two polynomial functions

$$p_1, p_2 : [0,1] \to [0,1] \text{ such that:}$$
$$(i)\ p_1(x) > 1-\varepsilon,\ \forall x \in [0, a+\delta]$$
$$(ii)\ p_1(x) < \varepsilon,\ \forall x \in [a+2\cdot\delta, 1]$$

and:

$$(iii)\ p_2(x) > 1-\varepsilon,\ \forall x \in [a-\delta, 1]$$
$$(iv)\ p_2(x) < \varepsilon,\ \forall x \in [0, a-2\cdot\delta].$$

Obvious, the product function $p = p_1 \cdot p_2 : I \to I = [0,1]$ has the properties:

$$(1)\ p(x) > (1-\varepsilon)^2,\ \forall x \in [a-\delta, a+\delta]$$
$$(2)\ p(x) < \varepsilon,\ \forall x \in [0, a-2\cdot\delta] \cup [a+2\cdot\delta, 1].$$

3. Connection with Uryson families

The Uryson family concept was introduced by I. Bucur in [5].

Definition 3.1. *A family $U \subset C(X,I)$ is called **Uryson family** on X if for any two disjoint closed subset A, B* of X *and any $\varepsilon \in (0,1)$ there exists a function $u \in U$ such that:*

$$u(x) \geq 1-\varepsilon,\ \forall x \in A \text{ and } u(x) \leq \varepsilon,\ \forall x \in B.$$

The main result regarding Uryson families is Theorem 3.3 from [5] which establishes that if $U \subset C(X,I)$ is a Uryson family then:

$$\overline{\text{co}}(U) = C(X;I).$$

Proposition 3.1. *The set of function $\mathcal{P}_1$ is an Uryson family on I.*

Proof. Let $A, B \subset I$ be two closed disjoint closed subsets such that $d(A,B) = 2\cdot r > 0$. If for any point $a \in A$ we denote by $D_k = [a-r, a+r]$, then we have $D_k \cap B = \phi$.

Taking into account that the subset A is a compact subset it results that there is a finite number of points $a_k \in A,\ k = \overline{1,n}$ such that:

$$A \subset \bigcup_1^n D_k.$$

According to Corollary 2, for any $\varepsilon > 0$, there exists $p_k \in \mathcal{P}_1$ such that:

$$p_k > \left(1-\frac{\varepsilon}{2\cdot n}\right)^2 \text{ on } D_k \text{ and } p_k < \frac{\varepsilon}{2\cdot n} \text{ on } [0, a-2\cdot\delta] \cup [a+2\cdot\delta, 1] \supset B.$$

If we denote by $q_0 = (1-p_1)(1-p_2)...(1-p_n)$, then $q_0 \in \mathcal{P}_1$ and we have:

for any $a \in A$ there is a $k \in \{1,2,...,n\}$ such that $a \in D_k$, hence $p_k(a) > \left(1-\frac{\varepsilon}{2\cdot n}\right)^2 \geq 1-\frac{\varepsilon}{n}$.

Therefore we have:

$$q_0(a) \leq 1-p_k(a) < \frac{\varepsilon}{n}, \text{ so:}$$
$$q_0 < \varepsilon \text{ on } A.$$

On the other hand, if $b \in B \Rightarrow q_0(b) > \left(1-\frac{\varepsilon}{2\cdot n}\right)^n \geq 1-n\cdot\frac{\varepsilon}{2\cdot n} = 1-\frac{\varepsilon}{2} > 1-\varepsilon$, so:

$$q_0 > 1-\varepsilon \text{ on } B.$$

Remark 3.1. In fact, from the proof of the Proposition 1, it follows that there is a proper subset $H \subset \mathcal{P}_1$ which is an Uryson family on I.

Indeed, let us we denote, for any $m, n \in \mathbb{N}^*$, with $g_{m,n}$ the function:

$$g_{m,n}(x) = \left(1-x^m\right)^n,\ \forall x \in I,$$

and by:

$$G=\left\{g:I\to I;\ g(x)=1-g_{k,l}(x)\cdot\left(1-g_{m,n}(x)\right),\ \forall k,l,m,n\in N^*\right\}.$$

From the proof of Proposition 1 it follows that the family:

$$H=\left\{\prod_{i=1}^{N} g_i;\ g_i\in G,\ \forall i=\overline{1,N},\ \forall N\in\mathbb{N}^*\right\} \qquad (*)$$

is an Uryson family on I.

Remark 3.2. *According to Theorem3.3 from [5] we have:*

$$\overline{\mathrm{co}}(H)=C(I;I).$$

Lemma 3.1. *For any point* $(a,b)\in I^2=[0,1]\times[0,1]$ *and any* $\varepsilon>0,\ \delta>0$ *there exists a polynomial function* $p\in\mathcal{P}_2$ *with the properties:*

$$p(x,y)>(1-\varepsilon)^4,\ \forall(x,y)\in[a-\delta,a+\delta]\times[b-\delta]\times[b+\delta]$$

$$p(x,y)<\varepsilon^2,\ \forall(x,y)\in\left(I\times I\right)\setminus\left([a-2\delta,a+2\delta]\times[b-2\delta,b+2\delta]\right).$$

Proof. From Lemma 1 it follows that there are two polynomial functions $p_1,p_2:[0,1]\to[0,1]$ such that:

$$(i)\ \ p_1(x)>1-\varepsilon,\ \forall x\in[0,a+\delta]$$

$$(ii)\ \ p_1(x)<\varepsilon,\ \forall x\in[a+2\cdot\delta,1]$$

and:

$$(iii)\ \ p_2(x)>1-\varepsilon,\ \forall x\in[a-\delta,1]$$

$$(iv)\ \ p_2(x)<\varepsilon,\ \forall x\in[0,a-2\cdot\delta].$$

Obvious, the product function $p_1\cdot p_2:K\to I=[0,1]$ has the properties:

$$(1)\ \ \left(p_1\cdot p_2\right)(x)>(1-\varepsilon)^2,\ \forall x\in[a-\delta,a+\delta]$$

$$(2)\ \ \left(p_1\cdot p_2\right)(x)<\varepsilon,\ \forall x\in[0,a-2\cdot\delta]\cup[a+2\cdot\delta,1].$$

Analogously, there are two other polynomial functions $p_3,p_4:I\to I$ such that:

$$(3)\ \ \left(p_3\cdot p_4\right)(y)>(1-\varepsilon)^2,\ \forall y\in[b-\delta,b+\delta]$$

$$(4)\ \ \left(p_3\cdot p_4\right)(y)<\varepsilon,\ \forall y\in[0,b-2\cdot\delta]\cup[b+2\cdot\delta,1].$$

It is clear now that the product function $p=p_1\cdot p_2\cdot p_3\cdot p_4:K\to I$ has the properties:

$$p(x,y)>(1-\varepsilon)^4,\ \forall(x,y)\in[a-\delta,a+\delta]\times[b-\delta]\times[b+\delta]$$

$$p(x,y)<\varepsilon^2,\ \forall(x,y)\in\left(I\times I\right)\setminus\left([a-2\delta,a+2\delta]\times[b-2\delta,b+2\delta]\right).$$

Proposition 3.2. *The set of function* $\mathcal{P}_2$ *is an Uryson family on* I^2.

Proof. Let $A,B\subset I\times I$ be two closed disjoint closed subsets such that $d(A,B)=4\cdot r>0$. For any point $(a',a'')\in A$ we have

$$D_k\cap B=\phi,\ \text{where}\ D_k=[a'-2\cdot r,a'+2\cdot r]\times[a''-2\cdot r,a''+2\cdot r].$$

Taking into account that the subset A is compact it results that there is a finite number of points $(a'_k,a''_k)\in A,\ k=\overline{1,n}$ such that:

$$A\subset\bigcup_{1}^{n}D_k.$$

According to Lemma 2, for any $\varepsilon>0$, there exists $p_k\in\mathcal{P}_2$ such that:

$$p_k>\left(1-\frac{\varepsilon}{4\cdot n}\right)^4\ \text{on}\ D_k\ \text{and}$$

$$p_k < \left(\frac{\varepsilon}{4\cdot n}\right)^2 \text{ on } I^2 \setminus \left([a-2\delta, a+2\delta]\times[b-2\delta, b+2\delta]\right) \supset B.$$

If we denote by $q_0 = (1-p_1)(1-p_2)...(1-p_n)$, then $q_0 \in \mathcal{P}_2$ and we have:

$(a', a'') \in A \Rightarrow \exists k \in \{1, 2, ..., n\}$ such that $(a', a'') \in D_k \Rightarrow p_k(a', a'') > \left(1 - \frac{\varepsilon}{4\cdot n}\right)^4 \geq 1 - \frac{\varepsilon}{n}$.

Further we have:

$$q_0(a', a'') < 1 - p_k(a', a'') < \frac{\varepsilon}{n} \leq \varepsilon,$$

so:

$$q_0 < \varepsilon \text{ on } A.$$

On the other hand, if $(b', b'') \in B \Rightarrow q_0(b', b'') > \left(1 - \frac{\varepsilon}{4\cdot n}\right)^{2\cdot n} \geq 1 - 2\cdot n \cdot \frac{\varepsilon}{4\cdot n} = 1 - \frac{\varepsilon}{2} > 1 - \varepsilon.$

so:

$$q_0 > 1 - \varepsilon \text{ on } B.$$

Remark 3.3. As in Remark 2.3, from the proof of Proposition 2.2, we find that there is a proper subset $H \subset \mathcal{P}_2$ which is an Uryson family on I^2.

Indeed, let us we denote, for any $m, n \in \mathbb{N}^*$, with $g_{m,n}$ the function:

$$g_{m,n}(x) = \left(1 - x^m\right)^n, \ \forall x \in I,$$

and by

$$G = \left\{ g : I \times I \to I;\ g(x, y) = 1 - g_{k,l}(x)\cdot\left(1 - g_{m,n}(x)\right)\cdot g_{p,q}(y)\cdot\left(1 - g_{r,s}(y)\right)\right\}.$$

We notice now that the family:

$$H = \left\{ \prod_{i=1}^{N} g_i;\ \mathrm{g}_i \in G,\ \forall i = \overline{1, N},\ \forall N \in \mathbb{N}^* \right\} \qquad (**)$$

is an Uryson family on I^2.

Remark 3.4. *According to Theorem3.3 from [5] we have:*

$$\overline{\mathrm{co}}(H) = C\left(I^2; I\right).$$

Clearly, Lemma 3.1 can be generalized.

Lemma 3.2. *For any point* $a = (a_1, ..., a_n) \in I^n$ *and any* $\varepsilon > 0,\ \delta > 0$ *there exists a polynomial function* $p \in \mathcal{P}_n$ *with the properties:*

$$p(x) > (1-\varepsilon)^{2\cdot n},\ \forall x = (x_1, ..., x_n) \in [a_1 - \delta, a_1 + \delta]\times ...\times[a_n - \delta]\times[a_n + \delta]$$

$$p(x) < \varepsilon^n,\ \forall x = (x_1, ..., x_n) \in I^n \setminus \left([a_1 - 2\delta, a_1 + 2\delta]\times ...\times[a_n - 2\delta, a_n + 2\delta]\right).$$

It can also be proved the following result:

Proposition 3.3. *The set of function* $\mathcal{P}_n$ *is an Uryson family on* I^n.

4. The main results

Lemma 4.1. *Let* $A, B \subset I \times I$ *be two closed disjoint closed subsets.*

Then for any $\varepsilon > 0$ and any $p \in \mathcal{P}_2$ there exists $q \in \mathcal{P}_2$ with the properties:

$$(a)\quad q \geq p,\ \forall (x, y) \in I \times I$$

$$(b)\quad q > 1 - \varepsilon,\ \ \forall (x, y) \in A$$

$$(c)\quad q < p + \varepsilon,\ \ \forall (x, y) \in B.$$

Proof. Let $A, B \subset I \times I$ be two closed disjoint closed subsets. According to Proposition 1 there exists a function $q_0 \in \mathcal{P}_2$ with the properties:

$$q_0 < \varepsilon \text{ on } A \text{ and } q_0 > 1-\varepsilon \text{ on } B.$$

Let $p \in P_2$ be arbitrary and let $q = 1-(1-p)\cdot q_0$. Then we have:

$$q = 1-(1-p)\cdot q_0 \geq 1-(1-p) = p .$$

If $a = (a', a'') \in A$, then:

$$q(a) = 1-(1-p(a))\cdot q_0(a) \geq 1-q_0(a) > 1-\frac{\varepsilon}{n} > 1-\varepsilon .$$

On the subset B we have:

$$q-p = 1-q_0 + p\cdot q_0 - p = (1-q_0)\cdot(1-p) < (1-q_0) < \varepsilon .$$

Lemma 4.2. *For any $a_k, b_k \in [0,1]$, $k \in 1,2,...,n$, we have:*

$$\left|\prod_1^n a_k - \prod_1^n b_k\right| \leq \sum_1^n |a_k - b_k| .$$

Proof. We assume by mathematical induction that:

$$\left|a = \prod_1^{n-1} a_k - b = \prod_1^{n-1} b_k\right| \leq \sum_1^{n-1} |a_k - b_k| .$$

Further we have:

$$|a\cdot a_n - b\cdot b_n| \leq |a\cdot a_n - b\cdot a_n| + |b\cdot a_n - b\cdot b_n| = |a-b|\cdot|a_n| + |b|\cdot|a_n - b_n| \leq |a-b| + |a_n - b_n| = \sum_1^n |a_k - b_k| .$$

Theorem 4.1. *For any $\varepsilon > 0$ there exists a function $p \in \mathcal{P}_2$ such that:*

$$|x \wedge y - p(x,y)| < \varepsilon, \ \forall (x,y) \in I \times I .$$

Proof. Let $0 < \varepsilon < \frac{1}{4}$ and $C = \{(x,y) \in I \times I;\ \varepsilon \leq x \wedge y \leq 1-\varepsilon\}$.

We notice that if $(x,y) \in C$, then $x > 0,\ y > 0$ and $x < 1-\varepsilon$ or $y < 1-\varepsilon$. Then there is $m > 0$ such that:

$$x^m \cdot y^m < \varepsilon .$$

If we denote by $p(x,y) = 1 - x^m \cdot y^m$, then we have:

$$p \in \mathcal{P}_2 \text{ and } 1-\varepsilon < p(x,y) < 1,\ \forall (x,y) \in C . \quad (1)$$

For any $k \in \mathbb{N}$ we will denote by:

$$A_k = \{(x,y) \in C;\ p^k(x,y) \geq x \wedge y\},$$

$$B_k = \{(x,y) \in C;\ p^k(x,y) \leq x \wedge y\}.$$

Clearly we have:

$$A_1 = C,\ A_k \supset A_{k+1},\ B_k \supset C \setminus A_k,\ A_{k+1} \cap B_k = \phi \text{ and } B_k \subset B_{k+1}.$$

Since $\bigcap_0^\infty A_k = \phi$ and C is a compact subset, it follows that there is $n > 1$ such that $A_n = \phi$.

Therefore:

$$C = \bigcup_1^{n-1} \left(A_k \setminus A_{k-1}\right).$$

According to Lemma 4.1 there exists $q_k \in \mathcal{P}_2$ with the properties:

$$q_k \geq p, \ \forall (x,y) \in I \times I$$
$$q_k > 1 - \frac{\varepsilon}{n}, \ \ \forall (x,y) \in B_{k-1} \tag{2}$$
$$q_k < p + \frac{\varepsilon}{n}, \ \ \forall (x,y) \in A_k \ .$$

We notice that if $(x,y) \in A_k \setminus A_{k+1}$, then on the one hand:

$$0 \leq p^k - x \wedge y < p^k - p^{k+1} = p^k \cdot (1-p) \leq 1 - p < \varepsilon$$

and on the other hand:

$$(x,y) \in \bigcap_1^k A_j, \ (x,y) \in C \setminus A_{k+1} \subset C \setminus A_{k+3} \subset B_{k+2} \subset \ldots \subset B_n \ .$$

If we denote by $q = q_1 \cdot q_2 \cdot \ldots \cdot q_n$, then from Lemma 4 and the inequalities (1) and (2) we deduce:

$$\left|p^k - q\right| \leq \left|p^k - p^{k+1}\right| + \left|p^{k+1} - q\right| < \varepsilon + \left|p^{k+1} - \prod_1^n q_j\right| \leq \varepsilon + \sum_1^k \left|p - q_j\right| + \left|p - q_{k+1}\right| + \sum_{k+2}^n \left|1 - q_j\right| \leq$$
$$\leq \varepsilon + k \cdot \frac{\varepsilon}{n} + (1-p) + (n-k-1) \cdot \frac{\varepsilon}{n} < \varepsilon + (n-1) \cdot \frac{\varepsilon}{n} + \varepsilon < 3 \cdot \varepsilon.$$

Therefore, if $(x,y) \in C$, we have:

$$\left|q - x \wedge y\right| \leq \left|q - p^k\right| + \left|p^k - x \wedge y\right| < 3 \cdot \varepsilon + \varepsilon = 4 \cdot \varepsilon \ .$$

Next we denote by:

$$D = \left\{(x,y) \in I \times I; \ x \wedge y \geq 1 - \varepsilon\right\}$$

and by:

$$F = \left\{(x,y) \in I \times I; \ x \wedge y \leq 1 - 2 \cdot \varepsilon\right\} \ .$$

From Lemma 4.1 it results that there exists $q' \in \mathcal{P}_2$ with the properties:

$$q' \geq q \text{ on } I \times I$$
$$q' > 1 - \varepsilon \text{ on } D,$$
$$q' < q + \varepsilon \text{ on } F.$$

Let us observe that if $x \wedge y \geq \varepsilon$, then $\left|q' - x \wedge y\right| < 6 \cdot \varepsilon$.

Indeed, if $(x,y) \in D$, we have:

$$1 \geq q'(x,y) > 1 - \varepsilon, \text{ and } 1 \geq x \wedge y > 1 - \varepsilon \ \Rightarrow \left|q'(x,y) - x \wedge y\right| \leq \varepsilon \ .$$

On the set $\left\{(x,y); \ \varepsilon \leq x \wedge y \leq 1 - 2 \cdot \varepsilon\right\} \subset C \cap F$ we have:

$$\left|q' - x \wedge y\right| \leq \left|q' - q\right| + \left|q - x \wedge y\right| < \varepsilon + 4 \cdot \varepsilon = 5 \cdot \varepsilon \ . \tag{**}$$

On the set $\left\{(x,y); \ 1 - 2 \cdot \varepsilon \leq x \wedge y \leq 1 - \varepsilon\right\} \subset C$ we have:

$$-4 \cdot \varepsilon \leq q - x \wedge y \leq q' - x \wedge y \leq 1 - x \wedge y \leq 2 \cdot \varepsilon \leq 4 \cdot \varepsilon \ ,$$

whence we deduce that:

$$\left|q' - x \wedge y\right| \leq 4 \cdot \varepsilon \ .$$

Let $A = \left\{(x,y) \in I \times I; \ x \wedge y \leq \varepsilon\right\}$ and $B = \left\{(x,y) \in I \times I; \ x \wedge y \geq 2 \cdot \varepsilon\right\}$.

Applying Lemma 4.1 for A, B and $p = 1 - q'$ it results that there exists $q'' \in \mathcal{P}_2$ with the properties:

$$q" \geq 1-q' \text{ on } I \times I$$
$$q" > 1-\varepsilon \text{ on } A$$
$$q" < 1-q'+\varepsilon \text{ on } B .$$

If we denote by $\tilde{q} = 1-q"$ then $\tilde{q} \in \mathcal{P}_2$ and:

$$\tilde{q} \leq q' \text{ on } I \times I$$
$$\tilde{q} < \varepsilon \text{ on } A$$
$$\tilde{q} > q'-\varepsilon \text{ on } B .$$

If $(x,y) \in A \Rightarrow \left|\tilde{q}(x,y) - x \wedge y\right| \leq \tilde{q}(x,y) + x \wedge y < 2 \cdot \varepsilon$.

If $(x,y) \in B \Rightarrow \left|\tilde{q}(x,y) - x \wedge y\right| \leq \left|\tilde{q}(x,y) - q'(x,y)\right| + \left|q'(x,y) - x \wedge y\right| < \varepsilon + 5 \cdot \varepsilon = 6 \cdot \varepsilon$.

If $\varepsilon \leq x \wedge y \leq 2 \cdot \varepsilon$ and $x \wedge y \leq \tilde{q} \;\Rightarrow\; \left|\tilde{q} - x \wedge y\right| \leq \left|q' - x \wedge y\right| < 6 \cdot \varepsilon$.

If $\tilde{q} < x \wedge y \;\Rightarrow\; \left|\tilde{q} - x \wedge y\right| \leq \tilde{q} + x \wedge y < 2 \cdot \varepsilon + 2 \cdot \varepsilon = 4 \cdot \varepsilon$.

Therefore we have:

$$\left|\tilde{q} - x \wedge y\right| \leq 6 \cdot \varepsilon, \ \forall (x,y) \in I \times I \ .$$

Since $x \vee y = 1 - (1-x) \wedge (1-y)$ it follows that the function $(x,y) \to x \vee y : I \times I \to I$ can be also uniformly approximated with a function of $\mathcal{P}_2$.

Theorem 4.2. *Any closed subset* $F \subset C(X;[0,1])$ *that has the property* (V) *is a lattice.*

Proof. According to Theorem 4.1, for any $n \in \mathbb{N}^*$ there exists $p_n \in \mathcal{P}_2$ such that:

$$\left|p_n(x,y) - x \wedge y\right| < \frac{1}{n}, \ \forall (x,y) \in [0,1] \times [0,1] .$$

In particular, for any $f, g \in F$ and any $x \in X$ we have:

$$\left|p_n[f(x), g(x)] - f(x) \wedge g(x)\right| < \frac{1}{n}, \ \forall x \in X .$$

On the other hand, from Remark 2, it follows that the function $h_n : X \to [0,1]$ defined by:

$$h_n(x) = p_n[f(x), g(x)], \ \forall x \in X ,$$

belongs to F .

Since $h_n \xrightarrow[X]{u} f \wedge g$, and the subset F is closed, it results that $f \wedge g \in F$, so F is lattice.

Theorem 4.3. *Let* X *be a compact Hausdorff space and let* $F \subset C(X;\ [0,1])$ *be a closed subset which has the property* (V) *and separates the points of* X *. If we denote by:*

$$S = \left\{x \in X;\ g(x) \in \{0,1\},\ \forall g \in F\right\}$$

then:

$$F = \left\{f \in C(X;\ [0,1]);\ f(S) \subset \{0,1\}\right\}.$$

Proof. We shall prove that if $f \in C(X;\ [0,1])$ has the property $f(S) \subset \{0,1\}$, then $f \in F$.

According to Lemma 16.3 of [2], it is sufficient to show that any such function can be approximated by a function of F at any two points $u, v \in X$.

Let $f \in C(X;\ [0,1])$ with the property $f(S) \subset \{0,1\}$ and let $u, v \in X$, $u \neq v$ be two arbitrary points.

Since F separates the points of X it results that there exists $g \in F$ such that $g(u) \neq g(v)$.

Case 1. If $u, v \in S$, then $g(u), g(v) \in \{0,1\}$ and $f(u), f(v) \in \{0,1\}$.

If $g(u) = 0; g(v) = 1; f(u) = 0; f(v) = 0$ then $h = g \cdot (1-g) \in F$ and $h(u) = f(u); h(v) = f(v)$.

If $g(u) = 0; g(v) = 1; f(u) = 0; f(v) = 1$ then $g(u) = f(u); g(v) = f(v)$.

If $g(u) = 0; g(v) = 1; f(u) = 1; f(v) = 1$ then $h' = 1 - g \cdot (1-g) \in F$ and

$$h'(u) = f(u); h'(v) = f(v).$$

If $g(u) = 0; g(v) = 1; f(u) = 1; f(v) = 0$ then $h'' = 1 - g \in F$ and $h''(u) = f(u); h''(v) = f(v)$.

The situation $g(u) = 1; g(v) = 0; f(u), f(v) \in \{0,1\}$ is similar.

Therefore, if $u, v \in S$, at last one of the functions: $g;\ 1-g;\ g \cdot (1-g);\ 1 - g \cdot (1-g)$ is equal to f at the points u, v.

Case 2. If $u \in S$ and $v \notin S$ then there is $g \in F$ such that $g(u) = f(u)$ and $g(v) \in (0,1)$.

Obvious, there exist $m \in \mathbb{N}^*$ such that $g^m(v) < \varepsilon$. If we denote by $h(x) = 1 - g^m(x)$ then $h \in F$ and $h(v) > 1 - \varepsilon$.

If $h(v) < f(v)$ then $|h(v) - f(v)| < \varepsilon$.

If $h(v) \geq f(v)$ then there is $n \in \mathbb{N}^*$ such that $h^{n+1}(v) < f(v) \leq h^n(v)$ an we have:

$$h^n(v) - h^{n+1}(v) = h^n(v) \cdot (1 - h(v)) \leq 1 - h(v) = g^m(v) < \varepsilon.$$

Therefore:

$$|h^n(v) - f(v)| < \varepsilon \text{ si } h^n \in F.$$

Case 3. If $u \notin S$ and $v \notin S$ there are $g_1, g_2, g_3 \in F$ with the properties:

$$g_1(v) \leq g_1(u) \in (0,1);\ g_2(u) \leq g_2(v) \in (0,1) \text{ and } g_3(v) < g_3(u).$$

Indeed, there exists $g_1 \in F$ such that $g_1(u) \in (0,1)$. If $g_1(v) \leq g_1(u)$ then the statement is verified. If $g_1(v) > g_1(u)$, then $(1-g_1)(v) < (1-g_1)(u)$ and $(1-g_1) \in F$.

We have a similar situation for g_2.

Let $g_3 \in F$ be so that $g_3(u) \neq g_3(v)$.

If $g_3(u) < g_3(v)$, then $(1-g_3)(v) < (1-g_3)(u)$ and $1 - g_3 \in F$.

Let $h_1 = g_1 \cdot g_3$ and $h_2 = g_2 \cdot (1-g_3)$. Then we have:

$$h_1(v) < h_1(u) < 1; \quad h_2(u) < h_2(v) < 1.$$

We notice that we can suppose $h_1(v) > 0$ because, if $h_1(v) = 0$, then for any $\varepsilon > 0$ there is $n \in \mathbb{N}^*$ such that $h_1^n(u) < \varepsilon$ and, obvious $h_1^n(v) = 0 < \varepsilon$, and $h_1^n \in F$.

Similarly we can assume that $h_2(u) > 0$.

If we denote by $a = h_1(v)$ and by $b = h_1(u)$ then, according to Lemma 2.1, it follows that there exists $\varphi \in F$, $\varphi(x) = (1 - x^m)^n$, $\forall x \in [0,1]$ with the properties:

$$\varphi(x) > 1 - \varepsilon,\ \forall x \in [0, a]$$

$$\varphi(x) < \varepsilon,\ \forall x \in [b, 1].$$

We notice that $\psi_1 = (1 - h_1^m)^n \in F$ and has the properties:

$$\psi_1(v) = (1 - h_1^m(v))^n = (1 - a^m)^n > 1 - \varepsilon$$

$$\psi_1(u) = (1 - h_1^m(u))^n = (1 - b^m)^n < \varepsilon.$$

Similarly, if we denote by $c = h_2(u)$ and by $d = h_2(v)$ then there exists a function $\psi_2 = \left(1 - h_2^r\right)^s \in F$ with the properties:

$$\psi_2(u) = \left(1 - h_2^r(u)\right)^s = \left(1 - c^r\right)^s > 1 - \varepsilon$$

$$\psi_2(v) = \left(1 - h_2^r(v)\right)^s = \left(1 - d^r\right)^s < \varepsilon .$$

We now observe that the function $\psi = \psi_1 \cdot \psi_2 \in F$ and has the following properties:

$$\psi(u) = \psi_1(u) \cdot \psi_2(u) \le \psi_1(u) < \varepsilon$$

$$\psi(v) = \psi_1(v) \cdot \psi_2(v) \le \psi_2(v) < \varepsilon .$$

Corollary 4.1. *Let X be a compact Hausdorff space and let $F \subset C(X;\ [0,1])$ be a closed subset which has the property (V), separates the points of X and contains at least function constant $c \in (0,1)$. Then $F = C(X, I)$.*

The statement follows from Theorem 4.3 because, in this case, $S = \phi$.

Theorem 4.4. (Von Neumann). *The smallest closed subset of $C\left(I^n;[0,1]\right)$ having property (V), containing the projections and at least one constant function $c \in (0,1)$ is $C\left(I^n;[0,1]\right)$ itself.*

Proof. Here we denote by $I^n = \underbrace{I \times I \times \ldots \times I}_{n-time}$. The statement follows now from Corollary 2 because the family $\mathcal{P}_n -$ the smallest closed subset of $C\left(I^n;[0,1]\right)$ having property (V), and containing the projections, separates the points of $I^n = \underbrace{I \times I \times \ldots \times I}_{n-time}$.

Theorem 4.5. (see Theorem 4.18 from [5]). *Let X be a compact Hausdorff space and let $F \subset C(X;\ [0,1])$ be a closed subset which has the property (VN), separates the points of X* **,** *contains the constant functions 0 and 1 at least one constant function $c \in (0,1)$. Then:*

$$F = C(X, I).$$

The statement follows from Remark 2.1 and Corollary 4.1.

Remark4.1. From Theorem 4.4 it follows that $\mathcal{P}_2 = C\left(I^2, I\right)$ and on other hand from Remark 3.4 we have $\overline{\mathrm{co}}(H) = C\left(I^2, I\right)$. *Therefore we deduce that:*

$$\overline{\mathrm{co}}(H) = \mathcal{P}_2 .$$

The Remark 4.1 shows us the structure of the family $\mathcal{P}_2$.

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A CONVERSE OF TCHEBYSHEV INEQUALITY. APPLICATIONS

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Abstract. In this paper we prove a lower bound of the measure of the set

$$[f \geq \alpha] = \{x \in X; f(x) \geq \alpha\},$$

where f is a positive, bounded, measurable function of the probability space $(X, \mathcal{B}, \mu)$.

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1. Introduction

Giving a probability space $(X, \mathcal{B}, \mu)$ and a positive, $\mathcal{B}$-measurable function f on X, then the famous Tchebyshev inequality offer an upper bound for the measure of the subset $[f \geq \alpha]$, namely $\alpha > 0 \Longrightarrow \mu([f \geq \alpha]) \leq \frac{1}{\alpha} \int f d\mu$.

If the function f is also a bounded one, then for any strictly positive number $\alpha, \alpha < \int f d\mu$, we have

$$\mu([f \geq \alpha]) \geq \frac{\int f d\mu - \alpha}{\|f\| - \alpha},$$

where $\|f\|$ means the uniform norm of the function f.

An application of this inequality to the product of an infinite family of probability spaces shows how the above result may be used on probability theory or processes theory as well.

2. The Converse of Tchebyshev Inequality

Throughout this paper $(X, \mathcal{B})$ will be a measurable space and for any positive, $\mathcal{B}$-measurable function f on X and any positive measure μ on $\mathcal{B}$ we denote by $\mu(f)$ the Lebesgue integral of the function f with respect to the measure μ

$$\mu(f) = \int f(x) d\mu(x).$$

As usually we use the notations $[f \geq \alpha], [f > \alpha], [f \leq \alpha], \ldots$ for the sets: $\{x \in X; f(x) \geq \alpha\}$;
$\{x \in X; f(x) > \alpha\}, \{x \in X; f(x) \leq \alpha\}, \ldots$.

Lemma 2.1. *If μ is a probability measure on $\mathcal{B}$ and $f : X \to \mathbb{R}$ is a $\mathcal{B}$-measurable, positive function $f(x) \leq 1$, for all $x \in X$, then for any positive, real number α, $\alpha < \mu(f)$*

we have

$$\mu([f \geq \alpha]) \geq \frac{\mu(f) - \alpha}{1 - \alpha}.$$

Proof. Obviously the function on X

$$x \mapsto 1 - x$$

is positive, $\mathcal{B}$-measurable with values in the interval $[0,1]$ since $\mu(f) \leq 1$ and $\alpha < \mu(f)$ we have $\alpha < 1$, $1 - \alpha > 0$.

Using Tchebyshev inequality for the positive, measurable function $1 - f$ we have

$$\mu[1 - f \geq 1 - \alpha] \leq \frac{1}{1 - \alpha} \cdot \mu(1 - f) = \frac{1 - \mu(f)}{1 - \alpha}. \tag{2.1}$$

Obviously we have the relation

$$[1 - f \geq 1 - \alpha] = [f \leq \alpha]$$

and therefore using relation (2.1) we deduce

$$\mu([f \leq \alpha]) \leq \frac{1 - \mu(f)}{1 - \alpha}. \tag{2.2}$$

Since μ is a probability on $\mathcal{B}$ we have

$$\mu([f > \alpha]) = 1 - \mu([f \leq \alpha])$$

and using relation (2.2) we get $\alpha < \mu(f) \leq M$

$$\mu([f \geq \alpha]) \geq \mu([f > \alpha]) = 1 - \mu([f \leq \alpha]) \geq 1 - \frac{1 - \mu(f)}{1 - \alpha},$$

$$\mu[f \geq \alpha] \geq \frac{\mu(f) - \alpha}{1 - \alpha}.$$

□

Theorem 2.2. *If μ is a probability on $\mathcal{B}$ and $f : X \to \mathbb{R}_+$ is a positive, $\mathcal{B}$-measurable and μ-essentially bounded function on X, then for any positive number $\alpha, \alpha < \mu(f)$ we have*

$$\mu[f \geq \alpha] \geq \frac{\mu(f) - \alpha}{\|f\|_\infty - \alpha},$$

where $\|f\|_\infty$ denotes the essential supremum of the function f with respect to the measure μ.

Proof. With no lost of generality we may suppose that $f(x) \leq \|f\|_\infty$, for all $x \in X$. Further we denote by g th positive $\mathcal{B}$-measurable function on X given by

$$g(x) = \frac{f(x)}{\|f\|_\infty}, \forall x \in X.$$

Obviously the function g satisfies all required condition of Lemma 2.1. Taking into account that $0 < \frac{\alpha}{\|f\|_\infty} < \mu(g)$, applying Lemma 2.1 we get

$$\mu\left(\left[g \geq \frac{\alpha}{\|f\|_\infty}\right]\right) \geq \frac{\mu(g) - \frac{\alpha}{\|f\|_\infty}}{1 - \frac{\alpha}{\|f\|_\infty}} = \frac{\mu(f) - \alpha}{\|f\|_\infty - \alpha}$$

and therefore

$$\mu([f \geq \alpha]) = \mu\left(\left[g \geq \frac{\alpha}{\|f\|_\infty}\right]\right) \geq \frac{\mu(f) - \alpha}{\|f\|_\infty - \alpha}.$$

□

Theorem 2.3. *If μ is a probability on $\mathcal{B}$ and f is a positive, $\mathcal{B}$-measurable function on X for which $\mu(f) < \infty$, then*

a) For any positive number $\alpha, \alpha < \mu(f)$, there exists a number $M_\alpha, M_\alpha \geq \mu(f)$ such that

$$\mu([f \geq \alpha]) \geq \frac{\mu(f) - \alpha}{2(M_\alpha - \alpha)}.$$

b) If $\mathcal{F}$ is a relatively compact subset of $L_1^+(\mu)$ with respect to the weak topology $\sigma(L_1(\mu), L_\infty(\mu))$ on $L^1(\mu)$, then for any positive number α

$$\alpha < \inf\{\mu(f); f \in \mathcal{F}\}$$

there exists a number M_α

$$M_\alpha \geq \sup\{\mu(f); f \in \mathcal{F}\}$$

such that

$$\mu([f \geq \alpha]) \geq \frac{\mu(f) - \alpha}{2(M_\alpha - \alpha)}, \forall f \in \mathcal{F}.$$

Proof. a) Since by hypotheses $f \in L^1(\mu)$ if follows that the sequence $\left(\int\limits_{[f>n_0]} f(x)d\mu(x)\right)_{n\in\mathbb{N}}$ converges to 0. Hence there exists $n_0 \in \mathbb{N}$ such that $n_0 > \mu(f)$ and

$$\int\limits_{[f>n_0]} f(x)d\mu(x) < \frac{1}{2}(\mu(f) - \alpha).$$

We consider now the positive, bounded function f_0 on X given by

$$f_0(x) = \min(n_0, f(x)).$$

Obviously we have $[f_0 \geq \alpha] = [f \geq \alpha]$ and applying Theorem 2.2 of the bounded function f_0 we get

$$\mu[f \geq \alpha] = \mu[f_0 \geq \alpha] \geq \frac{\mu(f_0) - \alpha}{n_0 - \alpha} = \frac{(\mu(f) - \alpha) - (\mu(f - f_0))}{n_0 - \alpha}. \tag{2.3}$$

But $f - f_0 = 0$ onthe set $[f \leq n_0]$ and therefore

$$\mu(f - f_0) = \int_{[f>n_0]} (f - f_0) d\mu \leq \int_{[f\geq n_0]} f d\mu < \frac{1}{2}(\mu(f) - \alpha). \tag{2.4}$$

The assertion a) follows now from the relations (2.3) and (2.4). □

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THE COMPUTATION OF THE AREA OF A POLYGON AND OF THE VOLUME OF A POLYHEDRON BY THE MONTE CARLO METHODS

DANIEL CIUIU

Abstract. In this paper we will compute the area of a polygon by the Monte Carlo methods. The area of a convex polygon can be computed using two approaches. First one consists in computing the areas of the including thriangles using one point for all (and computing the sum). We use our $C++$ function to compute the area of a thriangle knowing its three points.

Another approach is to include the polygone in a rectangle, and we simulate $nrsim$ points inside the rectangle. The aire of the rectangle multiplied by the ratio of simulated points inside the polygon is the desired area of polygon.

The second approach is also used for a concave polygon, and in the first approach we simulate points insode its convex hull, i.e. we simulate probabilistic weights for the points of the convex hull. The ratio of inside generated points is now multiplied by the area of convex hull.

Similar two approaches are made for volume of a polyhedron, but the common point is now the gravity center, and the faces of the polyhedron are divided into triangles as in the case of polygons (for computing the sum of the volumes of the tetrahedra). In the first approach, we include the polyhedron in a rectangular parallelepiped.

Mathematics Subject Classification (2010): see http://www.ams.org/msc/
Key words: Areas and volumes, convex hull, Monte Carlo.

1. Introduction

If the polygon is a thrianle we compute the area as we know from classic algebra as the norm of cross product of $\overrightarrow{AB} \times \overrightarrow{AC}$ divided by two. In the same way we compute the volume of the tetrahedron $ABCD$ as the absolute value of the mixed product of $\overrightarrow{AB}$, $\overrightarrow{AC}$ and $\overrightarrow{AD}$ divided by 6.

Monte Carlo methods to compute integrals are presented in [5]. We compute

$$\int_a^b f(x)dx \tag{1.1}$$

by simulating a big number $nsim$ of uniform random variables on $[a, b]$, and

Key words and phrases. Areas and volumes, convex hull, Monte Carlo.

$$\int_a^b f(x)dx = (b-a) \cdot E[f(X)] \approx (b-a) \cdot \overline{f(X)}. \tag{1.1'}$$

Remark 1.1. *If we have to compute a double/ triple integral (f has two/ three parameters) in the above formula we multiply the expectation with the area/ volume of the domain.*

2. Methodology

For a polygon we consider first the four points that have the minimum and maximum coordinates X and Y. Of course, we can have less than four points as in Fig. 1 , but there are essential four values: X_{min}, X_{max}, Y_{min} and Y_{max}.

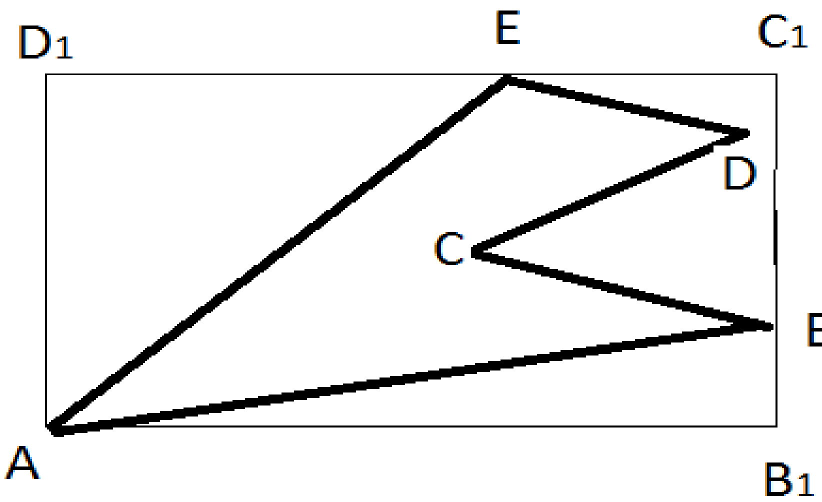

Figure 1. The polygon $ABCDE$ included in the rectangle $AB_1C_1D_1$

In the above picture we obtain the minimum for x and y in the same point A, but it is essential the rectangle we have obtained by parallels to Ox and Oy.

We simulate *nrsim* pairs (X, Y), where X is uniform on $[X_A, X_{B_1}]$ and Y is uniform on $[Y_A, Y_{D_1}]$. The area of $ABCDE$ is the area of $AB_1C_1D_1$ $\left(\|\vec{AB_1}\| * \|\vec{AD_1}\|\right)$ multiplied by the ratio of points that belong to the polygon.

For the second approach we consider the convex hull of the polygon: $ABDE$ in Fig. 1.

We generate weights for the points A, B, D and E such that

$$\sum_{i=1}^{n} p_i = 1. \tag{2.1}$$

For this we generate for each $(p_1, ...p_n)$

(1) p_1 uniform $[0, 1]$.

(2) p_i uniform on $\left[0, 1 - \sum_{j=1}^{i-1} p_j\right]$ for $i = \overline{2, n-1}$.

(3) $p_n = 1 - \sum_{i=1}^{n-1} p_i$.

We compute the area of the convex hull by dividing in thriagles with a common point (and two sides and diagonals), and we multiply this area by the ratio of inside points of polygon.

To test if a given point M is inside the polygon we start with a point P that we know that it is outside. We count the number of intersection of the line PM and the polygon sides A_iA_{i+1} and A_1A_n between the points A_i and A_{i+1}, and between P and M as in Fig. 2, where the intersection of PM with $ABCDE$ is $\{N_1, N_2\}$.

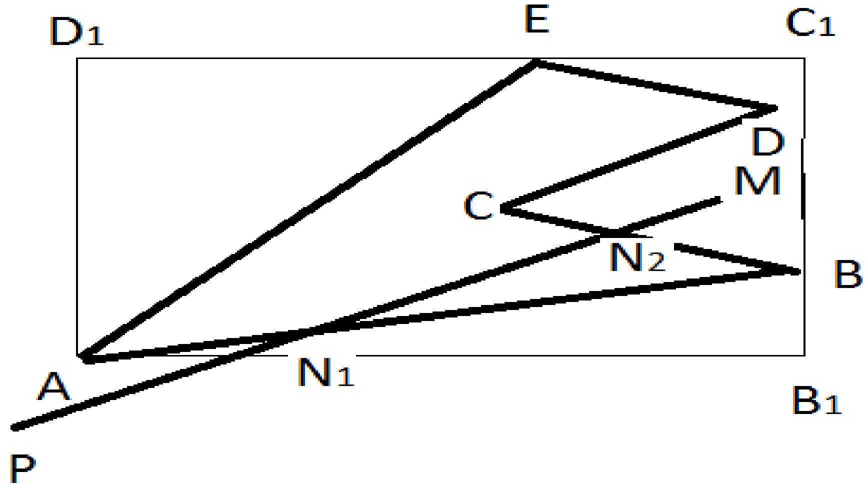

FIGURE 2. The intersection between PM and the polygon $ABCDE$

Because the number of points of intersection is even (two), we conclude that M is outside.

For testing if PM and A_iA_{i+1} have a common point, we consider the equation of PM

$$\alpha_0 X + \beta_0 Y + \gamma_0 = 0, \ \textit{and} \tag{2.2}$$

$$\alpha_i X + \beta_i Y + \gamma_i = 0. \tag{2.4'}$$

The intersection of the lines PM and A_iA_{i+1} is between P and M if

$$(\alpha_0 X_i + \beta_0 Y_i + \gamma_0) * (\alpha_0 X_{i+1} + \beta_0 Y_{i+1} + \gamma_0) < 0, \tag{2.3}$$

where X_i, Y_i and X_{i+1}, Y_{i+1} are the coordinates of A_i and A_{i+1}. Similar, if we denote by X_M and Y_M the coordinates of M and by X_P and Y_P the coordinates of P, we obtain

$$(\alpha_i X_P + \beta_i Y_P + \gamma_i) * (\alpha_i X_M + \beta_i Y_M + \gamma_i) < 0. \tag{2.3'}$$

In our $C++$ program we test if the product of the last two relations (the product of the four parentheses) is positive ("minus times minus=plus"), and onee of the products from above formulae is negative.

In the $3D$ case we determine the eight points with minimum and maximum of the three coordinates.

Instead of computing the area of a rectangle as in $2D$ case, we compute the volume of the parallelipiped that contains the polyhedron.

To check if the simulated point M in this parallelipiped is in the interior of the polyhedron, we count the number of faces that are intersected by the line PM (P is in the exterior of polyhedron, as in $2D$ case) in the interior of the face. If the number is odd M is interior, as in $2D$ case.

Using our $C++$ function we determine the intersection of the line

$$\begin{cases} X = x_0 + l * t \\ Y = y_0 + m * t \\ Z = z_0 + n * t \end{cases} \quad \text{with the plane} \tag{2.4}$$

$$A * X + B * Y + C * Z + D = 0. \tag{2.5}$$

To check if the intersection between PM and the plane that contains a face of the polyhedron, we count the number of the interior intersection of line P_iM_i and the lines of the face, where M_i is the intersection between PM and face i, and P_i a point on the plane of face i that is exterior.

Our $C++$ function that check if P_iM_i and a line of the face is interior for both lines uses the rotations of faces such that all points are in xOy and the origin for each face is the determined point exterior to the face. More exactly, if tha face is the polygon $M_1M_2....M_n$ and the exterior point is P we consider the coplanar vectors in $\mathbb{R}^3$

$$\overrightarrow{v}_i = \overrightarrow{PM_i}. \tag{2.6}$$

If the equation of the plane of the face is $AX + BY + CZ + D = 0$, the new vectors $\overrightarrow{i}_1$, $\overrightarrow{j}_1$ and $\overrightarrow{k}_1$ are

$$\begin{cases} \overrightarrow{k}_1 = \frac{1}{\sqrt{A^2+B^2+C^2}} \left(A\overrightarrow{i} + B\overrightarrow{j} + C\overrightarrow{k} \right) \\ \overrightarrow{i}_1 = \frac{1}{\|v_1\|} v_1 \\ \overrightarrow{j}_1 = \overrightarrow{k}_1 \times \overrightarrow{i}_1 \end{cases} . \tag{2.7}$$

The rotation of a thriangle and an exterior point P which becomes the new origin is presented in Fig. 3 that follows.

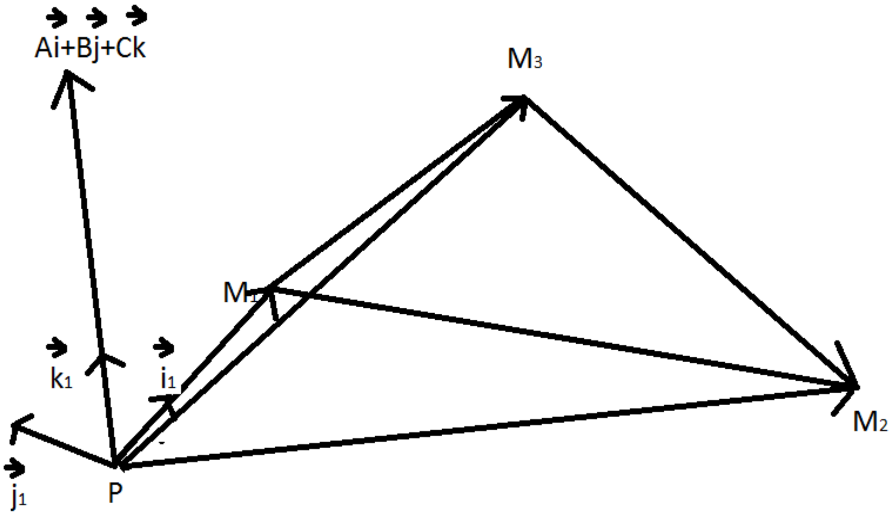

FIGURE 3. The rotation of the thriangle M_1M_2M3 with new origin P

3. Applications

Example 3.1. *Consider $ABCD$ with $A(1,2)$, $B(5,3)$, $C(3,0)$ and $D(3,1.67)$. We notice that D is the gravity center of ABC. Compute the areas of triangle ABC and the area of $ABCD$.*

The area of thriangle is 5.

The rectangle is $A_1B_1C_1D_1$ with $A_1(1,0)$, $B_1(5,0)$, $C_1(5,3)$ and $D_1(1,3)$.

The area of thriangle is 12, and using it and Monte Carlo methods simulating 10000 points is 3.3216.

The real value is $5 - \frac{5}{3} = \frac{10}{3} = 3.33333$.

Example 3.2. *Consider the tetrahedron $ABCD$ with $A(7,7,6)$, $B(9,8,6)$, $C(6,8,6)$ and $D(8,7,8)$. Compute the volume of $ABCD$.*

Consider also the gravity center of the tetrahedron, $G(7.5,7.5,6.5)$. We remove the face BCD and we add the faces GBC, GBD and GCD. Compute the volume of the polyhedron by Monte Carlo methods.

The volume of the tetrahedron is 1. Because we remove one quarter of it, the volume of the polyhedron is 0.75.

We include the polyhedron in the parallelipiped with $X \in [6,9]$, $Y \in [7,8]$ and $Z \in [6,8]$. The volume of the parallelipiped is 6.

The matrix of faces with the coordinates of points in columns, equations of faces, exterior points and rotated faces such that $Z = 0$ are presented in the following table.

Table 1. The six faces of the polyhedron, the equations of them, point in the planes of the polyhedron outside the faces and the rotated faces

Points	Equations and exterior points	Rotated faces
$\begin{pmatrix} 7 & 9 & 6 \\ 7 & 8 & 8 \\ 6 & 6 & 6 \end{pmatrix}$	$\begin{pmatrix} 0 & 5 \\ 0 & 0 \\ 3 & 6 \\ -18 & \end{pmatrix}$	$\begin{pmatrix} 5.196 & 2.888 \\ 6.928 & 2.309 \\ 5.196 & 4.041 \end{pmatrix}$
$\begin{pmatrix} 7 & 9 & 8 \\ 7 & 8 & 7 \\ 6 & 6 & 8 \end{pmatrix}$	$\begin{pmatrix} 2 & 2 \\ -4 & 6 \\ -1 & 0 \\ 20 & \end{pmatrix}$	$\begin{pmatrix} 6.928 & 2.268 \\ 8.66 & 1.134 \\ 8.66 & 3.401 \end{pmatrix}$
$\begin{pmatrix} 7 & 6 & 8 \\ 7 & 8 & 7 \\ 6 & 6 & 8 \end{pmatrix}$	$\begin{pmatrix} 2 & 5 \\ 2 & 6 \\ -1 & 0 \\ -22 & \end{pmatrix}$	$\begin{pmatrix} 5.196 & 0.577 \\ 5.196 & -0.577 \\ 6.928 & 1.155 \end{pmatrix}$
$\begin{pmatrix} 7.5 & 9 & 6 \\ 7.5 & 8 & 8 \\ 6.5 & 6 & 6 \end{pmatrix}$	$\begin{pmatrix} 0 & 5 \\ 1.5 & 14 \\ 1.5 & 0 \\ -21 & \end{pmatrix}$	$\begin{pmatrix} 1.443 & -5.307 \\ 2.309 & -4.899 \\ 0.577 & -4.899 \end{pmatrix}$
$\begin{pmatrix} 7.5 & 9 & 8 \\ 7.5 & 8 & 7 \\ 6.5 & 6 & 8 \end{pmatrix}$	$\begin{pmatrix} 0.5 & -10.5 \\ -2.5 & 6.5 \\ -1 & 0 \\ 21.5 & \end{pmatrix}$	$\begin{pmatrix} 14.722 & -1.897 \\ 15.588 & -2.846 \\ 15.588 & -0.949 \end{pmatrix}$
$\begin{pmatrix} 7.5 & 6 & 8 \\ 7.5 & 8 & 7 \\ 6.5 & 6 & 8 \end{pmatrix}$	$\begin{pmatrix} 0.5 & 18 \\ 2 & 6.5 \\ 0.5 & 0 \\ -22 & \end{pmatrix}$	$\begin{pmatrix} -1.732 & -6.94 \\ -2.598 & -7.348 \\ -0.866 & -7.348 \end{pmatrix}$

In the second column of the above table in the first column of the matrix we have A, B, C and D, while in the second column of the matrix we have the coordinates of the exterior points in the planes of the faces.

After simulating 10000 points we obtain the volume 0.74114.

If we compute the areas of the faces ABC, ABD, ACD, GBC, GBD and GDC we obtain the areas 1.5, 2.29129, 1.5, 1.06066, 1.36931 and 1.06066. For these computations we have computed the areas of thriangles for rotated faces. The total area is 8.78191.

4. Conclusions

In the approach of this paper (including the polygon in as rectangle and the polyhedron in a parallelipiped) and both cases (areas and volumes) we do not take into account the cases when one or more points of the polygon belong to PM. But this involves the points arround the points of the polygon/ polyhedron, which are neglectible for Monte Carlo methods.

We have not presented another approach, namely including the polygon/ polyhedron in its the convex hull. A method to determine the convex hull of a polygon is the Graham's scan algorithm [4]. An open problem is to determine the convex hull of the polygon/ polyhedron, and to compute the area/ volume of the convex hull. This area/ volume multiplied by the share of simulated points in the interior of the convex hull is the area of the polygon/ volume of the polyhedron.

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THE SETTING OF NORMED INTERVAL-SPACES

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Abstract. In this paper we will mainly introduce and will study *normed interval-spaces*. Then in the category of these spaces, we will consider *interval-bounded operators* and *interval-bounded functionals* and we will formulate results of the Hahn-Banach Extension Theorem type. As is known, this theorem is one of the fundamental results of Functional Analysis. But when we move from vector spaces to interval-spaces, the proofs can become difficult because these spaces are not vector spaces. The main reason is the non-existence, for some intervals, of the symmetric element with respect to the addition introduced in the interval-spaces. In 2013, the first author introduced the *interval-spaces* (abbreviated as *i-spaces*) and studied the problem of the extension of some *i-linear functionals.* Later, we studied the extension of *i-linear operators* with values in a Dedekind complete Riesz space. The next step will be to study the extension of the i-linear operators between two i-spaces. And this is precisely what we begin in this paper. *Interval-bounded operators* which we will introduce are some operators that act between two interval-spaces, more precisely between two *normed interval-spaces.* In the setting of normed interval-spaces we will give two results of the Hahn-Banach Extension Theorem type *for interval-bounded functionals.*
Mathematics Subject Classification: Primary 65G40; Secondary 46A22.
Keywords: interval-bounded operators, interval-bounded functionals, i-linear functionals, i-linear operators.

1. Introduction

In the last years, we studied *interval-spaces*, introduced by the first author, in 2013, at the Seventh Positivity Conference (a Zaanen Centennial Conference), Leiden University, the Netherlands, July 22-26 (see [2]). The initial purpose of this study took shape from the observation that for more than a century, Functional Analysis has been developed in the classical way, the basic structure being the *real vector spaces*. But an irrational (real) number x has an infinite number of decimals, none of which or any group repeats itself periodically. The number x can be placed in an infinite sequence of intervals, having the endpoints in the set of rational numbers, with increasingly smaller lengths. In this way, a sequence of intervals is obtained that approximates the respective number. Taking the sequence of the left endpoints of these intervals, we obtain the *sequence of approximations of the number x by missing.* Taking the sequence of the right endpoints of the same intervals, *the sequence of approximations of the number x through addition* is obtained. From here came the idea of considering *intervals of real numbers* instead of *real numbers.*

The next step was to consider *interval-spaces* instead of *vector spaces* and study Functional Analysis using such interval-spaces. Let's add to this, the observation as in Functional Analysis, we work, among other things, with *equalities* and *inequalities.* If we were wondering what is the fundamental tool in the calculations with these inequalities, respectively equalities, we remember that we have studied since elementary school that, for example, in any inequality the terms that appear can be passed from one member to another, provided that their signs to be changed. But the motivation of this 'rule' is given later, in the high school years, when algebraic structures are studied, including the *additive group* structure, on which the notion of *vector space* is based.

One of the methods of transforming these (in)equalities in an equivalent way, namely the method we refer to when we pass a term *x* (which is a real number) from one member to another, is based on the property of an (in)equality to remain true when to both members we add -*x*, that is, the *opposite* of *x*, which is the *symmetric element* of *x* in relation to the addition. But -*x* exists in $\mathbb{R}$, because $\mathbb{R}$ is a vector space. Which is no longer the case when we replace $\mathbb{R}$ with $I\mathbb{R}$, i.e. with the space of its closed intervals. And, even more so, when instead of $\mathbb{R}$ we consider some ordered vector space *E*, and therefore, instead of $I\mathbb{R}$, we consider the space *IE* of all closed intervals of *E*. We then understand why it can be difficult to study Functional Analysis in the setting of interval-spaces. We must mention that the development of the branch of mathematics known as *Interval Analysis* began with *Interval Arithmetic*, which was developed as a tool in *numerical computing.*

We will refer only to the beginnings of this field. For the mathematicians who contributed to the birth and development of Interval Analysis, see, for example [2] and the included references, and [6] and [5], too. In [1] we find some observations regarding the history of the subject.

So, it is mentioned that:

1) one of the first references to Interval Arithmetic as a tool of numerical computing was [8], written in 1967 by H. Grell, K. Maruhn, W. Rinow;

2) probably the most important paper for the development of *Interval Arithmetic* has been published earlier, in 1958, by the Japanese mathematician T. Sunaga – see [10];

3) although written in English the above cited paper of Sunaga did not find much attention until the first book on *Interval Analysis* appeared, which was written by R. E. Moore in 1966 – see [9], being the outgrowth of his Ph. D. Thesis.

In 2013, the first author of this paper introduced (see [2]) the *interval-spaces* (abbreviated as *i-spaces*) and studied (see [2], [3] and [4]) the problem of the the extension of some *i-linear functionals*. In two previous papers, (see [5] and [6]) we studied the extension of *i-linear operators* with values in a Dedekind complete Riesz space.

In [7], we specified that our next goal will be to extend this study to the case where the i-linear operators act between two i-spaces. This is exactly what we will begin in this paper, introducing first the *normed interval-spaces,* framework in which we will start with the *extension of the interval-bounded functionals.*

We specify that in what follows, all *scalars* that appear will be *real numbers.*

2. Preliminaries

2.1. Interval-Spaces

An *interval-space* (in short, *i-space*) is associated to an arbitrary real ordered vector space $(E,\leq)$.

An *order interval* in *E with the endpoints* $\underline{a},\overline{a}\in E$ where $\underline{a}\leq\overline{a}$, denoted by $[a]=[\underline{a},\overline{a}]$, is defined as the set: $\{x\in E \mid \underline{a}\leq x\leq\overline{a}\}$.

If $\underline{a}=\overline{a}$ is the element $a\in E$, then the interval $[a]=[\underline{a},\overline{a}]$ consists only of the element a, and we say that $[a]$ is a *degenerate interval*. If $\underline{a}<\overline{a}$, then the order interval

$[a]=[\underline{a},\overline{a}]$ is a *non-degenerate interval*. We can identify the element $a\in E$ with the degenerate interval $[a]=[a,a]$.

We endow IE with the following algebraic operations:

1) The *addition*:

$$[a]\oplus[b]=\{a+b \mid a\in[a],\, b\in[b]\},$$

that is,

$$[a]\oplus[b]=[\underline{a}+\underline{b},\overline{a}+\overline{b}],$$

if $[a]=[\underline{a},\overline{a}]$ and $[b]=[\underline{b},\overline{b}]$.

2) The *scalar multiplication* with reals:

$$\alpha[a]=\{\alpha a \mid a\in[a]\},\ \alpha\in\mathbb{R},$$

that is,

$$\alpha[a]=\begin{cases}[\alpha\underline{a},\alpha\overline{a}], \text{ if } \alpha\in\mathbb{R},\ \alpha\geq 0\\ [\alpha\overline{a},\alpha\underline{a}], \text{ if } \alpha\in\mathbb{R},\ \alpha<0\end{cases},$$

where $[a]=[\underline{a},\overline{a}]$.

We mention that, endowed with the previous algebraic operations, IE *is not* a real vector space; see, for example, [2].

Indeed, *the additive inverse does not* always *exist* and the so-called *second-distributive law* $((\alpha+\beta)[a]=\alpha[a]\oplus\beta[a])$ is certainly true only when $\alpha,\beta\in\mathbb{R}$ are such that $\alpha\beta>0$. (It should be mentioned that the other axioms of vector spaces are also true in i-spaces.)

More precisely $(IE,\oplus)$ *is* a commutative semigroup (monoid) with a neutral element $\mathbf{0}$ $(\mathbf{0}=[0,0])$ but *is not* a group, because a non-degenerate interval $[a]$ has no inverse with respect to addition. Indeed, suppose that, for $[a]=[\underline{a},\overline{a}]$ with $\underline{a}<\overline{a}$, there exists an inverse $[b]=[\underline{b},\overline{b}]$. Then:

$[\underline{a},\overline{a}]\oplus[\underline{b},\overline{b}]=\mathbf{0}\Rightarrow[\underline{a}+\underline{b},\overline{a}+\overline{b}]=[0,0]\Rightarrow\underline{a}+\underline{b}=0$ and $\overline{a}+\overline{b}=0\Rightarrow\underline{b}=-\underline{a}$ and $\overline{b}=-\overline{a}$.

But $\underline{b}\leq\overline{b}$ and therefore $-\underline{a}\leq-\overline{a}\Leftrightarrow\overline{a}\leq\underline{a}$, which is in contradiction with $\underline{a}<\overline{a}$.

We mention that we can define the *subtraction* in IE:

$$[a]\ominus[b]=[a]\oplus(-[b]),$$

where $[a],[b]\in IE$ and $-[b]$ means $(-1)[b]$.

Let us denote by $\mathcal{O}$ the set of all symmetric intervals of IE:

$$\mathcal{O}=\{[-a,a] \mid a\geq 0,\, a\in E\}.$$

Obviously, the set $\mathcal{O}$ is closed under the algebraic operations on IE.

Hence we say that an interval $[-a,a]$, where $a\in E$, is a *symetric interval* and we denote such an interval by $[o]$. If $[a]=[\underline{a},\overline{a}]$ and $[b]=[\underline{b},\overline{b}]$, are in *IE*, then $[a]\ominus[b]=[a]\oplus(-[b])$, where $-[b]=(-1)\cdot[b]$. It follows that:

$$[a]\ominus[b]=[\underline{a},\overline{a}]\oplus(-[\underline{b},\overline{b}])=[\underline{a}-\overline{b},\overline{a}-\underline{b}].$$

In particular $[a]\ominus[a]=[\underline{a}-\overline{a},\overline{a}-\underline{a}]$, so $[a]\ominus[a]=[-(\overline{a}-\underline{a}),\overline{a}-\underline{a}]$, that is, $[a]\ominus[a]$ is a symmetric interval. Hence, $[a]\ominus[a]\in\mathcal{O}$ for each $[a]\in IE$. Remark that if $[a]$

is a *degenerate interval*, that is, $[a]=[a,a]$, then $[a]\ominus[a]=\mathbf{0}$. For this reason, we will say that the set $\mathcal{O}$ is the *null set* of *IE*. Obviously, $[a]\ominus[a]\neq\mathbf{0}$ if $[a]$ is *not* a *degenerate interval* $[a,a]$. Thus we conclude again that IE is *not* a *real vector space*. Sometimes we will write $\mathcal{O}_{IE}$ instead of $\mathcal{O}$.

2.2. Interval-subspaces

An *interval-subspace* of IE or, in short, an *i-subspace* of IE, is a nonempty set IG of IE, closed under the algebraic operations (this meaning that for any $[u],[v]\in IG$ and $\alpha\in\mathbb{R}$, we have $[u]\oplus[v]\in IG$ and $\alpha[u]\in IG$.) Obviously $\mathbf{0}=[0,0]\in IG$ (because for any $[u]\in IG$, taking $\alpha=0$, it follows that $\mathbf{0}=0\cdot[u]\in IG$). Also the null set $\mathcal{O}$ of IE is an i-subspace of IE. (Recall that $\mathcal{O}=\{[-v,v]\mid v\geq 0,\ v\in E\}$.) We can define the *null part* $\mathcal{O}_{IG}$ of IG as the set $\mathcal{O}\cap IG$. It follows that $\mathcal{O}_{IG}=\{[u]\ominus[u]\,|\,[u]\in IG\}$. (Indeed, note that for all $[v]\in\mathcal{O}_{IG}$, $[v]=\left[\frac{1}{2}v\right]\ominus\left[\frac{1}{2}v\right]$ and $\left[\frac{1}{2}v\right]\in IG$, because IG is an i-subspace of IE.)

2.3. Interval-linear functionals (operators). Interval-sublinear functionals (operators)

If IE is an i-space, we say (see [2]) that a functional $f:IE\to\mathbb{R}$ is an *interval-linear functional* (in short, *i-linear functional*) if:

1. $f([x]\oplus[y])=f([x])+f([y])$, for all $[x],[y]\in IE$ (that is, f is an *interval-additive functional,* or in short, an *i-additive functional*);
2. $f(\alpha[x])=\alpha f([x])$, for all $[x]\in IE$ and $\alpha\in\mathbb{R}$ (that is, f is an *interval-homogeneous functional,* or in short, an *i-homogeneous functional*).

Example 1. (An *i-linear functional*) Define $f:I\mathbb{R}\to\mathbb{R}$ by $f([x])=\underline{x}+\overline{x}$, where $[x]=[\underline{x},\overline{x}]$. Then f is an *i-linear functional*.

If IE is an i-space, and IG is an i-subspace of IE we say that a real-valued function $s:IG\to\mathbb{R}$ is an *interval-sublinear functional* (in short an *i-sublinear functional*) on IG if:

1) $s([x]\oplus[y])\leq s([x])+s([y])$, for all $[x],[y]\in IG$ (that is, s is an *i-subadditive functional*);
2) $s(\alpha[x])=\alpha s([x])$, for all $[x]\in IG$ and $\alpha>0$ (that is, s is an *i-positively homogeneous functional*);
3) $s([x]\oplus[o])=s([x])$, for all $[x]\in IG$ and $[o]\in\mathcal{O}_{IG}$.

Example 2. (An *i-sublinear functional*) Define $s:IE\to\mathbb{R}$ by $s([x])=|\underline{x}+\overline{x}|$, where $[x]=[\underline{x},\overline{x}]$. Then s is an *i-sublinear functional.*

If IE is an i-space and F is an arbitrary Dedekind complete Riesz space, we say (see [5]) that an operator $L : IE \to F$ is an *interval-linear operator* (in short, *i-linear operator*) if:

1) $L([x]\oplus[y]) = L([x]) + L([y])$, for all $[x], [y] \in IE$ (that is, L is an *interval-additive operator*, or in short, an *i-additive operator*);

2) $L(\alpha[x]) = \alpha L([x])$, for all $[x] \in IE$ and $\alpha \in \mathbb{R}$ (that is, L is an *interval-homogeneous operator,* or in short, an *i-homogeneous operator*).

Remark 1. (Properties of i-linear operators) If $L : IE \to F$ is an i-linear operator, then:

3) $L([o]) = 0$, for all $[o] \in \mathcal{O}$;

4) $L([x]\oplus[o]) = L([x])$ for all $[x] \in IE$ and $[o] \in \mathcal{O}$.

Indeed, "4)" follows from "3)", since L is an i-additive operator. To prove "3)", take any $[o] \in \mathcal{O}$. From Lemma 1 below $[o] = -[o] \Rightarrow L([o]) = -L([o]) \Rightarrow 2L([o]) = 0$.

If IE is an i-space and F is an arbitrary Dedekind complete Riesz space, we say (see [5]) that an operator $S : IE \to F$ is an *interval-sublinear operator* or, in short, an *i-sublinear operator*, if

a) $S([x]\oplus[y]) \le S([x]) + S([y])$, for all $[x],[y] \in IE$ (that is, S is an *i-subadditive operator*);

b) $S(\alpha[x]) = \alpha S([x])$ for all $[x] \in IE$ and $\alpha > 0$ (that is, S is an *i-positively homogeneous operator*);

c) $S([x]\oplus[o]) = S([x])$, for all $[x] \in IE$ and $[o] \in \mathcal{O}$.

Remark 2. Notice that we assume "c)" for an i-sublinear operator since any i-linear operator, which obviously is an i-sublinear operator, satisfies "4)", from the above Remark 1.

Remark 3. (Properties of i-sublinear operators) Let IG be an i-subspace $(IG \subseteq IE)$, F an arbitrary Dedekind complete Riesz space and $S : IG \to F$ an i-sublinear operator. Then:

d) $S([o]) = 0$, for all $[o] \in \mathcal{O}_{IG}$;

e) $S(0\cdot[x]) = 0$, for all $[x] \in \mathcal{O}_{IG}$;

3. Main part

3.1. Interval-linear operators between two interval-spaces

Definition 1. Let IE and IF be two i-spaces and $T : IE \to IF$ an operator. We say that T is an:

1) *interval-additive operator* (in short, an *i-additive operator*) if

$$T([x]\oplus[y]) = T([x])\oplus T([y]), \text{ for all } [x], [y] \in IE;$$

2) *interval-homogeneous operator* (in short, an *i-homogeneous operator*) if

$$T(\alpha[x]) = \alpha T([x]), \text{ for all } [x] \in IE \text{ and } \alpha \in \mathbb{R}.$$

(Notice that $T([x]\oplus[y])$, $T([x])$ and $T([y])$ are from *IF*, that is, all these sets are closed intervals in *F*.)

Lemma 1. *An interval* $a \in IE$ *is symmetric if and only if* $[a] = -[a]$ (that is, $[a] = (-1) \cdot [a]$).

Proof. If $[a]$ is symmetric, then $[a] = [-a, a]$. But $-[a] = [-a, a]$. Hence $[a] = -[a]$. Conversely, if $[a] = -[a]$ and $[a] = [\underline{a}, \overline{a}]$, then $[\underline{a}, \overline{a}] = [-\overline{a}, -\underline{a}]$. Denoting $\overline{a} = a$, it follows that $[a] = [-a, a]$. ■

Proposition 2. *If* $[o] \in \mathcal{O}_{IE}$ *and* $T : IE \to IF$ *is an i-additive operator and, simultaneous, an i-homogeneous operator, then* $T([o]) \in \mathcal{O}_{IF}$.

Proof. Let $[o]$ be in $\mathcal{O}_{IE}$. According to Lemma 1, it follows that $[o] = -[o]$. Consequently, $T([o]) = -T([o])$. Applying Lemma 1 again we deduce that $T([o]) \in \mathcal{O}_{IF}$. ■

Definition 2. We say that an operator $T : IE \to IF$ is an *interval-linear operator* (in short, *an i-linear operator*) if:

1) $T([x] \oplus [y]) = T([x]) \oplus T([y])$, for all $[x], [y] \in IE$ (T is an *i-additive operator*);
2) $T(\alpha[x]) = \alpha T([x])$, for all $[x] \in IE$ and $\alpha \in \mathbb{R}$ (T is an *i-homogeneous operator*);
3) $T([o]) = \mathbf{0}$ $(= [0_F, 0_F])$, for all $[o] \in IE$.

We remark that according to Proposition 2, the hypothesis "3)" in Definition 2, makes sense, because $\mathbf{0} \in \mathcal{O}_{IF}$.

Example 3. (An *i-linear operator between two i-spaces*) For any fixed λ in $\mathbb{R}$, define $T_\lambda : I\mathbb{R} \to I\mathbb{R}$ by

$$T_\lambda([x]) = \begin{cases} \lambda[x], \ if\ [x] \in I\mathbb{R} \setminus \mathcal{O}_{I\mathbb{R}} \\ \mathbf{0} \quad\ , \ if\ [x] \in \mathcal{O}_{I\mathbb{R}} \end{cases}.$$

Then T_λ is an *i-linear operator.*

Proposition 3. If $T : IE \to IF$ *is an i-linear operator, then* $T([x] \oplus [o]) = T([x])$, *for all* $[x] \in IE$ *and* $[o] \in \mathcal{O}_{IE}$.

Proof. $T([x] \oplus [o]) = T([x]) \oplus T([o]) = T([x])$ since $T([o]) = \mathbf{0}$, from Definition 2. ■

We specify that in sections 3.2., 3.3. and 3.4. that follow, we will consider the i-space IF ordered with the so-called *weak order* "≤ " (see, for example, [7]). More precisely, if $[a] = [\underline{a}, \overline{a}]$ and $[b] = [\underline{b}, \overline{b}]$ are in IF, then

$$[a] \le [b] \text{ if } \underline{a} \le \underline{b} \text{ and } \overline{a} \le \overline{b}.$$

3.2. Interval-sublinear operators between two interval-spaces

Definition 3. Let IE and IF be two i-spaces and $S : IE \to IF$ an operator. We say that S is an:

1) *interval-subadditive operator* (in short, an *i-subadditive operator*) if

$$S([x]\oplus[y])\le S([x])\oplus S([y]), \text{ for all } [x],[y]\in IE;$$

2) *interval-positively homogeneous operator* (in short, an *i-homogeneous operator*) if

$$S(\alpha[x])=\alpha S([x]), \text{ for all } [x]\in IE \text{ and } \alpha>0.$$

(Notice that $S([x]\oplus[y])$, $S([x])$ and $S([y])$ are from IF, that is, all these sets are closed intervals in *F*.)

Definition 4. We say that an operator $S:IE\to IF$ is an *interval-sublinear operator* (in short, *an i-sublinear operator*) if:

1) $S([x]\oplus[y])\le S([x])\oplus S([y])$, for all $[x],[y]\in IE$ (*S* is an *i-subadditive operator*);

2) $S(\alpha[x])=\alpha S([x])$, for all $[x]\in IE$ and $\alpha>0$ (*S* is an *i-positively homogeneous operator*);

3) $S([x]\oplus[o])=S([x])$, for all $[x]\in IE$ and $[o]\in\mathcal{O}_{IE}$.

We notice that the hypothesis " 3) " in the Definition 4 is related to the remark that any i-linear operator *T* between two i-spaces *IE and IF* (that obviously must be an i-sublinear operator) satisfies the equality

$$T([x]\oplus[o])=T([x]),$$

for all $[x]\in IE$ and $[o]\in\mathcal{O}_{IE}$, according to the Proposition 3.

Example 4. (An *i-sublinear operator between two i-spaces*) Define $S:I\mathbb{R}\to I\mathbb{R}$ by

$$S([x])=\left[\underline{x}+\overline{x},|\underline{x}+\overline{x}|\right] \text{ if } [x]=[\underline{x},\overline{x}]\in I\mathbb{R}.$$

Then *S* is an *i-sublinear operator***.**

Proposition 4. (*Properties of i-sublinear operators between two i-spaces*)
If $S:IE\to IF$ *is an i-sublinear operator, then:*

4) $S([o])=\mathbf{0}$ $(\in\mathcal{O}_{IF})$, *for all* $[o]\in\mathcal{O}_{IE}$;

5) $S(0[x])=\mathbf{0}$, *for all* $[x]\in IE$.

Proof. 4) According to Definition 4, we can write: $2S([o])=S(2[o])=S([o]\oplus[o])=S([o])$. Then, if $S([o])=[\underline{a},\overline{a}]\in IF$, then $[2\cdot\underline{a},2\cdot\overline{a}]=[\underline{a},\overline{a}]$ and so, $\underline{a}=0=\overline{a}$. Then $S([o])=\mathbf{0}$.

5) If $[x]=[\underline{x},\overline{x}]$, then $S(0[x])=S([0,0])=\mathbf{0}$, according to 4). ■

3.3. Normed interval-spaces

Definition 5. Let *IE* be an i-space. We say that a function $\|\cdot\|:IE\to\mathbb{R}_+$ is an *interval-norm* (in short, an *i-norm*) on *IE*, if:

1) $\|[x]\oplus[y]\|\le\|[x]\|+\|[y]\|$, for all $[x],[y]\in IE$ ($\|\cdot\|$ is *interval-subadditive* or, in short, *i-subadditive*);

2) $\left\|\alpha[x]\right\| = |\alpha|\left\|[x]\right\|$, for all $\alpha \in \mathbb{R}$, and $[x] \in IE$ ($\|\cdot\|$ is *interval-absolute homogeneous*, or in short, *i-absolute homogeneous*);

3) $\left\|[x]\right\| = 0$, if and only if $[x] \in \mathcal{O}\ (= \mathcal{O}_{IE})$;

4) $\left\|[x] \oplus [o]\right\| \geq \left\|[x]\right\|$, for all $[x] \in IE$ and $[o] \in \mathcal{O}_{IE}$.

Sometimes, we will mark the *i-norm* $\|\cdot\|$ on *IE* with $\|\cdot\|_{IE}$.

Definition 6. If the i-space *IE* is endowed with an i-norm $\|\cdot\| : IE \to \mathbb{R}_+$, we say that ($IE, \|\cdot\|$) is a *normed interval-space* (in short, a *normed i-space*). It is also denoted by ($IE, \|\cdot\|_{IE}$).

Example 5. (A *normed i-space*) Define $\|\cdot\| : IE \to \mathbb{R}_+$, by $\left\|[x]\right\| = |\underline{x} + \overline{x}|$, if $[x] = [\underline{x}, \overline{x}]$. Then $\|\cdot\|$ is an *i-norm* on IE and $\left(IE, \|\cdot\|_{IE}\right)$ is a *normed i-space***.**

Example 6. (Another *normed i-space*) Define $\|\cdot\|_{I\mathbb{R}^2} : I\mathbb{R}^2 \to \mathbb{R}_+$ by the following equality:

$$\left\|[x]\right\|_{I\mathbb{R}^2} = |\underline{x}_1 + \overline{x}_1| + |\underline{x}_2 + \overline{x}_2|,$$

where $[x] = \left[(\underline{x}_1, \underline{x}_2), (\overline{x}_1, \overline{x}_2)\right] \in I\mathbb{R}^2$. Then $\|\cdot\|_{I\mathbb{R}^2}$ is an *i-norm* on $I\mathbb{R}^2$ and $\left(I\mathbb{R}^2, \|\cdot\|_{I\mathbb{R}^2}\right)$ is a *normed i-space*.

Proposition 5. *If* ($IE, \|\cdot\|$) *is a normed i-space, then* $\left\|[x] \oplus [o]\right\| = \left\|[x]\right\|$, *for all* $[x] \in IE$ *and* $[o] \in \mathcal{O}$.

Proof. $\left\|[x] \oplus [o]\right\| \leq \left\|[x]\right\| + \left\|[o]\right\| = \left\|[x]\right\|$, since $\left\|[o]\right\| = 0$. But $\left\|[x] \oplus [o]\right\| \geq \|x\|$, and therefore

$$\left\|[x] \oplus [o]\right\| = \|x\|.$$ ■

3.4. Interval-bounded operators between two normed interval-spaces.

Remark 5. After introducing the notion of *i-linear operator* between two *normed i-spaces*, we could consider the set *L*(*IE*, *IF*) of all i-linear operators between *IE* and *IF* and we could introduce the usual algebraic operations into it. Then we should study the resulting algebraic structure. But we will not do so because this is not the purpose of this work. We will associate a linear operator *T*, if possible, with a non-negative number so that this association has certain properties. If there exists such a number, we could call it, for the time being, the *i-norm of the operator T,* and we could say about the operator *T* that it is *i-bounded.* (See the next two definitions.)

Definition 7. If ($IE, \|\cdot\|_{IE}$) and ($IF, \|\cdot\|_{IF}$) are two normed i-spaces and $T : IE \to IF$ is an i-linear operator, we say that *T* is an *interval-bounded operator* (in short, an *i-bounded operator*), if there exists $M > 0$ such that

$$\left\|T\left([x]\right)\right\|_{IF} \leq M \left\|[x]\right\|_{IE} \text{ for all } [x] \in IE. \qquad (1)$$

Definition 8. Let ($IE, \|\cdot\|_{IE}$) and ($IF, \|\cdot\|_{IF}$) be two normed i-spaces, such that IE is not reduced to $\mathcal{O}_{IE}$ (that is, $IE \setminus \mathcal{O}_{IE} \neq \varnothing$). Let also $T : IE \to IF$ be an i-bounded operator. We say that the following nonnegative number

$$\|T\| = \sup_{[x]\in IE\setminus\mathcal{O}_{IE}} \frac{\|T([x])\|_{IF}}{\|[x]\|_{IE}} \tag{2}$$

is the *interval-norm of T* or, in short, the *i-norm* of *T*. (This number exists in $[0,+\infty)$, according to (1) and "3)" in Definition 5).

Proposition 6. *Let* ($IE, \|\cdot\|_{IE}$) *and* ($IF, \|\cdot\|_{IF}$) *be two normed i-spaces, such that* $IE \setminus \mathcal{O}_{IE} \neq \varnothing$.*Let also* $T : IE \to IF$ *be an i-bounded operator. Then it follows that*

$$\|T([x])\|_{IF} \leq \|T\| \|[x]\|_{IE}, \text{ for all } [x] \in IE. \tag{3}$$

Proof. Case 1. $[x] \in \mathcal{O}_{IE}$. Then $\|[x]\|_{IE} = 0$ and, from (1),

$$\|T([x])\|_{IF} = 0,$$

and therefore (3) is checked with equality.

Case 2. $[x] \in IE \setminus \mathcal{O}_{IE}$. Then $\|[x]\|_{IE} \neq 0$. From (2), it follows that

$$\|T([x])\|_{IF} \leq \|T\| \|[x]\|_{IE}.$$ ■

Proposition 7. (Equivalence of i-norm expressions for i-bounded operators) *Let* ($IE, \|\cdot\|_{IE}$) *and* ($IF, \|\cdot\|_{IF}$) *be two normed i-spaces, such that* $IE \setminus \mathcal{O}_{IE} \neq \varnothing$. *Let also* $T : IE \to IF$ *be an i-bounded operator. Then*:

$$\|T\| = \sup_{[x]\in IE\setminus\mathcal{O}_{IE}} \frac{\|T([x])\|_{IF}}{\|[x]\|_{IE}} = \sup_{\|[x]\|_{IE}=1} \|T([x])\|_{IF} = \sup_{\|[x]\|_{IE}\leq 1} \|T([x])\|_{IF}. \tag{4}$$

All these i-norm expressions for $\|T\|$ *describe the least possible* $M > 0$ *such that*

$$\|T([x])\|_{IF} \leq M \|[x]\|_{IE}, \text{ for all } [x] \in IE, \tag{5}$$

that is, describe the number $\inf\left\{M > 0 \,\middle|\, \|T([x])\|_{IF} \leq M \|[x]\|_{IE}, \text{ for all } [x] \in IE\right\}$.

Proof. Since *T* is an i-linear operator and any i- norm is i-positively homogeneous, we get:

$$\frac{\|T([x])\|_{IF}}{\|[x]\|_{IE}} = \left\|\frac{1}{\|[x]\|_{IE}} T([x])\right\|_{IF} = \left\|T\left(\frac{1}{\|[x]\|_{IE}} \cdot [x]\right)\right\|_{IF}, \text{ for all } [x] \in IE \setminus \mathcal{O}_{IE}. \tag{6}$$

Note that if we denote by S_1 and S_ρ the following sets

$$S_1 = \left\{[x] \in IE \,\middle|\, \|[x]\|_{IE} = 1\right\}, \text{ and}$$

$$S_\rho = \left\{[x] \in IE \,\middle|\, \|[x]\|_{IE} = \rho\right\},$$

then the mapping $\varphi : S_\rho \to S_1$, $\varphi([x]) = \frac{1}{\|[x]\|_{IE}}[x]$ is bijective.

(Indeed it suffices to prove that for any $[y] \in S_1$, there exists a unique $[x] \in S_\rho$ such $\varphi([x]) = [y]$. Let $[y] \in S_1$, that is, $\|[y]\| = 1$. Consider $[x] \in S_1$ given by $[x] = \rho[y]$.)

Thus, since $\rho > 0$ we have

$$\sup_{\|[x]\|_{IE} = \rho} \|T([x])\|_{IF} = \rho \sup_{\|[x]\|_{IE} = 1} \|T([x])\|_{IF}.$$

By considering all fixed, but different, $\rho > 0$, we see that

$$\sup_{\|[x]\| = \rho > 0} \frac{\|T([x])\|_{IF}}{\|[x]\|_{IE}} = \sup_{\|[x]\|_{IE} = 1} \|T([x])\|_{IF},$$

so that,

$$\sup_{[x] \in IE \setminus \mathcal{O}_{IE}} \frac{\|T([x])\|_{IF}}{\|[x]\|_{IE}} = \sup_{\|[x]\|_{IE} = 1} \|T([x])\|_{IF}.$$

Then, consider $\rho \le 1$ to see that

$$\sup_{\|[x]\|_{IE} \le 1} \|T([x])\|_{IF} \overset{\rho \le 1}{\le} \sup_{\|[x]\|_{IE} = 1} \|T([x])\|_{IF} \le \sup_{\|[x]\|_{IE} \le 1} \|T([x])\|_{IF},$$

were the last inequality follows from the definition of the supremum. This proves the assertion. ∎

Example 7. (An *i-bounded operator between two normed i-spaces* and its *i-norm*)
Let $(I\mathbb{R}^2, \| \|_{I\mathbb{R}^2})$ and $(I\mathbb{R}, \| \|_{I\mathbb{R}})$ be the normed i-spaces endowed with the following i-norms. $\|[x]\|_{I\mathbb{R}^2} = |\underline{x}_1 + \overline{x}_1| + |\underline{x}_2 + \overline{x}_2|$, where $[x] = [(\underline{x}_1, \underline{x}_2), (\overline{x}_1, \overline{x}_2)] \in I\mathbb{R}^2$, and $\|[y]\|_{I\mathbb{R}} = |\underline{y} + \overline{y}|$, where $[y] = [\underline{y}, \overline{y}] \in I\mathbb{R}$. Let $P : I\mathbb{R}^2 \to I\mathbb{R}$ be the canonical projection on the $Ox-$ axis, that is,

$$P([(\underline{x}_1, \underline{x}_2), (\overline{x}_1, \overline{x}_2)]) = [\underline{x}_1, \overline{x}_1],$$

for all $[x] = [(\underline{x}_1, \underline{x}_2), (\overline{x}_1, \overline{x}_2)] \in I\mathbb{R}^2$.
Then P is an *i-bounded operator* and $\|P\| = 1$.

From now on, we will not always write out the indices for the norms we just have to remember that $[x] \in IE$ and $T([x]) \in IF$.

3.5. Interval-bounded functionals on normed interval-spaces

Remark 6. We can easily rewrite those stated in Remark 5 from the previous section, replacing the *i-linear operators* with *i-linear functionals*. We thus arrive at the following two definitions.

Definition 9. If (*IE*, $\| \|$) is a normed i-space and $f : IE \to \mathbb{R}$ is an i-linear functional, we say that *f* is an *interval-bounded functional* (in short, *i-bounded functional*), if there exists $m > 0$, such that

$$|f([x])| \le m\|[x]\|, \text{ for all } [x] \in IE. \tag{7}$$

Definition 10. If (*IE*, $\| \|$) is a normed i-space such that *IE* is not reduced to $\mathcal{O}_{IE}$

(that is, $IE \setminus \mathcal{O}_{IE} \neq \varnothing$), and $f : IE \to \mathbb{R}$ is an i-bounded functional, we say that the following nonnegative number

$$\| f \| = \sup_{[x] \in IE \setminus \mathcal{O}_{IE}} \frac{\left| f\left([x]\right)\right|}{\left\|[x]\right\|} \tag{8}$$

is the *i-norm* of *f*.

Example 8. (An *i-bounded functional on an normed i-space and* its *i-norm*) Let $\left(I\mathbb{R}, \| \ \|_{I\mathbb{R}}\right)$ be the normed i-space endowed with the i-norm

$$\left\|[x]\right\|_{I\mathbb{R}} = \left|\underline{x} + \overline{x}\right|,$$

where $[x] = [\underline{x}, \overline{x}] \in I\mathbb{R}$.

Let $f : I\mathbb{R} \to \mathbb{R}$ be the following i-linear functional:

$$f\left([x]\right) = \underline{x} + \overline{x},$$

where $[x] = [\underline{x}, \overline{x}]$.

Then f is an *i-bounded functional* on $I\mathbb{R}$ and $\| f \| = 1$.

Proposition 8. *Let* (*IE*, $\| \ \|$) *be a normed i-space such that IE is not reduced to* $\mathcal{O}_{IE}$ *and* $f : IE \to \mathbb{R}$ *is an i-bounded functional. Then it follows that*:

$$\left| f\left([x]\right)\right| \leq \| f \| \left\|[x]\right\|, \textit{ for all } [x] \in IE. \tag{9}$$

Proposition 9. (Equivalence of i-norm expressions for i-bounded functionals) *Let* (*IE*, $\| \ \|$) *be a normed i-space such that IE is not reduced to* $\mathcal{O}_{IE}$ *and let* $f : IE \to \mathbb{R}$ *be an i-bounded functional*. *Then*:

$$\| f \| = \sup_{[x] \in IE \setminus \mathcal{O}_{IE}} \frac{\left| f\left([x]\right)\right|}{\left\|[x]\right\|} = \sup_{\left\|[x]\right\| = 1} \left| f\left([x]\right)\right| = \sup_{\left\|[x]\right\| \leq 1} \left| f\left([x]\right)\right|.$$

All these i-norm expressions for $\| f \|$ *describe the least possible* $m > 0$ such that

$$\left| f\left([x]\right)\right| \leq m \left\|[x]\right\|, \textit{ for all } [x] \in IE,$$

that is, they describe the number

$$\inf\left\{ m > 0 \,\middle|\, \left| f\left([x]\right)\right| \leq m \left\|[x]\right\|, \textit{ for all } [x] \in IE \right\}.$$

3.6. Hahn-Banach Extension type Theorems in the setting of normed i-spaces

Theorem 10. (Hahn-Banach Extension type Theorem in the setting of i-spaces; see Corollary 5 in [2]) *Let* IE *be an arbitrary i-space and* $IG \subseteq IE$ *an i-subspace. Let also* $s : IE \to \mathbb{R}$ *be an i-sublinear functional and* $t : IG \to \mathbb{R}$ *an i-linear functional. Then the following are equivalent:*

i) there exists an i-linear functional $l : IE \to \mathbb{R}$, *such that:*

a) $l \leq s$ *on* IE, *and*

b) $l = t$ *on* IG, *that is,* l *is an i-linear extension of* t;

ii) $t \leq s$ *on* IG.

Theorem 11. *Let* (*IE*, $\| \ \|$) *be a normed i-space, and* $IG \subseteq IE$ *an i-subspace of IE. Let also* $t : IG \to \mathbb{R}$ *an i-bounded functional such that* $\left|t\left([v]\right)\right| \le \left\|[v]\right\|$, for all $[v] \in IG$. *Then there exists an i-bounded functional* $l : IE \to \mathbb{R}$ *such that*:

a) $l\left([v]\right) = t\left([v]\right)$, *for all* $[v] \in IG$, *that is, l is an i-linear extension of t;*

b) $\left|l\left([x]\right)\right| \le \left\|[x]\right\|$, *for all* $[x] \in IE$.

Proof. Define $s\left([x]\right) = \left\|[x]\right\|$, for all $[x] \in IE$. According to Proposition 5, *s* is an i-sublinear functional on *IE*. Also for all $[v] \in IG$ it follows:

$$t\left([v]\right) \le \left|t\left([v]\right)\right| \le s\left([v]\right).$$

Applying Theorem 10, it follows that there exists an i-linear functional $l : IE \to \mathbb{R}$ such that:

1) $l\left([v]\right) = t\left([v]\right)$, for all $[v] \in IG$;

2) $l\left([x]\right) \le s\left([x]\right)$, for all $[x] \in IE$.

But $-l\left([x]\right) = l\left(-[x]\right) \le s\left(-[x]\right) = \left\|-[x]\right\| = |-1|\left\|[x]\right\| = \left\|[x]\right\|$.

So $-l\left([x]\right) \le \left\|[x]\right\|$, for all $[x] \in IE$. Therefore $\left|l\left([x]\right)\right| \le \left\|[x]\right\|$, for all $[x] \in IE$.

Consequently, *l* is an i-bounded extension of *t*. ■

Theorem 12. *Let* (*IE*, $\| \ \|$) *be a normed i-space and IG an i-subspace of IE, such that* $IE \setminus \mathcal{O}_{IE} \ne \emptyset$ *and* $IG \setminus \mathcal{O}_{IG} \ne \emptyset$ *Let also* $t : IG \to \mathbb{R}$, *an i-bounded functional. Then there exists an i-bounded functional* $l : IE \to \mathbb{R}$, *such that*:

1) $l\left([v]\right) = t\left([v]\right)$, *for all* $[v] \in IG$, *that is, l is an i- bounded extension of t*;

2) $\|l\| = \|t\|$, *where* $\|l\| = \sup\limits_{\substack{[x]\in IE\setminus\mathcal{O}_{IE} \\ \|[x]\|=1}} \left|l\left([x]\right)\right|$ *and* $\|t\| = \sup\limits_{\substack{[x]\in IG\setminus\mathcal{O}_{IG} \\ \|[v]\|=1}} \left|t\left([v]\right)\right|$.

Proof. From Proposition 8, it follows that $\left|t\left([v]\right)\right| \le \|t\|\left\|[v]\right\|$, for all $[v] \in IG$.

Define $s : IE \to \mathbb{R}$ by: $s\left([x]\right) = \|t\|\left\|[x]\right\|$. Applying Theorem 11, we obtain an i-bounded functional $l : IE \to \mathbb{R}$ such that:

a) $l\left([v]\right) = t\left([v]\right)$, for all $[v] \in IG$;

b) $\left|l\left([x]\right)\right| \le s\left([x]\right) = \|t\| \cdot \left\|[x]\right\|$, for all $[x] \in IE$.

Applying b) and taking the supremum over all $[x] \in IE$ with $\left\|[x]\right\| = 1$, we obtain the inequality:

$$\|l\| = \sup_{\substack{[x]\in IE\setminus\mathcal{O}_{IE} \\ \|[x]\|=1}} \left|l\left([x]\right)\right| \le \|t\|. \tag{10}$$

Since *l* is an extension of *t*, the norm should increase, that is,

$$\|l\| \ge \|t\|.$$

Therefore, using (10), it follows that

$$\|l\| = \|t\|.$$

■

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ON THE SOLUTIONS OF SOME FRACTIONAL ORDINARY DIFFERENTIAL EQUATIONS

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Abstract. Solutions of some initial value problems for fractional ordinary diferential equations, with respect to the Caputo fractional derivative, are studied.

Mathematics Subject Classification (2010): 34A08, 26A33
Key words: Caputo fractional derivative, α-fractional power series

1. Introduction

Fractional differential equations are useful tools for modelling phenomena in science and engineering (see, for example, [2], [5], [6] or [8]). In order to define the Caputo fractional derivative we need the following definition.

Definition 1.1. *A function $y:(x_0,\infty)\to\mathbb{R}$ is said to be of class C_μ ($\mu\in\mathbb{R}$) if there exists $p>\mu$ such that $y(x)=(x-x_0)^p z(x),\forall x>x_0$, where $z:[x_0,\infty)\to\mathbb{R}$ is a continuous function. The function y is said to be of class $C_\mu^{(n)}$ ($n\in\mathbb{N}$) if $y^{(n)}\in C_\mu$.*

Consider α a positive real number and $n=\lceil\alpha\rceil$, where $\lceil\cdot\rceil$ is the *ceiling function*, that is $\lceil x\rceil=\min\{z\in\mathbb{Z}:z\geq x\}$.

Definition 1.2. *The Caputo fractional derivative of order $\alpha\geq 0$ of a function $y\in C_{-1}^{(n)}$ is defined as*

$$\hat{D}_{x_0}^{\alpha}y(x)=\begin{cases}\dfrac{1}{\Gamma(n-\alpha)}\displaystyle\int_{x_0}^{x}\frac{y^{(n)}(s)}{(x-s)^{1+\alpha-n}}ds\,, & \alpha\notin\mathbb{N}\\ y^{(\alpha)}(x)\,, & \alpha\in\mathbb{N}\,.\end{cases}$$

The Caputo fractional derivative has the following properties:

$$\hat{D}_{x_0}^{\alpha}\left(\lambda_1y_1(x)+\lambda_2y_2(x)\right)=\lambda_1\hat{D}^{\alpha}y_1(x)+\lambda_2\hat{D}^{\alpha}y_2(x)\,,\ \forall\lambda_1,\lambda_2\text{ constants},$$

$$\hat{D}_{a}^{\alpha}(x-x_0)^{\beta}=\begin{cases}0, & \text{if }\beta\in\{0,1,...,\lceil\alpha\rceil-1\}\\ \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)}(x-x_0)^{\beta-\alpha}, & \text{if }\beta\in\mathbb{N}\text{ and }\beta\geq\lceil\alpha\rceil\\ & \text{or }\beta\notin\mathbb{N}\text{ and }\beta>\lceil\alpha\rceil-1\end{cases}\tag{1.1}$$

2. Main results

Many results from usual calculus were extended, not necessary in the same manner, to fractional calculus. Thus a generalized Taylor's Formula, with respect to the Caputo fractional derivative, was proved in [7].

Theorem 2.1. *(Generalized Taylor's Formula) Suppose that* $(\hat{D}_a^\alpha)^k y(x) \in C(a,b]$ *for* $k = 1, 2, ..., n+1$, *where* $0 < \alpha \le 1$. *Then we have*

$$y(x) = \sum_{i=0}^{n} \frac{(x-a)^{i\alpha}}{\Gamma(i\alpha+1)}((\hat{D}_a^\alpha)^i y)(a^+) + R_n^\alpha(x,a),$$

where

$$R_n^\alpha(x,a) = \frac{((\hat{D}_a^\alpha)^{n+1} y)(\xi)}{\Gamma((n+1)\alpha+1)}(x-a)^{(n+1)\alpha},$$

with $a \le \xi \le x$, *for each* $x \in (a,b]$ *and* $((\hat{D}_a^\alpha)^n u)(x) = \underbrace{(D_0^\alpha \circ D_0^\alpha \circ \ldots \circ D_0^\alpha}_{n-times} u)(x)$ *is the Caputo fractional derivative of order* n.

Definition 2.2. *We call a series of the form*

$$\sum_{i=0}^{+\infty} a_i (x-x_0)^{i\alpha}, \tag{2.1}$$

where a_i *are real numbers,* $\alpha > 0$, $x, x_0 \in \mathbb{R}$, $x \ge x_0$ *an* α*-fractional power series at* x_0.

If $I = (a,b) \subset \mathbb{R}$ is an open interval, a function $y : I \to \mathbb{R}$, is called *fractional analytic on* I, written $y \in C^\omega(I)$, if, for each $x_0 \in I$, the function y may be represented into α-fractional power series at x_0 in some interval $[x_0, \tilde{x}) \subset I$.

Theorem 2.3. *(see [3]) Let* $\sum\limits_{i=0}^{\infty} a_i x^{i\alpha}$, $\alpha \in (0,1]$ *be a* α*-fractional power series, let* $R \ge 0$ *be the radius of convergence of the power series* $\sum\limits_{i=0}^{\infty} a_i x^i$ *and* $r = \begin{cases} R^{\frac{1}{\alpha}}, \text{ if } R < \infty \\ \infty, \text{ if } R = \infty \end{cases}$. *Then*

i) If $R > 0$, *then the series* $\sum\limits_{i=0}^{\infty} a_i x^{i\alpha}$ *converges absolutely and uniformly on* $[0,b]$, *for all* $b \in (0,r)$ *and there exists a positive integer* $N(b)$ *such that*

$$|a_i| \le b^{-i\alpha}, \text{ for all } i \ge N(b); \tag{2.2}$$

ii) if $u : [0,r) \to \mathbb{R}$ *is the sum of the* α*-fractional power series,* $u(x) = \sum\limits_{i=0}^{\infty} a_i x^{i\alpha}$, $\forall x \in [0,r)$, *then* u *is continuous and there exists the Caputo derivative,* $\hat{D}_0^\alpha u : [0,r) \to \mathbb{R}$. *Moreover, the series of the Caputo derivatives,* $\sum\limits_{i=0}^{\infty} a_i \hat{D}_0^\alpha (x^{i\alpha}) = \sum\limits_{i=1}^{\infty} a_i \frac{\Gamma(i\alpha+1)}{\Gamma((i-1)\alpha+1)} x^{(i-1)\alpha}$ *converges absolutely and uniformly on* $[0,b]$, $\forall b \in (0,r)$ *and*

$$\hat{D}_0^\alpha u(x) = \sum_{i=0}^{\infty} a_i \hat{D}_0^\alpha \left(x^{i\alpha}\right) = \sum_{i=1}^{\infty} a_i \frac{\Gamma(i\alpha+1)}{\Gamma((i-1)\alpha+1)} x^{(i-1)\alpha}, \forall x \in [0,r). \tag{2.3}$$

Corollary 2.4. *If u is the sum of an α-fractional power series* (2.1)*, with $x_0 = 0$ on the open interval I, then, for any $n \in \mathbb{N}$, there exists $(\hat{D}_0^\alpha)^n u \in C(I)$ and*

$$((\hat{D}_0^\alpha)^n u)(x) = \sum_{i=n}^{\infty} a_i \frac{\Gamma(i\alpha+1)}{\Gamma((i-n)\alpha+1)} x^{(i-n)\alpha}, \forall x \in I. \tag{2.4}$$

Consequently, the coefficients of the α - fractional Taylor series (2.1)*, where $x_0 = 0$, are given by*

$$a_i = \frac{D^{i\alpha}u(0)}{\Gamma(i\alpha+1)}.$$

Theorem 2.5. *Let* (2.1) *be an α-fractional power series at $x_0 \in I$ and $\mathcal{C} = [x_0, x_0 + r)$, where r is defined in Theorem 2.3. If $\lceil \alpha \rceil = 1$ and $u : \mathcal{C} \to \mathbb{R}$ is defined by*

$$u(x) = \sum_{i=0}^{\infty} a_i (x - x_0)^{i\alpha}, \tag{2.5}$$

then $u(x)$ is α-fractional analytic on $\mathcal{C}$.

Proof. Without loss the generality we may assume $x_0 = 0$. Let $x \in \mathcal{C} =]0, r)$ be arbitrary. We can choose $\delta > 0$ such that $(1+\delta)^{\frac{1}{\alpha}} x < r$. Then there exists a positive constant C_1 such that, for all i, $|a_i| \left| \left((1+\delta)^{\frac{1}{\alpha}} x \right)^{i\alpha} \right| < C_1$. Set $b_j = \frac{((\hat{D}_0^\alpha)^j y)(x)}{\Gamma(j\alpha+1)}$. Then, by (2.4), it follows that

$$b_j = \sum_{i=0}^{+\infty} a_{i+j} \frac{\Gamma((i+j)\alpha+1)}{\Gamma(i\alpha+1)\Gamma(j\alpha+1)} x^{i\alpha}.$$

Choose $0 < \rho < \delta$ and $y > x$ such that $y - x < \rho^{\frac{1}{\alpha}} x$. Since $\frac{\Gamma((i+j)\alpha+1)}{\Gamma(i\alpha+1)\Gamma(j\alpha+1)} \le \binom{i+j}{i}$ we get, for every m,

$$\sum_{j=0}^{m} |b_j| \left| (y-x)^{j\alpha} \right| \le \sum_{j=0}^{m} \sum_{i=0}^{+\infty} |a_{i+j}| \frac{\Gamma((i+j)\alpha+1)}{\Gamma(i\alpha+1)\Gamma(j\alpha+1)} x^{i\alpha} (y-x)^{j\alpha}$$

$$\le \sum_{j=0}^{m} \sum_{i=0}^{+\infty} |a_{i+j}| x^{(i+j)\alpha} \binom{i+j}{i} \rho^j \le C_1 \sum_{j=0}^{m} \sum_{i=0}^{+\infty} \binom{i+j}{i} \frac{\rho^j}{(1+\delta)^{i+j}}$$

$$\le C_1 \sum_{j=0}^{+\infty} \sum_{i=0}^{+\infty} \binom{i+j}{i} \cdot \left(\frac{\rho}{1+\delta} \right)^j \frac{1}{(1+\delta)^i}. \tag{2.6}$$

Since, for two real numbers a, b such that $|a| + |b| < 1$, it follows that

$$\sum_{j=0}^{+\infty} \sum_{i=0}^{+\infty} \binom{i+j}{i} a^i b^j = \frac{1}{1-a-b}$$

(see, for example, Lemma 2.2.6 from [4]) the series in (2.6) converges and its sum is equal to $C_1 \cdot \frac{1+\delta}{\delta-\rho}$. Hence it follows that the series $\sum\limits_{j=0}^{+\infty} b_j\,(y-x)^{j\alpha}$ converges absolutely, for every $y \in J = [x, (1+\rho^{\frac{1}{\alpha}})x) \subset \mathcal{C}$.

Finally, as in the case of usual analytic functions, by Theorem 2.1, it is enough to estimate

$$\left|u(x) - \sum_{j=0}^{m} b_j\,(y-x)^{j\alpha}\right| \le \frac{\left|((\hat{D}_x^\alpha)^{m+1}u)(\xi)\right|}{\Gamma((m+1)\alpha+1)}\left|(y-x)^{(m+1)\alpha}\right|$$

$$\le \sum_{i=0}^{+\infty} |a_{i+m+1}|\,|x^{i\alpha}| \begin{pmatrix} i+m+1 \\ i \end{pmatrix} \left|(y-x)^{(m+1)\alpha}\right|$$

$$\le C_1 \sum_{i=0}^{+\infty} \begin{pmatrix} i+m+1 \\ i \end{pmatrix} \cdot \frac{\rho^{m+1}}{(1+\delta)^{i+m+1}} \le 2C_1 \sum_{i=0}^{+\infty} \begin{pmatrix} i+m+1 \\ i \end{pmatrix} \cdot \left(\frac{\rho}{1+\delta}\right)^{m+1} \frac{1}{(1+\delta)^i},$$

where, from the convergence of the series in (2.6), the last series approaches 0 as m approaches $+\infty$. □

Consider $\alpha \in (0,1]$ and the fractional linear ordinary differential equation

(2.7)
$$((\hat{D}_0^\alpha)^n y)(x) + a_1(x)((\hat{D}_0^\alpha)^{n-1}y)(x) + \ldots + a_{n-1}(x)(\hat{D}_0^\alpha y)(x) + a_n(x)y(x) = f(x),\ x \in [0,b].$$

Theorem 2.6. *(see [3]) Suppose that $b' > b$ and $f, a_j,\ j = 1,2,\ldots,n$ are respresentable into α-fractional power series at 0 on $[0,b')$. If $y_i^{(0)}$, $i = 0,1,...,n-1$, are arbitrary real numbers, then there exists $y = y(x)$ respresentable into α-fractional power series at 0 on $[0,b]$ which is a solution of the equation (2.7), uniquely determined such that*

$$((\hat{D}_0^\alpha)^i y)(0) = y_i^{(0)},\ i = 0,1,...,n-1. \tag{2.8}$$

From Theorems 2.5 and 2.6 it followsis an extension of the Cauchy-Kowalevski theorem in the case of fractional linear ordinary differential equation.

3. Numerical examples

Let

$$\sum_{i=0}^{+\infty} b_i x^{i\alpha}, \tag{3.1}$$

where $x \ge 0$, $\alpha \in (0,1]$, a convergent α-fractional power series at 0 on $I = [0,b)$ having the sum equal to $\gamma(x)$. Consider the initial value problem for the α-fractional ordinary differential equation

$$((\hat{D}_0^\alpha)^2 y)(x) + \gamma(x)y(x) = 0,\ x \in [0,+\infty), \tag{3.2}$$

and

$$y(0) = \delta_0,\ ((\hat{D}_0^\alpha)y)(0) = \delta_1, \tag{3.3}$$

where $\delta_0,\ \delta_1 \in \mathbb{R}$.

Based on Theorem 2.6 we seek a soluion of the form

$$y(x) = \sum_{i=0}^{+\infty} a_i x^{i\alpha}. \tag{3.4}$$

Hence

$$\gamma(x)y(x) = \sum_{i=0}^{+\infty} c_i x^{i\alpha},$$

where

$$c_i = \sum_{j=0}^{i} a_j b_{i-j}.$$

Since, by Corollary 2.4, $((\hat{D}_0^\alpha)^2 y)(x) = \sum_{i=2}^{+\infty} \frac{\Gamma(i\alpha+1)}{\Gamma((i-2)\alpha+1)} a_i x^{(i-2)\alpha}$, by (3.2)-(3.4) we get $a_0 = \delta_0$, $a_1 = \frac{\Gamma(1)\delta_1}{\Gamma(\alpha+1)}$,

$$a_{i+2} = -\frac{\Gamma(i\alpha+1)}{\Gamma((i+2)\alpha+1)} \sum_{j=0}^{i} a_j b_{i-j},\ i \geq 0. \tag{3.5}$$

Example 3.1. Consider $\gamma(x) = 1$. In this case, by (3.4) and (3.5), we get $a_{i+2} = -\frac{\Gamma(i\alpha+1)}{\Gamma((i+2)\alpha+1)} a_i,\ i \geq 0$. Hence it follows that

$$a_{i+2} = \begin{cases} (-1)^j \frac{\Gamma(1)}{\Gamma((2j+2)\alpha+1)} \delta_0^j & \text{if } i = 2j \\ (-1)^j \frac{\Gamma(\alpha+1)}{\Gamma((2j+3)\alpha+1)} \delta_1^j & \text{if } i = 2j+1 \end{cases}$$

and

$$y(x) = \sum_{j=0}^{+\infty} \frac{(-1)^j \delta_0 x^{2j\alpha}}{\Gamma(2j\alpha+1)} + \sum_{j=0}^{+\infty} \frac{(-1)^j \delta_1 \Gamma(\alpha+1) x^{(2j+1)\alpha}}{\Gamma((2j+1)\alpha+1)}. \tag{3.6}$$

If $\delta_0 = 0$ and $\delta_1 = 1$ the solution of the initial value problem is

$$y(x) = \sum_{j=0}^{+\infty} \frac{(-1)^j \Gamma(\alpha+1) x^{(2j+1)\alpha}}{\Gamma((2j+1)\alpha+1)}. \tag{3.7}$$

For $\alpha = 0$ and $\alpha = 0.8$ the solutions are represented in Figure 1.

Example 3.2. Consider the equation (3.2), where $\gamma(x) = 1$, $x \in [0, b]$ with $b > 0$ and the boundary conditions

$$y(0) = 0,\ y(b) = s, \tag{3.8}$$

where s is a real number different from 0. In order to find solutions for the boundary value problem (3.2), (3.7), as in the classical case (see [1], p.88), consider the initial value problem (3.2) and

$$y(0) = 0,\ ((\hat{D}_0^\alpha)y)(0) = s_1,\ s_1 \neq 0. \tag{3.9}$$

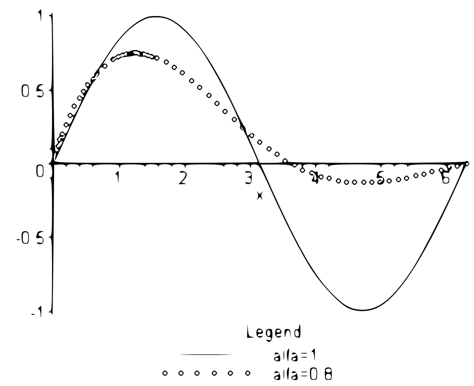

FIGURE 1. The solutions in the cases $\alpha = 1$ and $\alpha = 0.8$

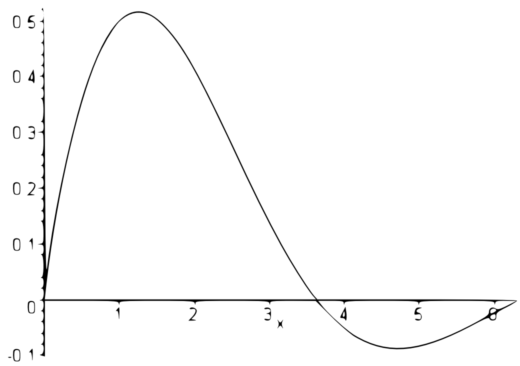

FIGURE 2. The solution of BVP for $\alpha = 0.8$

Then, by (3.6) we find the solution of the initial value problem (3.2), (3.9).

$$\tilde{y}(x) = w(x, s_1) = \sum_{j=0}^{+\infty} \frac{(-1)^j s_1 \Gamma(\alpha+1) x^{(2j+1)\alpha}}{\Gamma((2j+1)\alpha+1)}. \tag{3.10}$$

If we impose that $w(b, s_1) = s$, it follows that

$$\sum_{j=0}^{+\infty} \frac{(-1)^j \Gamma(\alpha+1) b^{(2j+1)\alpha}}{\Gamma((2j+1)\alpha+1)} = \frac{s}{s_1}. \tag{3.11}$$

Hence, for $s_1 = \frac{s}{y(b)}$, where y is defined by (3.7), $\tilde{y}(x) = w(x, s_1)$ defined by (3.10) is a solution of the boundary value problem (3.2), (3.9). As a numerical example, if we take $\alpha = 0.8$, $b = 1$, $s = 0.5$ we get $s_1 = .6908$ and the solution $\tilde{y}$ of the boundary value problem is represented in Figure 2.

Example 3.3. Consider the equation (3.2), where $\gamma(x) = \sum_{i=0}^{+\infty} x^{i\alpha}$, $x \in [0, 1)$ and the initial conditions

$$y(0) = \delta_0, \; ((\hat{D}_0^\alpha)y)(0) = \delta_1. \tag{3.12}$$

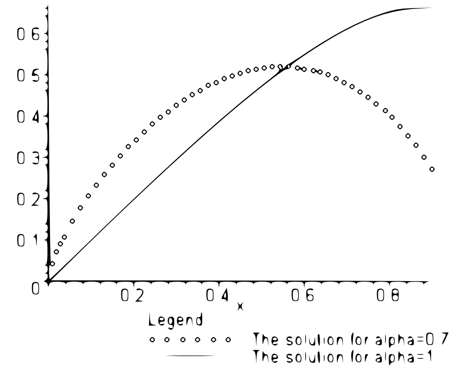

FIGURE 3. The solutions of the initial value problem for $\alpha = 0.7$ and $\alpha = 1$.

Then, by (3.5) we find the solution (3.4) of the initial value problem (3.2), (3.12) where $a_0 = \delta_0$, $a_1 = \frac{\Gamma(1)\delta_1}{\Gamma(\alpha+1)}$,

$$a_{i+2} = -\frac{\Gamma(i\alpha + 1)}{\Gamma((i+2)\alpha + 1)} \sum_{j=0}^{i} a_j, \ i \geq 0. \tag{3.13}$$

As a numerical example, if we take $\delta_0 = 0$, $\delta_1 = 1$, $\alpha = 0.7$ and $\alpha = 1.$, the solutions of the Cauchy problem are represented in Figure 3.

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CHROMATIC POLYNOMIALS FOR SOME FAMILIES OF GRAPHS

LEONARD DĂUŞ AND MARILENA JIANU

Abstract. In this paper, we compute the chromatic polynomials for several families of graphs. We first consider the family of support graphs for consecutive k-out-of-n networks, and then we study three types of ladder graphs. In the final part, using the deletion-contraction theorem, we find the chromatic polynomial for "circular ladder" graphs. All these families of graphs are related to the reliability theory of networks.

Mathematics Subject Classification (2010): 05C15, 05C31
Key words: chromatic polynomial, consecutive k-out-of-n graph, ladder graph.

1. Introduction

In 1852, Francis Guthrie, a former graduate of University College London, observed that the counties (regions) of England could be colored with four or fewer colors, such that neighboring counties were colored differently. In the same year, the following question (today known as the Four Color Problem) was communicated by Francis' brother to Augustus De Morgan:

Can every map be colored with at most four colors in such a way that neighboring countries are colored differently?

An article published by Alfred Kempe in 1879 contains a "proof" of the Four Color Problem. In 1890 Percy Heawood presented a map which was a counterexample to the technique used by Kempe. Moreover, Heawood was able to use Kempe's technique to prove that every map could be colored with five or fewer colors.

In 1976, so more than a century after it was stated, the theorem was finally proven by Kenneth Appel and Wolfgang Haken (University of Illinois). Their proof rests in part on 1,200 hours of computer calculation during which about ten billion logical decisions had to be made. It was the first major theorem to be proved using a computer. Initially, this proof was not accepted by all mathematicians because the computer-assisted proof was infeasible for a human to check by hand.

The chromatic polynomial was introduced by George Birkhoff in 1912, in an attempt to prove the famous four color theorem. He defined the function $P(M, \lambda)$ that gives the number of proper colorings (i.e. neighboring countries are colored differently) of a map M using λ colors (λ is any positive integer). Using this (polynomial) function and representing a map as an undirected graph that has a vertex for each country and an edge for every pair of countries with a common boundary, the four color theorem could be established by showing that $P(G, 4) > 0$ for any planar graph G. In 1932, Hassler Whitney generalized Birkhoff's polynomial from the planar case to general graphs and obtained a series of results regarding the chromatic polynomials, but none of them led to a proof of the four color theorem. However, the chromatic polynomial started to play a central role in the algebraic graph theory, especially after William Tutte discovered in 1954 a bivariate generalization, the Tutte polynomial [13].

2. Chromatic polynomials - basic results

In this section we present the most important results related to chromatic polynomials (see [2, 4, 7, 11] for more details).

Definition 2.1. If G is a graph and λ a positive integer, then $P(G, \lambda)$ denotes the number of different proper λ-colorings of G. The expression $P(G, \lambda)$ is a polynomial in λ whose degree is the order of the graph and it is called the *chromatic polynomial* of G.

Examples:

1. The chromatic polynomial of the empty graph of order n, $\emptyset_n$ (the graph with n vertices and no edge) is

$$P(\emptyset_n, \lambda) = \lambda^n. \tag{2.1}$$

2. The chromatic polynomial of the complete graph of order n, K_n (the graph with n vertices connected by all possible edges) is

$$P(K_n, \lambda) = \lambda(\lambda - 1)\dots(\lambda - n + 1) = \lambda^{(n)}, \tag{2.2}$$

where $\lambda^{(n)}$ denotes the falling factorial, or the Pochammer symbol (also denoted by $(\lambda)_n$).

3. The chromatic polynomial of the path graph of order n, P_n (see Figure 1) is

$$P(P_n, \lambda) = \lambda(\lambda - 1)^{n-1}. \tag{2.3}$$

Figure 1. The path graph of order n, P_n.

As a matter of fact, a more general result can be proved by induction on n (recall that a tree is a graph in which any two vertices are connected by exactly one path, or, equivalently, a connected graph with no cycles):

Theorem 2.2. *The chromatic polynomial of any tree of order n T_n is*

$$P(T_n, \lambda) = \lambda(\lambda - 1)^{n-1}.$$

Let A and B be two different vertices (adjacent or not) of the graph G and let e denote the edge AB. We denote by G/e the graph obtained by identifying the nodes A and B (we call this operation *contraction*). Note that multiple edges have no influence on the chromatic polynomial, so they can be replaced by simple edges. If A and B are adjacent nodes, then we denote by $G \setminus e$ the graph obtained by deleting the edge $e = AB$. If they are nonadjacent nodes, then we denote by $G \cup e$ the graph obtained from G by adding the edge $e = AB$.

Suppose that A and B are two nonadjacent nodes of the graph G. Then all the λ-colorings of G can be divided into two disjoint sets: the set of colorings where A and B have different colors (the λ-colorings of the graph $G \cup e$), and the set where A and B have the same color (the λ-colorings of the graph G/e). Thus, we obtain the following fundamental theorem (see Figure 2):

Theorem 2.3.

$$P(G, \lambda) = P(G \cup e, \lambda) + P(G/e, \lambda).$$

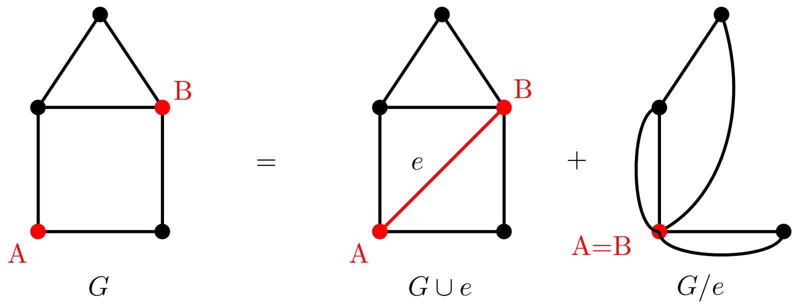

FIGURE 2. Illustration of Theorem 2.3.

This theorem can be also used in the following equivalent form (also known as *Deletion-Contraction Theorem* - see Figure 3):

Theorem 2.4.

$$P(G, \lambda) = P(G \setminus e, \lambda) - P(G/e, \lambda).$$

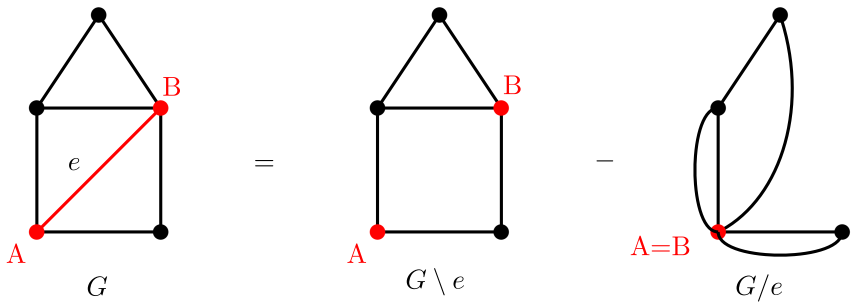

FIGURE 3. Illustration of Theorem 2.4.

The chromatic polynomial of the cycle graph C_n can be calculated by the recurrence relation that follows from Theorem 2.4 (see Figure 4):

$$P(C_n, \lambda) = P(P_n, \lambda) - P(C_{n-1}, \lambda) = \lambda(\lambda - 1)^{n-1} - P(C_{n-1}, \lambda).$$

Thus, using induction on n, the following explicit formula can be proved:

$$P(C_n, \lambda) = \lambda(\lambda - 1)^n + (-1)^n(\lambda - 1). \tag{2.4}$$

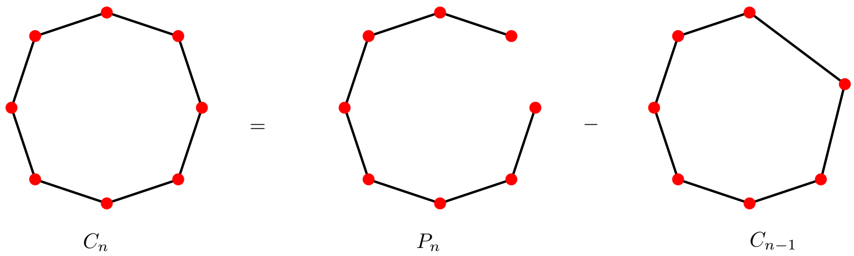

FIGURE 4. The cycle of order n, C_n.

Theorem 2.5. *If G is a disconnected graph formed by k connected components $G_1, \ldots, G_k$, then*

$$P(G, \lambda) = P(G_1, \lambda) \cdot P(G_2, \lambda) \cdot \ldots \cdot P(G_k, \lambda).$$

Theorem 2.6. *If the graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ "overlap" in a complete graph $K_n = (V_1 \cap V_2, E_1 \cap E_2)$, then the chromatic polynomial of their union, $G = (V_1 \cup V_2, E_1 \cup E_2)$ is*

$$P(G, \lambda) = \frac{P(G_1, \lambda) \cdot P(G_2, \lambda)}{\lambda^{(n)}}.$$

3. Chromatic polynomials for graphs inspired by reliability theory

3.1. **Chromatic polynomial of linear consecutive k-out-of-n graph.** The linear consecutive-k-out-of-n:F system was introduced by Chiang and Niu [5] and Kontoleon [10]. It is defined as a system formed by n components placed in a row which fails if and only if at least k consecutive components fail. A linear consecutive-k-out-of-n:F system can be represented as an undirected graph $G = (V, E)$, with n vertices, $V = \{1, 2, \ldots, n\}$ and having the set of edges $E = \{ij : 1 \leq i < j \leq n, j - i \leq k\}$. If the nodes $2, \ldots, n-1$ are considered identical and statistically independent, failing with probability $q = 1 - p$, while the terminals $S = 1, T = n$, as well as the edges, are always operational, then the reliability of the system (the probability that the system works) can be expressed as a polynomial in p, and was intensively studied (see [6, 9]). In this paper we study the chromatic polynomial of the graph corresponding to the consecutive-k-out-of-n:F system. An example (consecutive-3-out-of-9 graph) is presented in Figure 5.

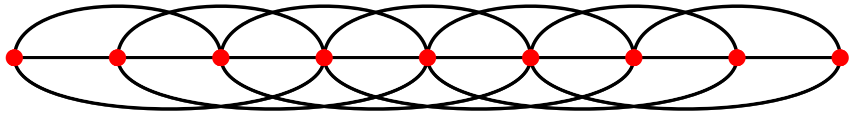

FIGURE 5. The graph of consecutive 3-out-of-9:F system.

If $k = 1$, the graph of the consecutive-1-out-of-n:F system is the path graph P_n, whose chromatic polynomial is

$$P(Con_n^1, \lambda) = \lambda(\lambda - 1)^{n-1}.$$

If $k = 2$, we can compute the chromatic polynomial using the definition, as follows: for the first vertex we can use λ colors, for the second one, the number of permitted colors is $(\lambda - 1)$, for the third one we can use $(\lambda - 2)$ colors since the 2 colors used for the first vertices are forbidden (because both are adjacent with the third). For each one of the next vertices the number of permitted colors is $(\lambda - 2)$, hence the chromatic polynomial of the consecutive-2-out-of-n graph is

$$P(Con_n^2, \lambda) = \lambda(\lambda - 1)(\lambda - 2)^{n-2}. \tag{3.1}$$

In the same way we can prove that, for any $k \geq 1$, the chromatic polynomial of the consecutive-k-out-of-n graph is

$$P(Con_n^k, \lambda) = \lambda(\lambda - 1) \dots (\lambda - k + 1)(\lambda - k)^{n-k}. \tag{3.2}$$

Note that for $k = n$ the graph of the consecutive-n-out-of-n:F system is the complete graph K_n and equation 3.2 becomes (2.2).

3.2. **Chromatic polynomials of ladder graphs.** We examine here three types of ladder graphs, which have already been analyzed in the light of reliability theory: the simple ladder (see [8]), the ladder with a diagonal (Brecht-Colbourn ladder [3]) and the ladder with two diagonals (K_4 ladder [12]).

1. The simple ladder

The simple ladder graph L_n, $n \geq 2$, is presented in Figure 6.

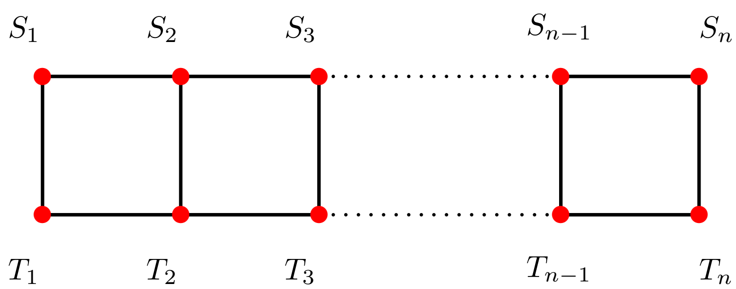

FIGURE 6. The simple ladder L_n.

For $n = 2$, L_2 is identical to the cycle C_4. Hence, by (2.4) we have:

$$P(L_2, \lambda) = (\lambda - 1)^4 + (\lambda - 1) = \lambda(\lambda - 1)(\lambda^2 - 3\lambda + 3)$$

For $n = 3$ the simple ladder L_3 is the union of two cycles C_4 that overlap in a complete graph K_2. By Theorem 2.6 one can write:

$$P(L_3, \lambda) = \frac{P(C_4, \lambda)^2}{\lambda^{(2)}} = \lambda(\lambda - 1)(\lambda^2 - 3\lambda + 3)^2.$$

Moreover, for any $n \geq 3$, the simple ladder L_n is formed by the union of two graphs, L_{n-1} and C_4, overlapping in a complete graph K_2. Thus, using Theorem 2.6, the following formula follows by induction on n:

$$P(L_n, \lambda) = \lambda(\lambda - 1)(\lambda^2 - 3\lambda + 3)^{n-1}.$$

2. The ladder with a diagonal

The ladder with a diagonal $L_n^{(1)}$, $n \geq 2$, is the graph presented in Figure 7.

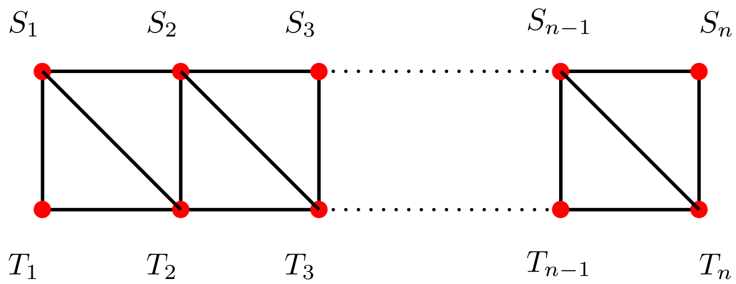

FIGURE 7. The ladder with 1 diagonal $L_n^{(1)}$.

It is easy to see that it corresponds to the consecutive-2-out-of-2n graph. By equation (3.1), we can write:

$$P(L_n^{(1)}, \lambda) = P(Con_{2n}^2, \lambda) = \lambda(\lambda - 1)(\lambda - 2)^{2n-2}.$$

3. The ladder with 2 diagonals

The ladder with 2 diagonals $L_n^{(2)}$, $n \geq 2$, is the graph presented in Figure 8.

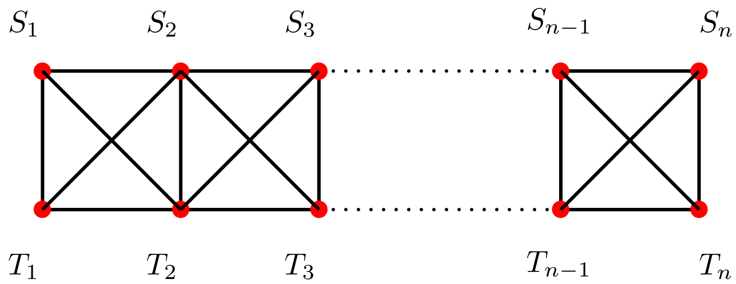

FIGURE 8. The ladder with 2 diagonals $L_n^{(2)}$.

For $n = 2$, it is the complete graph of order 4, so

$$P(L_2^{(2)}, \lambda) = \lambda^{(4)} = \lambda(\lambda - 1)(\lambda - 2)(\lambda - 3).$$

For $n \geq 3$, the 2-diagonal ladder $L_n^{(2)}$ is formed by the union of two graphs, $L_{n-1}^{(2)}$ and K_4, overlapping in a complete graph K_2. Thus, using Theorem 2.6, the following formula follows by induction on n:

$$P(L_n(2), \lambda) = \lambda(\lambda - 1)(\lambda - 2)^{n-1}(\lambda - 3)^{n-1}.$$

3.3. **Chromatic polynomial of the circular ladder graph.** Consider a simple ladder L_{n+1}, $n \geq 2$. If we identify the vertices S_1 and S_{n+1} on one hand, and T_1 and T_{n+1} on the other, then we obtain the circular ladder of order n, CL_n. We remark that it is a planar graph (see Figure 9).

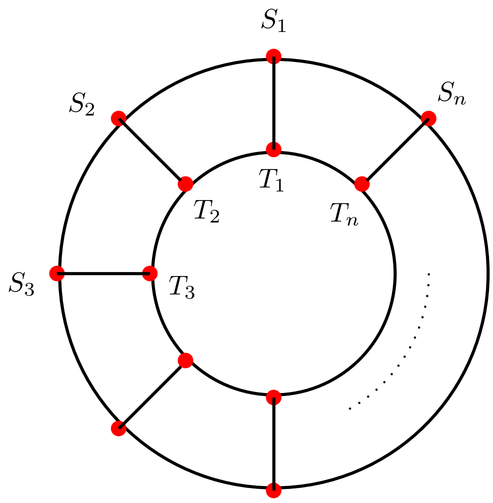

FIGURE 9. The circular ladder CL_n (for $n = 8$).

For $n = 2$, we notice that $CL_2 = C_4$, hence

$$P(CL_2, \lambda) = \lambda(\lambda - 1)(\lambda^2 - 3\lambda + 3).$$

For $n = 3$, we use Theorem 2.4 to calculate the chromatic polynomial (see Figure 10). Thus, we obtain that

$$P(CL_3, \lambda) = P(G_0, \lambda) - 3P(G_1, \lambda) + 3P(G_2, \lambda) - P(G_3, \lambda),$$

where:

- G_0 is the graph formed by two independent cycles C_3; by Theorem 2.5 we have:
$$P(G_0, \lambda) = P(C_3, \lambda)^2 = \lambda^2(\lambda - 1)^2(\lambda - 2)^2;$$
- G_1 is the graph formed by two cycles C_3 with a common vertex; by Theorem 2.6 we have:
$$P(G_1, \lambda) = \frac{P(C_3, \lambda)^2}{\lambda} = \lambda(\lambda - 1)^2(\lambda - 2)^2;$$
- G_2 is the graph formed by two cycles C_3 with 2 common vertices; by Theorem 2.6 we have:
$$P(G_2, \lambda) = \frac{P(C_3, \lambda)^2}{\lambda(\lambda - 1)} = \lambda(\lambda - 1)(\lambda - 2)^2;$$
- G_3 is the cycle C_3 (the two cycles coincide):
$$P(G_3, \lambda) = \lambda(\lambda - 1)(\lambda - 2).$$

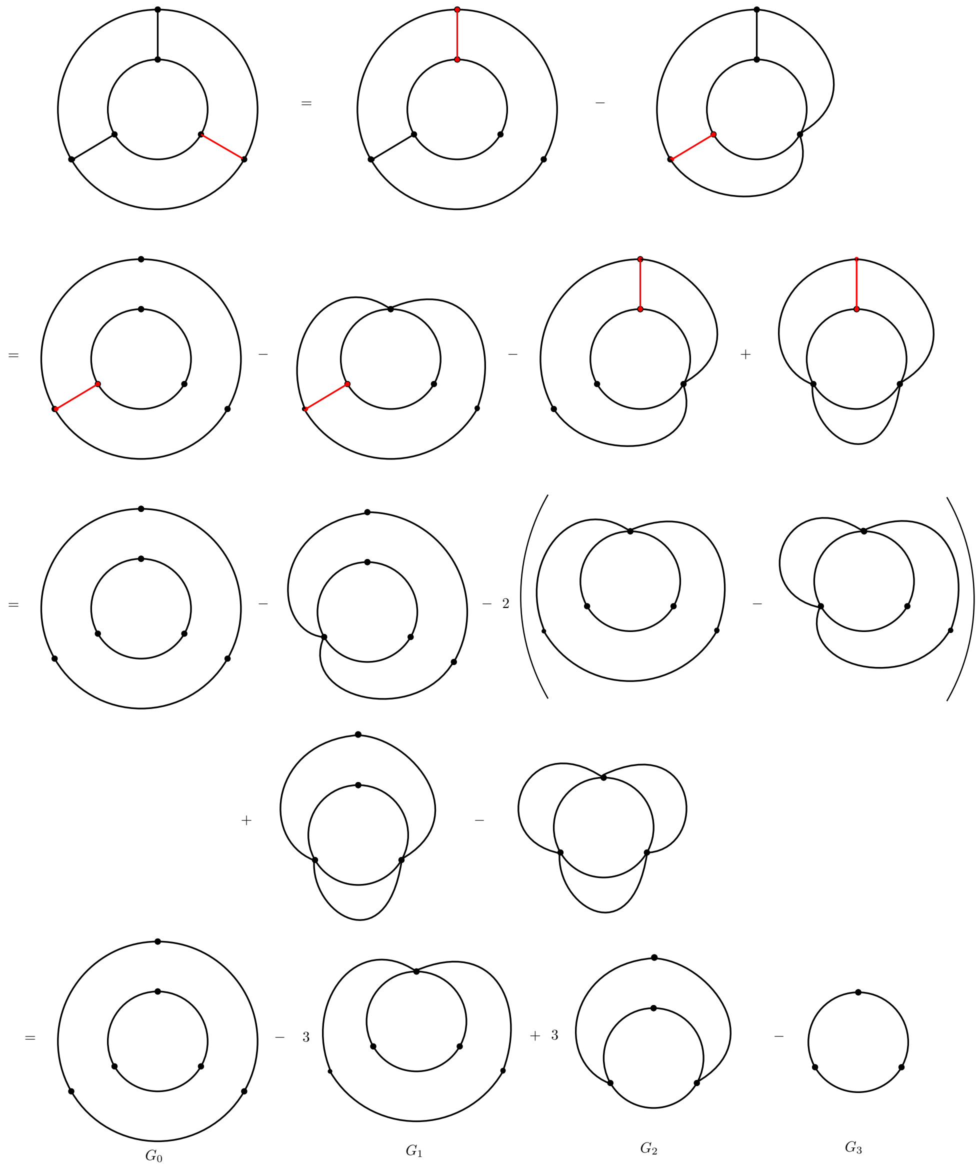

Figure 10. Calculation of the chromatic polynomial of CL_3.

Hence

$$P(CL_3, \lambda) = \lambda^{(3)} \left(\lambda^{(3)} - 3(\lambda - 1)^{(2)} + 3(\lambda - 2)^{(1)} - 1 \right).$$

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DIGISTEM EDUCATION

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Abstract: This paper delves into the significance, challenges, and potential solutions surrounding STEM (Science, Technology, Engineering, and Mathematics) education. STEM education has emerged as a vital interdisciplinary approach to learning, cultivating critical thinking, problem-solving, and innovation skills among students.

The paper highlights the importance of STEM education in preparing a skilled workforce to meet the demands of a technology-driven world and its role in fostering creativity and groundbreaking ideas. However, the paper also addresses the challenges faced by STEM education, including the gender and minority gap in STEM fields, inadequate resources in certain schools, and the need for better teacher training. These challenges hinder the full realization of STEM education's potential and limit the access and opportunities for some students. To overcome these hurdles, the paper outlines various initiatives and solutions. Promoting early exposure to STEM concepts through hands-on activities and workshops can spark interest and engagement at a young age. Moreover, investing in teacher professional development ensures educators are well-equipped to deliver effective STEM instruction. Public-private partnerships are advocated as a means to bridge the gap between education and real-world applications. Collaborations with private companies provide access to advanced technology and relevant experiences, making STEM education more compelling and relevant to students. Furthermore, the paper emphasizes the need for diversity and inclusion efforts. Encouraging participation from underrepresented groups through mentorship programs, scholarships, and promoting diverse role models can create a more inclusive and dynamic STEM community.

In conclusion, the paper highlights the transformative potential of STEM education and the importance of addressing challenges to maximize its impact by implementing the proposed initiatives and solutions, societies can empower students to excel in STEM fields, contribute to solving global challenges, and lead the way in technological advancements, ensuring a brighter and more sustainable future for all. The primary context of the DigiSTEM project is STEM education. The objective is to promote innovative utilization of educational technology, learning analytics and use of open educational resources (OERs) in online, classroom and blended learning, especially in HEIs STEM subjects. The project aims to support professional development of HEI educators by increasing their technological and pedagogical skills and competence. The objective is to build HEI educators' competence of such instructional design that improves students' active learning, self-regulated learning and learning engagement with the help of educational technology and learning analytics to provide more effective and personalized support of learners.

Mathematics Subject Classification (2010): 97B10, 97M50
Key words: Mathematics education, STEM, pedagogical skills, competence

1. Introduction

STEM education stands for Science, Technology, Engineering, and Mathematics. It is an interdisciplinary approach to learning that integrates these four key disciplines to foster critical thinking, problem-solving, and innovation among students. STEM education has gained significant attention in recent years due to its potential to prepare the next generation for the challenges of the modern world, where technology and scientific advancements play a central role in nearly every aspect of life.

Importance of STEM Education:

- Preparing a Skilled Workforce: STEM education equips students with the knowledge and skills necessary to pursue careers in various STEM-related fields. As industries continue to rely on technology and innovation, there is a growing demand for individuals with expertise in STEM disciplines.
- Enhancing Critical Thinking: STEM education encourages students to think critically and analytically, enabling them to approach complex problems with logical and evidence-based solutions.
- Fostering Innovation: Innovation is at the core of STEM education. By fostering creativity and encouraging experimentation, students are more likely to develop groundbreaking ideas and technologies that can drive progress and improve society.
- Addressing Real-world Challenges: Many of the world's most pressing issues, such as climate change, healthcare, and sustainable energy, require STEM solutions. STEM-educated individuals are better equipped to contribute to finding solutions to these global challenges.

Challenges in STEM Education:

- Gender and Minority Gap: There is a significant gender and minority gap in STEM fields. Encouraging more girls and underrepresented minorities to pursue STEM education and careers is crucial for diversity and equal opportunities.
- Lack of Resources: Many schools, particularly in low-income areas, lack the necessary resources and funding to offer comprehensive STEM programs. This disparity can hinder the development of STEM skills in certain communities.
- Teacher Training: Effective STEM education relies on well-trained teachers who can inspire and engage students. However, not all teachers receive sufficient training and professional development in STEM subjects.
- Perception and Stereotypes: Some students may perceive STEM subjects as difficult or uninteresting due to stereotypes or outdated teaching methods. Changing these perceptions is essential to attract more students to STEM fields.

Initiatives and Solutions:

- Promoting Early Exposure: Introducing STEM concepts at an early age can spark interest and curiosity. Schools and organizations should implement hands-on activities and workshops to engage young students in STEM topics.
- Teacher Professional Development: Providing ongoing training and support for teachers is crucial in ensuring they have the knowledge and skills to effectively teach STEM subjects.
- Public-Private Partnerships: Collaboration between educational institutions and private companies can offer students access to cutting-edge technology and real-world experiences, making STEM education more relevant and engaging.
- Diversity and Inclusion Efforts: Encouraging diversity in STEM fields can be achieved through mentorship programs, scholarships, and initiatives that promote role models from underrepresented backgrounds.

STEM education is a powerful tool that equips students with the skills needed to thrive in a technologically advanced world. By addressing challenges and promoting inclusivity, societies can harness the full potential of STEM education and empower the next generation to lead in innovation, problem-solving, and shaping a better future for all.

2. Objectives

The objective of the project is to increase digital and pedagogical competence of HEI educators and availability of digital resources in STEM subjects on a large scale to achieve long-lasting effects in everyday activity on the project partners and other European HEIs. By increasing such competences of educators, it gives them tools and knowledge to redesign their teaching and implement digital resources and activities (e.g. learning analytics, digital languaging, screencasts, visualizations and intelligent assessment) for different personalized learning scenarios.

The agenda for the modernization of Europe's higher education systems supports the project idea as it suggests the need to exploit new technologies and ICT to enrich teaching/learning experience and providing ubiquitous and personalized digital learning possibilities for students. Also, the Digital Education Action Plan set by the Commission has a priority to make better use of digital technology for teaching and learning. The project aims to enhance educators' technological and pedagogical competences by organizing different kind of pilot events and providing OERs and learning environment for competence development. By increasing such competencies of HEI educators (main target group), it gives them tools and knowledge to produce and implement digital resources for different personalized learning scenarios and resources that supports students' activation. Hence, the project aims to promote digital and pedagogical competence and skills of HEI educators nationally and transnationally in Europe. Simultaneously based on literature, this is expected to affect positively on HEI students' engagement and learning outcomes (secondary target group) but also support to recover from the Covid-19 pandemic consequences such as decrease of students' competence level.

3. Implementation

The main activities/results of the project are:

- developing DigiSTEM methodology that encapsulates innovative pedagogies, best practices and concrete examples for implementing digital learning/teaching of STEM and other similar subjects (PR1)
- building, maintaining and developing a digital platform for STEM subjects' digital teaching and learning to support educators' continuous professional development with high-quality resources (PR2)
- developing guidelines for European STEM educators to increase digital and pedagogical competence and implementing good practices and new methods smoothly into daily teaching activities and curricula (PR3)
- enhancing HEI educators' competence of such instructional design that improves students' active learning, self-regulated learning and learning engagement with the help of educational technology and learning analytics to provide more effective and personalized support of learners (PR1-PR3)
- to organize/participate 3 LTT (C1-C3) events and participate dissmination events (E1-E6) aimed at providing technological and pedagogical training/knowledge to STEM educators in the context of the project. Participants of training activities will be awarded with digital competence certificate.
- developing MOOC as a form of OER that will combine all the tangible results of the project to be able to promote and integrate developed good practices and innovative methods into daily activities of European HEIs. The MOOC is based on piloted PRs, which have been developed on the basis of feedback received during piloting. The

project's pedagogical innovations and technological choices will be made as sustainable solutions that can be utilized after the project, for example 10 years after the end of the project.

4. Importance of STEM Education

One of the most significant benefits of STEM education lies in its ability to prepare a skilled workforce that is well-equipped to meet the demands of a rapidly evolving technological landscape. As industries increasingly rely on advanced technology, data analysis, and innovation, there is a growing need for professionals with expertise in science, technology, engineering, and mathematics. STEM education plays a vital role in nurturing the next generation of professionals who can drive progress and fuel economic growth.

Enhancing Technical Knowledge and Skills: STEM education focuses on providing students with a strong foundation in science, technology, engineering, and mathematics disciplines. Students engage in hands-on learning, problem-solving activities, and experiments that enable them to grasp complex concepts and apply theoretical knowledge to practical scenarios. This exposure to real-world applications helps students develop technical skills that are highly valued in the job market.

Encouraging Specialization in High-Demand Fields: STEM education allows students to explore various STEM disciplines and discover their areas of interest and aptitude. As they progress through their education, students can specialize in specific STEM fields, such as computer science, data analytics, biotechnology, renewable energy, and more. This specialization prepares them to meet the specialized needs of industries and increases their employability in high-demand sectors.

Fostering Critical Thinking and Problem-Solving Abilities: STEM education places a strong emphasis on critical thinking and problem-solving skills. Students are encouraged to approach challenges analytically, identify patterns, and devise innovative solutions. These problem-solving abilities are not only valuable within STEM careers but also transferable to a wide range of professions, making STEM-educated individuals versatile and adaptable in the job market.

Nurturing a Culture of Innovation: Innovation is at the heart of STEM education. Through experimentation and project-based learning, students learn to think creatively and develop novel ideas. They are encouraged to explore unconventional approaches and embrace failure as an opportunity for growth. This culture of innovation not only prepares students for entrepreneurial endeavors but also instills an entrepreneurial mindset within the workforce, driving continuous improvement and progress in industries.

Meeting Industry Demands: Industries are increasingly seeking candidates with expertise in technology, automation, artificial intelligence, and other STEM-related fields. STEM-educated individuals are better equipped to understand and leverage advanced technologies, making them invaluable assets to companies seeking to remain competitive in a technology-driven world.

STEM education equips students with the knowledge and skills to tackle some of the world's most pressing challenges, such as climate change, healthcare, food security, and sustainable energy. Through STEM research and innovation, students can contribute to finding solutions that improve the quality of life for people around the globe.

STEM education plays a pivotal role in preparing a skilled workforce that can thrive in a technologically advanced and dynamic world. By enhancing technical knowledge, encouraging specialization, fostering critical thinking, nurturing a culture of innovation, meeting industry demands, and addressing global challenges, STEM education empowers individuals to make meaningful contributions to society and shape a prosperous and sustainable future.

Enhancing Critical Thinking through STEM Education: Critical thinking is a fundamental skill cultivated through STEM education, and it is essential for success not only in STEM fields but also in various aspects of life. STEM education provides a fertile ground for students to develop and refine their critical thinking abilities through various methods and approaches.

Problem-Based Learning: STEM education often employs problem-based learning approaches, where students are presented with real-world challenges and tasked with finding solutions. By engaging in these problem-solving scenarios, students learn to analyze information, identify relevant data, and devise effective strategies to address the issues at hand. This process hones their critical thinking skills as they evaluate multiple solutions, assess potential outcomes, and make informed decisions.

Inquiry-Based Learning: Inquiry-based learning is another prominent feature of STEM education. Students are encouraged to ask questions, explore, and investigate concepts independently. This process fosters a sense of curiosity and instills a deeper understanding of the subject matter. By seeking answers to their inquiries, students learn to think critically and develop their own conclusions based on evidence and logical reasoning.

Data Analysis and Interpretation: In STEM disciplines, data analysis is integral to drawing meaningful conclusions. Students are exposed to data sets and research findings, learning to analyze and interpret the information critically. They develop skills in recognizing patterns, identifying trends, and drawing valid conclusions from the data, which are crucial competencies in decision-making processes.

Application of Theory to Practice: STEM education emphasizes practical applications of theoretical knowledge. By connecting classroom learning to real-world situations, students are encouraged to think critically about how scientific principles, technology, engineering concepts, and mathematics can be applied to solve practical problems. This application-oriented approach reinforces critical thinking skills as students assess the feasibility and effectiveness of different solutions in practical scenarios.

Collaboration and Communication: In STEM projects, collaboration and communication are essential for success. Working in teams, students exchange ideas, discuss strategies, and challenge each other's assumptions. These interactions foster critical thinking as students evaluate the merits of different perspectives and refine their own arguments based on constructive feedback.

Dealing with Uncertainty and Complexity: The nature of many STEM problems involves uncertainty and complexity. Students must learn to navigate ambiguity and consider multiple variables that may influence outcomes. This aspect of STEM education encourages students to think critically about uncertainties and consider the potential implications of various factors when arriving at solutions.

Encouraging Innovation and Creativity: Critical thinking and creativity are interconnected. In STEM education, students are encouraged to think creatively, brainstorming new ideas and approaches to solve problems. They learn to think "outside the box" and challenge conventional wisdom, leading to innovative solutions that can drive progress in various fields.

STEM education serves as a catalyst for enhancing critical thinking skills in students. Through problem-based and inquiry-based learning, data analysis, application of theory to practice, collaboration, and dealing with uncertainty, students develop the ability to think critically, analyze information objectively, and make well-informed decisions. These skills not only equip them for success in STEM careers but also empower them to excel in any endeavor they pursue, making STEM education a transformative force in nurturing thoughtful, innovative, and adaptable individuals.

Fostering Innovation through STEM Education: Innovation lies at the core of STEM education, and it is a key factor that drives progress and advancements in various fields.

STEM education provides a nurturing environment that fosters creativity, curiosity, and the spirit of exploration, essential components for cultivating a culture of innovation.

Hands-on and Project-Based Learning: STEM education often emphasizes hands-on and project-based learning, where students actively engage in designing, building, and experimenting with real-world projects. By working on these projects, students are encouraged to explore innovative ideas, experiment with different approaches, and learn from both successes and failures. This iterative process nurtures their creative thinking and problem-solving abilities.

Encouraging Curiosity and Exploration: STEM education places a strong emphasis on curiosity-driven learning. Students are encouraged to ask questions, seek answers, and explore beyond the confines of textbooks. By nurturing curiosity, students develop a sense of wonder about the world around them, inspiring them to generate novel ideas and solutions.

Cultivating an Entrepreneurial Mindset: STEM education instills an entrepreneurial mindset in students. They are encouraged to think critically about real-world problems and develop solutions that can have a positive impact. This mindset fosters a culture of innovation, where students are motivated to take initiative, identify opportunities, and create their own ventures to address societal challenges.

Exposure to Cutting-Edge Technology: STEM education often provides access to cutting-edge technology and tools. By working with advanced equipment and software, students gain exposure to the latest innovations and research in their respective fields. This exposure inspires them to push boundaries, explore new possibilities, and contribute to the forefront of technology and knowledge.

Interdisciplinary Approach: STEM education brings together diverse disciplines, encouraging cross-pollination of ideas. Students are exposed to the intersections between science, technology, engineering, and mathematics, fostering a holistic understanding of complex problems. This interdisciplinary approach nurtures innovative thinking as students draw inspiration from different domains to develop innovative solutions.

Supportive and Open Learning Environment: STEM education provides a supportive and open learning environment where students are encouraged to share their ideas without fear of judgment. Teachers and peers provide constructive feedback, helping students refine their ideas and develop innovative concepts further. This supportive atmosphere allows students to take risks and explore unconventional approaches, essential elements of fostering innovation.

Embracing Failure as a Learning Opportunity: STEM education promotes a growth mindset where failure is viewed as a stepping stone towards improvement. Students are encouraged to learn from their mistakes and iterate on their ideas, fostering resilience and perseverance. Embracing failure as a learning opportunity allows students to take bolder risks, leading to breakthroughs and innovation.

STEM education serves as a catalyst for fostering innovation among students. Through hands-on learning, curiosity-driven exploration, an entrepreneurial mindset, exposure to advanced technology, interdisciplinary approaches, a supportive learning environment, and a positive attitude towards failure, students develop the creative thinking and problem-solving skills needed to drive innovation in various fields. By nurturing innovation through STEM education, societies can empower the next generation to tackle complex challenges, create novel solutions, and lead the way in advancing technology and knowledge for the betterment of humanity.

5. Results

The objective is to build HEI educators' competence of such instructional design that improves students’ active learning, self-regulated learning and learning engagement with the help of educational technology and learning analytics to provide more effective and personalised support of learners. In this way, it is possible to achieve the effectiveness of the project from the curriculum level to practice. The project is divided in three main Project Results (PR1-PR3). One of the partners is appointed as the coordinator of each output even though each output will be developed in close cooperation with all partners. The Project Results are described in previous section. As a consequence of the exploitation of Project's Results (PRs), it is expected to improve HEI students' motivation and to decrease dropouts (Kinnari-Korpela, 2019). Although the project focuses on HEI STEM subjects, the PRs can be applied to other disciplines with some adaptations. Hence, the project can contribute to improve educators' skills to apply modern methodologies, novel pedagogy and digital teaching/learning solutions on a large scale.

6. Conclusions

STEM education stands as a cornerstone in shaping a future that embraces progress, innovation, and sustainability. This comprehensive report highlights the significance of STEM education in preparing a skilled workforce, enhancing critical thinking, fostering innovation, and addressing real-world challenges. Through a multi-faceted approach, STEM education equips students with the knowledge, skills, and mindset required to thrive in a technologically advanced world. By preparing a skilled workforce, STEM education meets the demands of rapidly evolving industries. Graduates with expertise in science, technology, engineering, and mathematics are poised to lead advancements and drive economic growth in a highly competitive global landscape. The emphasis on critical thinking in STEM education cultivates individuals who can approach challenges with analytical and evidence-based reasoning. This vital skill empowers students to make informed decisions, solve complex problems, and adapt to dynamic circumstances in both professional and personal endeavors.

Fostering innovation is at the heart of STEM education, as it encourages creativity, curiosity, and a spirit of exploration. By nurturing an entrepreneurial mindset and providing access to cutting-edge technology, STEM education nurtures individuals who are unafraid to challenge the status quo and develop groundbreaking solutions to pressing global issues. Moreover, STEM education is a potent tool for addressing real-world challenges that affect humanity and the planet. From climate change to healthcare and sustainable energy, students are empowered to contribute to finding solutions that improve lives and secure a more sustainable future for generations to come. To fully realize the potential of STEM education, it is crucial to address existing challenges. This includes promoting diversity and inclusivity in STEM fields, ensuring access to resources for all communities, providing comprehensive teacher training, and challenging stereotypes and perceptions about STEM subjects. STEM education is a transformative force that empowers individuals to make a positive impact on society and the world. By embracing the multifaceted benefits of STEM education and implementing initiatives to overcome challenges, societies can unleash the potential of the next generation of thinkers, innovators, and problem-solvers.

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GENERATION FOR INNOVATION, RESILIENCE, LEADERSHIP AND SUSTAINABILITY (GIRLS PROJECT)

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Abstract: The paper goes into the imperative role of the current generation in driving transformative change across various domains. Focusing on innovation, resilience, leadership, and sustainability, the paper examines how these interconnected concepts play a crucial role in shaping the future of society, economy, and the environment.

The paper highlights the significance of innovation as a catalyst for progress, exploring how new ideas, technologies, and approaches are fundamental in addressing global challenges and driving economic growth. It emphasizes the need for fostering innovation ecosystems that support entrepreneurship, research, and development.

In the context of resilience, the paper analyzes the increasing frequency of disruptions, such as natural disasters, pandemics, and economic crises, and underscores the importance of equipping individuals and communities with the capacity to withstand and recover from such shocks. Resilience-building strategies, from adaptive governance to social safety nets, are explored to enhance societal preparedness.

Leadership emerges as a pivotal aspect, with the paper shedding light on the qualities and responsibilities of effective leaders in guiding organizations and societies towards a sustainable future. It emphasizes ethical decision-making, inclusivity, and the ability to inspire collective action as key traits for transformative leadership.

Sustainability forms a central theme throughout the paper, as it examines the pressing need to balance economic growth with environmental stewardship and social equity. The paper delves into sustainable practices in sectors like energy, agriculture, and transportation, emphasizing the importance of responsible resource management and reducing carbon footprints.

Furthermore, the paper addresses the vital role of education and intergenerational collaboration in fostering a sense of ownership and responsibility among the current generation to drive positive change. It emphasizes the need for mentorship and knowledge-sharing across age groups to ensure continuity and the passing on of wisdom.

"Generation for Innovation, Resilience, Leadership, and Sustainability" - GIRLS project calls for concerted efforts from individuals, communities, governments, and businesses to collaborate and embrace sustainable practices, innovative thinking, resilient approaches, and transformative leadership to create a more prosperous and sustainable world for future generations.

Mathematics Subject Classification (2010): 92F05, 91F99
Key words: Sustainability, pedagogical skills, Leadership

1. Introduction

The 21st century has presented humankind with numerous challenges, ranging from climate change and resource depletion to social inequality and economic instability. The current generation faces the responsibility of addressing these pressing issues while simultaneously laying the groundwork for future generations. This paper focuses on four key areas that are paramount to achieving a sustainable and prosperous future: innovation, resilience, leadership, and sustainability.

Innovation has long been the driving force behind human progress. It is the spark that ignites new ideas, propels breakthrough technologies, and creates novel approaches to age-old problems. In this dynamic section, we explore the significance of nurturing an innovation culture that encourages creativity, collaboration, and risk-taking. We shed light on how fostering innovation ecosystems where ideas are incubated and nurtured can propel us towards a future that surpasses even our wildest aspirations.

Throughout history, humanity has encountered numerous challenges, from natural calamities to societal upheavals. The current generation must equip itself with the resilience to withstand such shocks and emerge stronger. This section delves into the importance of building individual and communal resilience, exploring adaptive governance, the role of community support networks, and the power of forging connections in times of crisis. By drawing from the collective strength of resilience, we can navigate turbulent waters with grace and determination.

Leadership, in its true essence, transcends mere management; it is about inspiring others and creating a vision for a better world. This segment celebrates the transformative leaders of our time and examines the qualities that set them apart. Ethical decision-making, inclusivity, empathy, and the ability to envision a sustainable future are among the attributes we explore. Furthermore, we emphasize the need for intergenerational mentorship, as seasoned leaders pass on their wisdom to the next wave of change-makers.

At the heart of humanity's pursuits lies the essential quest for sustainability. In this section, we delve into the delicate balance between economic progress, environmental preservation, and social equity. The current generation is tasked with embracing sustainable practices across various domains, from transitioning to renewable energy sources to adopting circular economy principles. By harmonizing human endeavors with nature's rhythms, we can create a world that thrives for generations yet to come.

Education is the key that unlocks the full potential of the current generation. In this chapter, we explore the transformative role of formal education in fostering innovation, resilience, leadership, and sustainability. Moreover, we celebrate the strength of informal education through mentorship and knowledge-sharing as it bridges the generational divide and fosters a sense of ownership and responsibility in the pursuit of positive change.

The challenges we face are too monumental for solitary efforts. Collaborative action is the catalyst that propels us beyond our perceived limitations. In this comprehensive section, we advocate for cohesive partnerships between individuals, communities, governments, and businesses. Through collective action, we can create synergy and address the interconnected challenges of our time with unparalleled efficacy.

Armed with innovation, resilience, transformative leadership, and a commitment to sustainability, this generation holds the key to a brighter and more promising future. As we traverse this ever-changing landscape, let us heed the call to action and embark on a journey of cooperation, knowledge-sharing, and audacious dreams a journey that leads us toward a world where innovation thrives, resilience endures, leadership inspires, and sustainability reigns supreme. Together, we shall pioneer the path to a flourishing and sustainable future for all.

2. GIRLS project

This GIRLS project is presented as a partners’ proposal to promote several important aspects in Europe such as inclusion and diversity, equality, digital transformation and the sustainable development goals (SDG).

As was detailed throughout the proposal, we have proposed the project “GIRLS – Generation for innovation, resilience, leadership and sustainability. The game is on!” as a game since we think that the best way to learn is “by doing”, in this case “by playing”.
Currently, many teachers use innovative active methodologies in their classes to motivate and engage students in their own learning, but this is not the case in higher education, where it is not so common to change the classic system of lectures. The GIRLS project promotes the use of active methodologies in higher education and engage more teachers to use them.

Fig. 1 The game is on

The way to make this possible is to teach how to play by playing; i.e., we are going to focus on 4 methodologies (more may be included throughout the project): research-based learning (RBL), game-based learning (GBL), competency-based learning (CBL) and learning-service (LS), which will be used in different activities. For example, the entire project is organized as a game with work packages ranging from «the board» to «Game over!» going through «The rules of the game» and «The Game».
The first innovative aspect of the GIRLS project is that the project has been defined as a game, in which the work packages are specified by game elements: board, rules of the game, playing the game, and game over! The consortium that participates in this proposal wants to use active methodologies also in the design of all the activities that will be carried out during the project. In addition, this project will provide digital tools to higher education, from the approach of learning by doing. Each of the tools used will be duly documented and the necessary material will be provided to promote its use.
Another new aspect is the inclusion of anon-European entity from an associated country. The participation of the Vasco de Quiroga University of Morelia in Mexico will allow working on social, digital and educational innovation aspects, in a different environment. Additionally, this partner has experience working for the community. In Mexico they have established what they call internships, a year of social services that graduate students must give for community service. In this way they give to society what it has given them in the form of free university studies. The inclusion of this partner brings added value to the project, since university

students and teachers will be able to learn from the service-learning projects and projects related to the SDGs that are being developed and with new ones that will be proposed from a different perspective than the European one, since these communities present a reality that does not exist in Europe.
Finally, it is worth mentioning that the central part of the project is the goals of sustainable development and women. A specific objective is not established for these aspects since their use will be transversal and will be the background of the entire project.

3. Objectives

The GIRLS project has been proposed as a game about innovation, resilience, leadership, the sustainable development goals (SDGs) and sustainability. It is divided into 4 phases: the board, the rules of the game, the game, and Game over!
The game begins by drawing the roadmap, defining the setting and the instruction manual, 17 squares are covered, one for each SDG, and new sources of inspiration are sought. The winner's prize will be to share the experience and disseminate the results.
This GIRLS project is presented to promote several important aspects in Europe such as inclusion and diversity, equality, digital transformation, and the SDGs.
Currently, many teachers use innovative active methodologies in classes to motivate and engage students in their own learning, but this is not the case in higher education, where it is not so common to change the classic system of lectures. The GIRLS project promotes the use of active methodologies in higher education and engage more teachers to use them.
This proposal goal is to teach how to play by playing; that is, we are going to focus on 4 methodologies: research-based learning, game-based learning, competency-based learning and service-learning, which will be used in different activities all over the project.
The GIRLS project is presented to promote several important aspects in Europe such as inclusion and diversity, equality, digital transformation and the sustainable development goals (SDG).
We propose the project «GIRLS – Generation for innovation, resilience, leadership and sustainability. The game is on!» as a game, since we start from the assumption that the best way to learn is by «doing», in this case by «playing».
Currently, many teachers use innovative active methodologies in their classes to motivate and engage students in their own learning, but this is not the case in higher education, where it is not so common to change the classic system of lectures. The GIRLS project promotes the use of active methodologies in higher education.
We propose to teach how to play by playing; that is, we are focused on 4 methodologies: research-based learning, game-based learning, competency-based learning, and service-learning. The entire project is organized like a game with work packages ranging from «The Board» to «Game over!» going through «The rules of the game» and «The game».
The approach of the GIRLS project represents an interesting challenge in the Erasmus+ program, and with the proposed activities we want to support, through lifelong learning, the educational, professional and personal development of people in the fields of education, training and youth, within Europe and outside, thus contributing to sustainable growth, quality employment and social cohesion, in addition to promoting innovation and strengthening European identity and active citizenship.

- OB1: Train in digital skills and innovative pedagogies.
- OB2: Bring the university closer to society.
- OB3: Promote sustainable development in higher education.
- OB4: Promote transformation and individual change, as well as that of organizations, promoting improvements, new approaches and institutional changes

4. Results

- Teachers training in digital skills and active pedagogies.
- Promote the university approach to society. This is directly related to SDG 17, since alliances and action networks will be created, to implement the 2030 Agenda. Teachers, students and organizations will collaborate closely in students (and teachers) learning process, which is transformed in community service.
- Integrate the SDGs in higher education, activities will be proposed, and resources will be provided to make this possible in different disciplines.
- Promote innovation, resilience, leadership and sustainability starting individually and reach institutions and governments. It also seeks to promote real gender equality; that is to say, not with activities that have the title of equality but integrating this equality in all the activities that are developed in the project.

5. Conclusions

"Generation for Innovation, Resilience, Leadership, and Sustainability" delivers profound insights that emphasize the paramount importance of the current generation's role in shaping the world's trajectory. From its analysis of innovation as a catalyst for progress to its emphasis on intergenerational collaboration, the paper draws several broad conclusions that underscore the transformative potential of the present generation. The paper unequivocally establishes innovation as the driving force behind progress. It showcases how nurturing innovation ecosystems and promoting creative thinking are essential for overcoming global challenges and fostering economic growth (The Power of Innovation). By highlighting the increasing frequency of disruptions, the paper stresses the urgency of building resilience. It demonstrates that adaptive governance, community engagement, and preparedness are indispensable for navigating uncertainties and recovering from crises (Building Resilience for a Changing World).

Also, the paper lauds transformative leadership as a critical enabler of positive change. It showcases the qualities of visionary leaders who inspire collective action and promote sustainability, inclusivity, and ethical decision-making (Transformative Leadership for Positive Impact) and underscores the urgent need for sustainable practices in various sectors. By advocating for responsible resource management, renewable energy adoption, and circular economy principles, it promotes a harmonious balance between development and ecological preservation (Harmonizing Sustainability with Development). Education emerges as a cornerstone for empowering the current generation. The paper celebrates formal and informal education's transformative role in nurturing innovation, resilience, and sustainable practices (Education as a Catalyst for Change).

The paper emphasizes the power of collaborative action. It calls for unity among individuals, communities, governments, and businesses to address interconnected global challenges collectively (The Call for Collaborative Endeavors). Intergenerational collaboration is celebrated as a potent force for change. The paper recognizes the wisdom and guidance that previous generations can offer to empower and support the current generation's pursuits (The Potential of Intergenerational Collaboration). An urgent call to action for the current generation to step up and drive transformative change promptly. It reminds us that the decisions and actions taken today will have profound implications for the world's future (The Imperative for Immediate Action).

The "Generation for Innovation, Resilience, Leadership, and Sustainability" paper underscores the unprecedented opportunity that the present generation holds to shape a more prosperous and sustainable world. By embracing innovation, resilience, leadership, and sustainability, this generation can craft a legacy of positive change, leaving behind a brighter and more promising future for generations to come. The paper serves as a compelling blueprint for collective action and intergenerational collaboration, amplifying the call for the Generation for Innovation, Resilience, Leadership, and Sustainability to rise and make its indelible mark on human civilization.

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ON THE GENERALIZED FIBONACCI SEQUENCES AND APPLICATIONS

SEVER ANGEL POPESCU

Abstract. Let K be an arbitrary commutative field and let a, b be two elements in K. By a generalized Fibonacci sequence we mean a recurrent sequence $\{x_n\}_{n\geq 0}$ with the following recurrence relation: $x_{n+2} = bx_{n+1} + ax_n$, $n = 0, 1, ...$, where x_0, x_1 are two fixed elements in K. Using techniques of the modern Algebra we supply a general frame to study such sequences. For instance, we give formulas for the general term x_n of $\{x_n\}_{n\geq 0}$ in language of a, b, x_0 and x_1. We also provide a simple application of this theory in cryptography.

Mathematics Subject Classification (2010): Primary 11B37, 11B39; Secondary 11B83, 11B99.
Key words: Fibonacci sequences, Generalized Fibonacci sequences, cryptography

1. A general theory

Let K be an arbitrary commutative field and let us fix two elements a, b in K. For any x_0, x_1 in K we define a generalized Fibonacci sequence $F_{a,b}^K(x_0, x_1) = \{x_n\}_{n\geq 0}$ by the following recurrence formula:

$$x_{n+2} = bx_{n+1} + ax_n, \ n = 0, 1, ... \tag{1.1}$$

We see that $F_{1,1}^K(0, 1)$ is the classical Fibonacci sequence. We denote by $S_{a,b}^K$ the union of all $F_{a,b}^K(x_0, x_1)$ when x_0, x_1 run freely over K. It is easy to see that $S_{a,b}^K$ is a K-vector space.

Proposition 1.1. *For any a, b in K the K-vector space $S_{a,b}^K$ has dimension 2. A basis in $S_{a,b}^K$ is $\{F_{a,b}^K(1, 0), F_{a,b}^K(0, 1)\}$.*

Proof. Using the mathematical induction we can see that

$$F_{a,b}^K(x_0, x_1) = \{P_0(a, b)x_0 + Q_0(a, b)x_1, ..., P_n(a, b)x_0 + Q_n(a, b)x_1, ...\}, \tag{1.2}$$

where P_j, Q_j are polynomials in a and b with coefficients in the prime subfield K_0 of K. So $K_0 = \mathbb{Q}$, the rational number field, when the characteristic of K is zero or $K_0 = \mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$, the field with p elements for a prime number p, when the characteristic of K is p. For instance,

$$P_0(a, b) = 1, Q_0(a, b) = 0, P_1(a, b) = 0, Q_1(a, b) = 1,$$

$$P_2(a, b) = a, Q_2(a, b) = b, P_3(a, b) = ab, Q_3(a, b) = a + b^2, ...$$

For the moment we do not know to directly calculate $P_n(a,b)$ and $Q_n(a,b)$ for an arbitrary n. This calculation will be made later by an indirect way. From (1.2) we see that

$$F_{a,b}^K(x_0,x_1) = x_0F_{a,b}^K(1,0) + x_1F_{a,b}^K(0,1),$$

i.e. $\{F_{a,b}^K(1,0), F_{a,b}^K(0,1)\}$ is a generating system in $S_{a,b}^K$. It is easy to see that $\{F_{a,b}^K(1,0), F_{a,b}^K(0,1)\}$ is also a linear independent subset in $S_{a,b}^K$. □

If we assume that K has an infinite number of elements and if we look at x_0 and x_1 as free variables in K, from (1.1) and (1.2) we get:

$$P_{n+2} = bP_{n+1} + aP_n, \tag{1.3}$$

and

$$Q_{n+2} = bQ_{n+1} + aQ_n. \tag{1.4}$$

Since for now we cannot describe the sequences $F_{a,b}^K(1,0)$ and $F_{a,b}^K(0,1)$, we are not able to work with such a basis in $S_{a,b}^K$. Let us search for other particular sequences in $S_{a,b}^K$. For instance, let us search for geometrical progressions $x_n = r^n$, $n \geq 1$, $r \in K$. From (1.1) we get:

$$r^2 - br - a = 0. \tag{1.5}$$

If the characteristic of K is not 2, this equation in r has solutions in K if and only if $\Delta = b^2 + 4a$ is a square in K. If Δ is not a square in K, the polynomial $P(x) = x^2 - bx - a$ is irreducible over K and we can construct a field extension K' of K (of degree 2 over K) in the following way. The subset

$$M_P = \{gP \in K[x] : g \in K[x]\}$$

is a maximal ideal in the ring $K[x]$. So the quotient ring $K' = K[x]/M_P$ is a field and $K \subset K'$ (see [1], II, 2). Moreover, $K' = K[\sqrt{\Delta}]$, the least subring of K' generated by K and $\sqrt{\Delta}$. Why $\sqrt{\Delta} \in K'$? Since $\widehat{x}^2 - b\widehat{x} - a = 0$ in K' (here $\widehat{h}$ is the class of the polynomial h in K'), $\widehat{x} = r$ is a solution of the equation $P(x) = 0$ in K', i.e. this r verifies the equality (1.5). But $(2r-b)^2 = \Delta$, because $r^2 = br + a$ (see (1.5)), so $\pm\sqrt{\Delta} \in K'$. Since $\sqrt{\Delta} \notin K$, we see that $K[\sqrt{\Delta}] : K = 2$, so $K' = K[\sqrt{\Delta}]$. Thus any element of K' is of the form: $\alpha + \beta\sqrt{\Delta}$, where $\alpha, \beta \in K$. Therefore, in $S_{a,b}^{K'}$ one has two geometrical progressions:

$$x_n = r_1^n,\ y_n = r_2^n, \tag{1.6}$$

where $r_1 = (b + \sqrt{\Delta})/2$ and $r_2 = (b - \sqrt{\Delta})/2$ in K'.

Theorem 1.2. *Let us assume the above notation and definitions. a) if $r_1 \neq r_2$, i.e. if $\Delta \neq 0$, then $\{\{x_n\}_n, \{y_n\}_n\}$ is a basis in $S_{a,b}^{K'}$, b) if $r_1 = r_2$, i.e. if $\Delta = 0$, then $\{\{x_n\}_n, \{nx_n\}_n\}$ is a basis in $S_{a,b}^{K'}$.*

Proof. Since $\dim_{K'} S_{a,b}^{K'} = 2$, we see that it is sufficient to prove the linear independence of the above two binary sets and that $\{nx_n\}_n \in S_{a,b}^{K'}$. First of all let us assume that

$$c_1\{x_n\}_n + c_2\{y_n\}_n = \{0_n\}_n,\ c_1, c_2 \in K',$$

in the case a), $\Delta \neq 0$. For $n = 0, 1$ we get the following 2×2 system of linear equations:

$$\begin{cases} c_1 + c_2 = 0 \\ c_1 r_1 + c_2 r_2 = 0 \end{cases}.$$

Since its determinant is $r_2 - r_1 \neq 0$, we see that $c_1 = c_2 = 0$.

In the case b), it is easy to verify that the sequence $\{nx_n\}_n \in F_{a,b}^{K'}$ if $\Delta = 0$ (prove formula (1.1)). Let us prove now that the set $\{\{x_n\}_n, \{nx_n\}_n\}$ is linear independent over K'. Let us assume that

$$c_1\{x_n\}_n + c_2\{nx_n\}_n = \{0_n\}_n, \ c_1, c_2 \in K',$$

For $n = 0, 1$ we get: $c_1 = 0$, $c_1 r_1 + c_2 r_1 = 0$, so $c_2 = 0$. Thus the set $\{\{x_n\}_n, \{nx_n\}_n\}$ is linear independent if $\Delta = 0$. □

The mapping $\sigma : K' \to K'$, $\sigma(\alpha + \beta\sqrt{\Delta}) = \alpha - \beta\sqrt{\Delta}$ is a K-automorphism of K', i.e. it is closed to addition, subtraction, multiplication, $\sigma(1) = 1$, is one-to-one and onto and it fixes the elements of K. Moreover, $\sigma \circ \sigma = id.$ and $\sigma(r_1) = r_2$. Now theorem 1.2 gives us the complete structure of the space of sequences $S_{a,b}^{K'}$.

Corollary 1.3. *a) If $\Delta \neq 0$ and $\{z_n\}_n = F_{a,b}^K(z_0, z_1) \in S_{a,b}^K$, then there are C_1, $C_2 \in K'$ such that $C_2 = \sigma(C_1)$ and*

$$z_n = C_1 r_1^n + C_2 r_2^n, \ n = 0, 1, ... \tag{1.7}$$

Moreover

$$C_1 = \frac{z_1 - z_0 r_2}{r_1 - r_2}, \ C_2 = \frac{z_1 - z_0 r_1}{r_2 - r_1} \tag{1.8}$$

b) If $\Delta = 0$ (i.e. $r_2 = r_1$), $b \neq 0$ and $\{z_n\}_n = F_{a,b}^K(z_0, z_1) \in S_{a,b}^K$, then there are C_1, $C_2 \in K'$ such that $C_2 = \sigma(C_1)$ and

$$z_n = C_1 r_1^n + C_2 n r_1^n, \ n = 0, 1, ...$$

In this last case, $C_1 = z_0$ and $C_2 = (2z_1 - bz_0)/b$.

If in corollary 1.3 we put $r_1 = (b + \sqrt{\Delta})/2$ and $r_2 = (b - \sqrt{\Delta})/2$, we get another main consequence of theorem 1.2.

Corollary 1.4. *With the notation of formula (1.2), i.e. if $F_{a,b}^K(1, 0) = \{P_n(a, b)\}_n$ and if $F_{a,b}^K(0, 1) = \{Q_n(a, b)\}_n$ and if $\Delta \neq 0$, then*

$$P_n(a, b) = \frac{a}{2^{n-2}} \sum_{j=0}^{\left[\frac{n-2}{2}\right]} \binom{n-1}{2j+1} b^{n-2j-2} \Delta^j, n \geq 2, \tag{1.9}$$

$P_0(a, b) = 1$, $P_1(a, b) = 0$, and

$$Q_n(a, b) = \frac{1}{2^{n-1}} \sum_{k=0}^{\left[\frac{n-1}{2}\right]} \binom{n}{2k+1} b^{n-2k-1} \Delta^k, n \geq 1, Q_0(a, b) = 0. \tag{1.10}$$

2. A simple application in cryptography

Here is a general philosophy for applying the structure of the vector spaces $S_{a,b}^{K}$ and $S_{a,b}^{K'}$ of generalized Fibonacci sequences to make encoding in cryptography.

Let S be a sender and R be a receiver of a coded message. They make a secret convention at the beginning of a year about the encoding of the usual 50 signs namely, the English alphabet letters, the digits $0, 1, ..., 9$ and some other special signs as "." , ",", "!", "+", "−", etc. They want to construct a dynamic encoding, i.e. that the encoding system varies with the month M in this year, with the day D in the agreed month M and with the hour H in the prescribed day D. By a prescribed hour H we mean to prescribe one of the following 24 hour intervals: [5min, 55min], [1h 5min, 1h 55min], ..., [23h 5min, 23h 55min]. Here we left 10 minutes break between two consecutive hours so that the message from the previous interval (which is differently coded) to arrive at destination. We chose 12 prime numbers $p_1, p_2, ..., p_{12} > 50$ for encoding the twelve months of the year. If the sender S wants to send a coded message inside the i-th month he uses the k-th term of the generalized Fibonacci sequence $F_{a_j,b_j}^{\mathbb{F}_{p_i}}(0, s_l)$, where $\mathbb{F}_{p_i}$ is the finite field with p_i elements $\{\widehat{0}, \widehat{1}, ..., \widehat{p_i - 1}\}$, (a_j, b_j), $j = 1, 2, ..., 28, 29, 30$ or 31, is a couple of elements in $\mathbb{F}_{p_i}$ such that $\Delta_j = b_j^2 + 4a_j \neq 0$ (in $\mathbb{F}_{p_i}$) and $F_{a_j,b_j}^{\mathbb{F}_{p_i}}(0, s_l) \neq 0$ in $\mathbb{F}_{p_i}$ for any $l = 1, 2, ..., 50$), k encodes the number of one of the above 24 hour interval, and $s_l \in \mathbb{F}_{p_i}$, $l = 1, 2, ..., 40$. Here the couple (a_j, b_j) encodes the number of the day in the i-th month and s_l encodes a sign (a letter, a digit or special sign). The sender S sends to R a sequence of numbers $z_k(s_l)$, each depending on the encoded number s_l of a one of the above signs. In order to decode the actual sign encoded by s_l, the receiver R takes the k-th term $z_k = z_k(s_l)$ of the sequence $F_{a_j,b_j}^{\mathbb{F}_{p_i}}(0, s_l)$ (with s_l unknown for that moment) and by using formulas (1.2) and (1.10) he obtains:

$$z_k = s_l \frac{1}{2^{k-1}} \sum_{t=0}^{[\frac{k-1}{2}]} \binom{k}{2t+1} b_j^{k-2t-1} \Delta_j^t, \tag{2.1}$$

which is not zero in $\mathbb{F}_{p_i}$ (because of the above restriction)

Since $\mathbb{F}_{p_i}$ is a field, we can solve the equation (2.1) and find the value of s_l in $\mathbb{F}_{p_i}$. The table agreed with the sender S helps the receiver R to decode the actual character or special sigh which is behind the number s_l.

Here is an example.

Suppose that the month May is encoded by the prime number 53, the 27-th day of May is encoded by the couple $(1, 1)$ in $\mathbb{F}_{53}$. In this case $\Delta = 5 \bmod 53$ and it is not a square in $\mathbb{F}_{53}$. Indeed, using the Legendre symbol:

$$\left(\frac{n}{p}\right) = \begin{cases} 1, & \text{if } n \text{ is a square in } \mathbb{F}_p, \\ -1, & \text{otherwise,} \end{cases}$$

and Euler's quadratic reciprocity law for p, q two prime numbers distinct from 2:

$$\left(\frac{q}{p}\right) = (-1)^{\frac{(p-1)(q-1)}{4}} \left(\frac{p}{q}\right)$$

we get:

$$\left(\frac{5}{53}\right) = \left(\frac{53}{5}\right) = \left(\frac{3}{5}\right) = \left(\frac{5}{3}\right) = \left(\frac{2}{3}\right) = -1.$$

So $\Delta = 5$ is not a square in $\mathbb{F}_{53}$. For the parameter k we take $k = 10$ which corresponds to the hour interval [9h 5min, 9h 55min]. Suppose we want to send the message "YES" with the encoding: Y$\to \widehat{2}$, E$\to \widehat{1}$, S$\to \widehat{3}$. Here the hat means the corresponding class in $\mathbb{F}_{53}$. The receiver R knows all these keys. He receives the following sequence of numbers: $z_{10} = \widehat{15}, z^{*}_{10} = \widehat{34}$ and $z^{**}_{10} = \widehat{49}$, on which he knows that they are the 10-th terms in the sequences $F^{\mathbb{F}_{53}}_{1,1}(0, s_1)$, $F^{\mathbb{F}_{53}}_{1,1}(0, s_2)$ and $F^{\mathbb{F}_{53}}_{1,1}(0, s_3)$ respectively. These last 10-th terms can be easily calculated by the recurrence process of formula (1.1). We solve three equations in $\mathbb{F}_{53}$ of the type (2.1) and find $s_1 = 2$, $s_2 = 1$, $s_3 = 3$. From the table agreed with the sender he decodes the massage: "YES".

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VECTOR CALCULUS APPLICATIONS IN MATLAB. AREAS AND VOLUMES

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Abstract: Most of the engineering problems are solved using vector calculus. So, for understanding the algorithm behind any problem, the mathematical model is a must to know. Vector calculus is a very important part because it has many applications. For studying the vector calculus, the MatLAB program is an efficient instrument and many applications, like calculating areas and volumes, will be made with MatLAB.
Mathematics Subject Classification (2010): 97R20
Key words: vector calculus, MatLAB.

1. Introduction. Operations with vectors in MatLAB

In MatLAB for a vector it will be associated a line matrix which will contain the components of the vector for an orthonormal base:

In $\Re^3$: $\vec{a} = a_1 \vec{i} + a_2 \vec{j} + a_3 \vec{k}$ for the orthonormal canonic base $\{ \vec{i}, \vec{j}, \vec{k} \}$ so in MatLAB the vector will be define like this: $a = [a_1\ a_2\ a_3]$.

The sum of vectors: it will be used the sum operation of matrix, because they are defined as line matrix.

The multiplication with scalar: it will be used the multiplication with a scalar operation of a matrix.

The scalar product of 2 vectors can be made in MatLAB in 2 ways:

- using the command: **dot(a,b)**;
- by multiplying one vector with the other transpose**: a*b'.**

To calculate the norm of a vector in MatLAB it will be used the square root of the scalar product or the command **norm.**

The vector product in MatLAB can be calculate by using the command **cross(a,b)** only if the vectors have the dimension minimum 3.

The mixed product of vectors in MatLAB. By aligning 3 given vectors it will be formatted the square matrix with 3 lines and 3 columns. Then it will be calculated the determinant of this matrix. All this it will made in MatLAB by the command det([a;b;c]).

Collinearity, coplanarity and orthogonality with MatLAB. It can be determinate the collinearity of two vectors by making the vectorial product of them. The coplanarity of three vectors can be found from the mixed product. From the calculus of the scalar product of two vectors can determinate the orthogonality of those vectors.

2. Calculus of areas and volumes with MatLAB

To calculate the ***area of a triangle*** we will use the vector product of two vectors determined by the sides of the triangle.

So, if there are given the points $A(x_a, y_a, z_a), B(x_b, y_b, z_b), C(x_c, y_c, z_c)$, it will be calculated the vectors :

$$\overrightarrow{AB} = (x_b - x_a)\vec{i} + (y_b - y_a)\vec{j} + (z_b - z_a)\vec{k}$$

$$\overrightarrow{AC} = (x_c - x_a)\vec{i} + (y_c - y_a)\vec{j} + (z_c - z_a)\vec{k}$$

and then $Aria_{\Delta ABC} = \frac{1}{2}\|\overrightarrow{AB} \times \overrightarrow{AC}\|$.

In MatLAB we introduce the points coordonates a=$[x_a\ y_a\ z_a]$, b= $[x_b\ y_b\ z_b]$, $c = [x_c\ y_c\ z_c]$ after that we calculate the vectors

$$\overrightarrow{AB} = b - a$$

$$\overrightarrow{AC} = c - a$$

To calculate the area of ABC we will use one of these formulas

```
Area=0.5*sqrt(dot(cross(AB,AC),cross(AB,AC)))
Area=0.5*norm(cross(AB,AC))
```

If Area is zero then the points are colinear.

To calculate the ***area of a quadrilateral or a polygon,*** we will divide the polygon in two or more triangles and use the algorithm above. We can verify if the quadrilateral is a parallelogram. For four points $A(x_a, y_a, z_a), B(x_b, y_b, z_b), C(x_c, y_c, z_c)$ *and* $D(x_d, y_d, z_d)$, if *d=c-b+a*, then *ABCD* is a parallelogram and to calculate area we will use the formula:

$$S_{\Delta ABC} = \frac{1}{2} S_{ABCD} = \frac{1}{2}\|\vec{a} \times \vec{b}\| = \frac{1}{2}\|\overrightarrow{AB} \times \overrightarrow{AC}\|$$

In MatLAB

```
Area=0.5*norm(cross(AB,AC)).
```

To calculate the volume of a parallelepiped formed by three vectors with the same origin we used the mixt product and the formula: $V_{paralelipiped} = |(\vec{a}, \vec{b}, \vec{c})|$

To calculate the volume of a tetrahedron we will use the mixt product of three vectors and the formula $V_{ABCD} = \frac{1}{6} V_{paralelipiped} = \frac{1}{6}|(\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD})|$

Applications

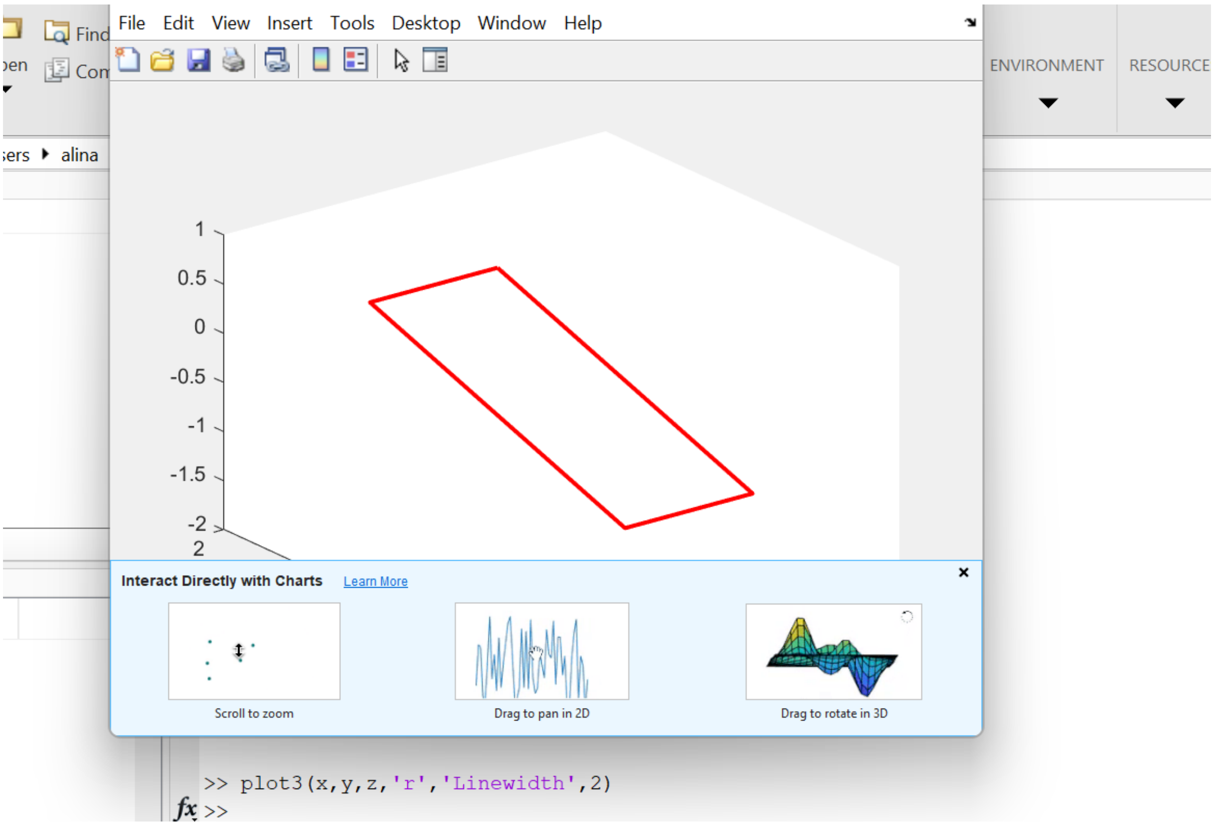
In $\mathbb{R}^3$ let be the points A(-2,2,1), B(2,1,-2), C(0,1,-2), D(1,1,0), E(4,2,-4), M(5,-3,6), N(8,0,3)

1) Write the vectors AB, AC, BC;
2) Calculate the perimeter and area of the triangle ABC;
3) Calculate the length of a median from B to AC in the triangle ABC;
4) Calculate the length of a height from C to AB in the triangle ABC;
5) Study if the points A, B, C and D are coplanar;
6) Study if the points A, B and E are colinear;
7) Find a point T colinear with the points A and B;
8) Calculate the volume of the tetrahedral ABCM;
9) Calculate the distance from M to the plane ABC;
10) Calculate the volume of the parallelepiped formed with the vectors AB, AC, AN;
11) Calculate the length of a diagonal of the parallelepiped formed with the vectors AB, AC, AN;
12) Find a point P such as ABCP is a parallelogram and draw an graph.

```
a=[-2 1 1]
a =
    -2     1     1
>> b=[2 1 -2]
b =
     2     1    -2
>> c=[0 1 -2]
c =
     0     1    -2
>> ab=b-a
ab =
     4     0    -3
>> ac=c-a
ac =
     2     0    -3
>> bc=c-b
bc =
    -2     0     0
>> perim=norm(ab)+norm(ac)+norm(bc)
perim =
   10.6056
>> Aabc=0.5*norm(cross(ab,ac))
Aabc =
     3
>> k=(a+c)/2
k =
   -1.0000    1.0000   -0.5000
>> med=norm(k-b)
med =
    3.3541
>> Hab=2*Aabc/norm(ab)
```

```
Hab =
   1.2000
>> d=[1 1 0]
d =
     1     1     0
>> ad=d-a
ad =
     3     0    -1
>> pm=det([ab;ac;ad])
pm =
     0
>> if pm==0 disp('COPLANARE')
else disp('NECOPLANARE')
end
COPLANARE
>> e=[4 2 -4]
e =
     4     2    -4
>> ae=e-a
ae =
     6     1    -5
>> cross(ae,ab)
ans =
    -3    -2    -4
>> t=2*b-a
t =
     6     1    -5
>> ta=t-a
at =
     8     0    -6
>> cross(ab,at)
ans =
     0     0     0
>> m=[5 -3 6]
m =
     5    -3     6
>> am=m-a
am =
     7    -4     5
>> VolT=abs(det([ab;ac;am]))/6
VolT =
   4.0000
>> hM=3*VolT/Aabc
hM =
```

```
    4.0000
>> n=[8 0 3]
n =
     8     0     3
>> an=n-a
an =
    10    -1     2
>> VolP=abs(det([ab;ac;an]))
VolP =
     6
>> lungDIAG=norm(ab+ac+an)
lungDIAG =
   16.5227
p=a-b+c
p =
    -4     1     1
```



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ON POPULATION MODELS AND THE LOGISTIC MAP

DANIEL TUDOR AND DAN CARAGHEORGHEOPOL

Abstract. In this paper, we explore population growth model and we observe how recurrent sequences for such models can have a great sensitivity on the initial condition.
Mathematics Subject Classification (2010): 34H10, 65P20
Key words: chaos, recursion, discrete dynamical systems.

1. Introduction

Many phenomena occuring in the nature can be described by time-dependent, discrete or continuous, differential equations. One such discrete model is the logistic map, which models the population growth of a species with non-overlapping generations and living in an environment with limited resources. Such a species can be, for example, mosquitoes and its population growth can be described by the logistic map.

2. Model

Suppose we are interested in analysing the long-term behaviour of the population of a species over a large number of generations. Denote p_n to be the population in generation n. Here, p_0 represents the size of the population upon starting to observe it.

In this model, we take into effect three phenomena occuring in the natural world: birth, death and competition. Here, the competition refers to a pair of animals from the same species competing for the same natural resource. We also assume that any two pairs of animals within the same species can theoretically compete for the same resource. That is, for a population of N animals in a species, there are $\binom{N}{2}$ possible pairs of competitors for the same resource. Denote the birth rate by by b, the death rate by d and the competition rate by c.Then the evolution of the population from one generation to the next can be described by the following recurrence relation

$$p_{n+1} = (1+b-d)p_n - c\binom{p_n}{2} = (1+b-d)p_n - c(p_n-1)p_n.$$

As this equation looks rather complicated, let us introduce the new variable $x_n = p_n \cdot \frac{c/2}{1+b-d+c/2}$. This variable can be interpreted as the fraction of the maximum population supported by the environment. Under this new variable, the recurrence relation becomes

$$x_{n+1} = rx_n(1-x_n), \tag{2.1}$$

where $r = 1+b-d+\frac{c}{2}$. Equation (2.1) is commonly referred to as the *logistic map*.[1]

3. Conditions on growth coefficient

In order for the sequence (x_n) to retain its meaningful description, we ought to ensure that $0 \leq x_n \leq 1$, for all $n \in \mathbb{N}$. To do this, further conditions on the coefficient r have to be imposed. Assume that $x_k \in [0,1]$, and our goal is to ensure that $x_{k+1} \in [0,1]$. Evidently, one condition that needs to be imposed is that $r \geq 0$, otherwise the next term in the sequence will take negative values. Define the function $f : [0,1] \to \mathbb{R}$ by $f(x) = rx(1-x)$. This function exhibits a zero minimum at its boundary (provided that $r \geq 0$), and a maximum of $\frac{r}{4}$ at $x = 1/2$. Therefore, we need to impose the additional condition $\frac{r}{4} \leq 1$. Therefore, the condition on r in order to satisfy $0 \leq x_n \leq 1$, for all $n \in \mathbb{N}$ is $r \in [0,4]$. In the following plot, the one-step dependency between x_n and x_{n+1} is highlighted.

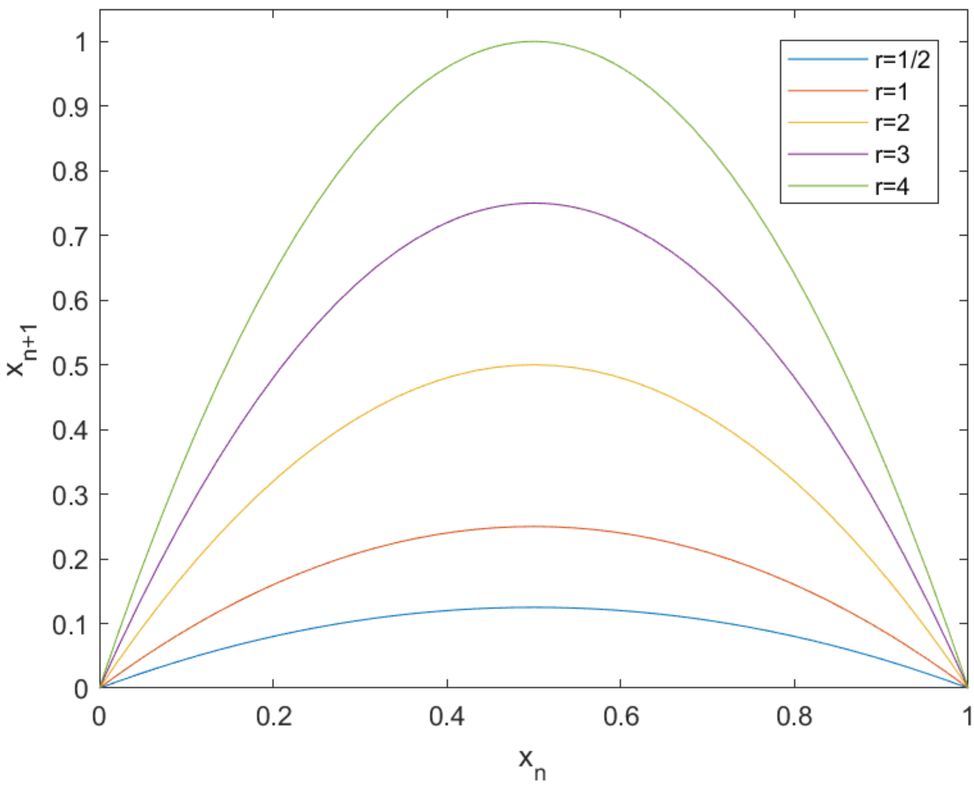

Figure 1. Plot of x_n against x_{n+1} for different values of r

We are, however, not only interested in the short-term behaviour of this model, that is, from one-step to the next, but we also wish to see the limiting behaviour of 2.1 as $n \to \infty$.

4. Limiting behaviour

We are now interested in analysing the limiting behaviour of the sequence defined as in (2.1), depending on the parameter r, and assuming that $x_0 \in (0,1)$. It has been previously shown that for $r \in [0,4]$, $x_n \in [0,1]$ for all $n \in \mathbb{N}$. This behaviour can only easily be described for $0 \leq r \leq 1$.

Theorem 4.1. *If* $x_0 \in [0,1]$ *and* $r \in [0,1]$*, then* $\lim_{n\to\infty} x_n = 0$.

Proof. As seen above, $x_n \in [0,1]$, for all values of n. Furthermore, we have $x_{n+1} = rx_n(1-x_n) \leq x_n(1-x_n) \leq x_n$ for all $n \in \mathbb{N}$. Therefore, the sequence is decreasing and bounded, and thus the limit exists. Denote $\lim_{n\to\infty} x_n = l$. Then, plugging this value into (2.1), we obtain $l = rl(1-l)$, and so $l = 0$, or $l = \frac{r-1}{r}$. As $\frac{r-1}{r} \leq 0$ and $0 \leq x_n \leq 1$, we need to have $l = 0$.

It turns out that the limiting behaviour of the sequence is much harder to anticipate when r falls outside of this interval. Below you can find a plot of the limiting values of (2.1).

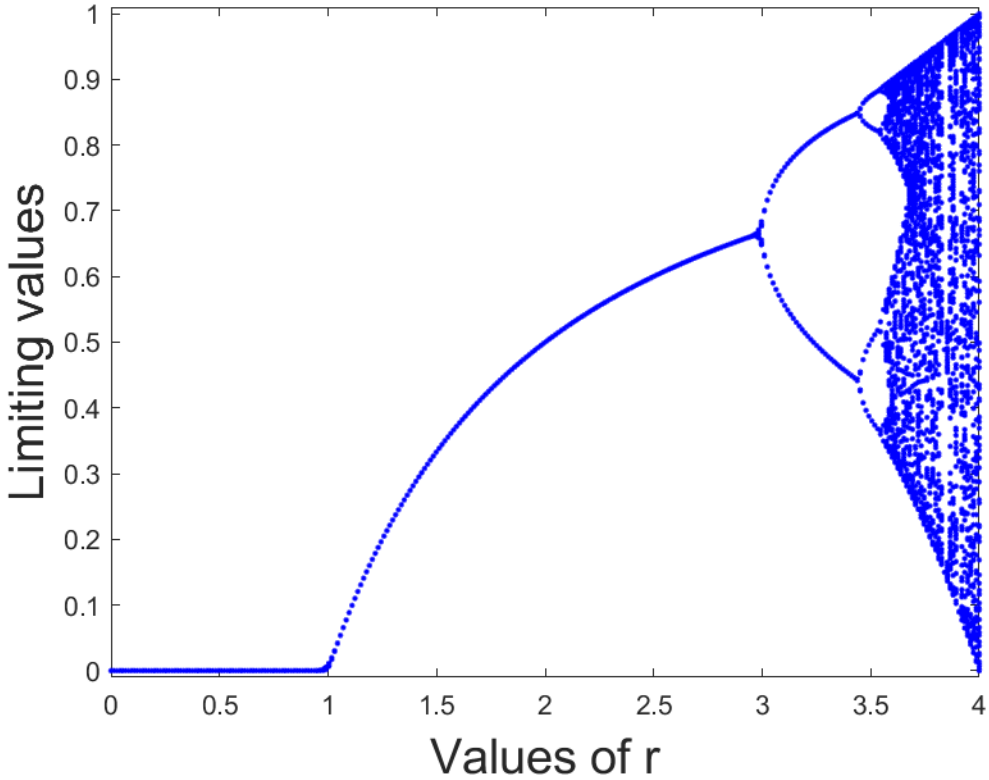

FIGURE 2. Limiting values of the logistic map based on the values of r

As can be observed from the picture, for $r \in (1,3]$, the sequence still converges to a non-zero value. Setting in the limit in 2.1, this value has to be $\frac{r-1}{r}$. For r between 3 and 3.449 (approximately), the sequence approaches permanent oscillations between two values. For values of $r \in (3.449, 3.544)$, the sequence approaches oscillations between 4 values. On a small interval between 3.544 and 3.569, the sequence approaches oscillations of length 8, then 16, then 32, and so on, with the period doubling each time.

The most striking feature observed in this graph is seen from 3.566 onwards where chaos emerges. Apparently, there is no longer limiting behaviour that can be clearly predicted and the sequence displays a chaotic long-term behaviour. It is to be noted that some values might still exhibit periodic behaviour after 3.566. For instance, at $1 + \sqrt{8}$, a 3-period cycle emerges, but for the vast majority of values beyond 3.566, the system is seen to show a chaotic behaviour.

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ABOUT BASIC PROPERTIES OF $\mathcal{A}_S -$ SCALAR OPERATORS

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Abstract: The concept of $\mathcal{A}_S -$scalar operator (]8]) is a naturally generalization of the notion of $\mathcal{A}-$scalar operator studied in [2].
In this work, we present some basic properties concerning the $\mathcal{A}_S -$scalar operators and some examples of $\mathcal{A}_S -$scalar operators obtained as restrictions and quotients of $\mathcal{A}-$scalar operators.
Mathematics Subject Classification (2010): 47B47, 47B40.
Key words: $\mathcal{A}_S -$scalar ($\mathcal{A}-$scalar) operator; $\mathcal{A}_S -$spectral ($\mathcal{A}-$spectral) function; restriction and quotient of an operator.

1. Introduction

The aim of this work is to present some basic results of the theory of $\mathcal{A}_S -$scalar operators in arbitrary complex Banach spaces in a systematic way. The restrictions and the quotients of $\mathcal{A}-$scalar operators with respect to closed invariant subspaces are presented here, and it is shown that they are $\mathcal{A}_S -$scalar operators.

In the first section, we briefly recall several notions and basic definitions that will be employed throughout this paper.

In the following, let X be a complex Banach space, let $\mathbf{B}(X)$ be the algebra of all linear bounded operators on X and let $\mathbf{C}$ be the complex plane.

As usual, if $T \in \mathbf{B}(X)$ and $Y \subset X$ is a linear closed subspace invariant to T (i.e. $TY \subseteq Y$), let us denote by $T|Y$ the restriction operator of T to Y, respectively by $\dot{T}$ the quotient operator induced by T on the quotient space $\dot{X} = X/Y$.

Moreover, the *spectrum* of an operator $T \in \mathbf{B}(X)$ is denoted by $\sigma(T)$ and it is defined as the set of all complex numbers $\lambda \in \mathbf{C}$ such that the operator $\lambda I - T$ is no inversable in $\mathbf{B}(X)$.

Definition 1.1. ([2]) A closed linear subspace $Y \subset X$ is called *spectral maximal space* of $T \in \mathbf{B}(X)$ if Y is invariant to T and for any other closed subspace $Z \subset X$, also invariant to T, the inclusion $\sigma(T|Z) \subset \sigma(T|Y)$ implies the inclusion $Z \subset Y$.

Definition 1.2. ([7]) An open set $\Omega \subset \mathbf{C}$ is said to be a *set of analytic uniqueness* for $T \in \mathbf{B}(X)$ if for any open set $D \subset \Omega$ and any analytic function $f : D \to X$ satisfying the equation $(\lambda I - T)\, f(\lambda) \equiv 0$, it follows that $f(\lambda) \equiv 0$ in D.

For $T \in \mathbf{B}(X)$, there is a unique maximal open set Ω_T of analytic uniqueness (2.1., [7]).

The set $S_T = \mathbf{C} \setminus \Omega_T$ is called the *analytic spectral residuum of* T.

Definition 1.3. ([7]) For an operator $T \in \mathbf{B}(X)$ and $x \in X$, we denote by $\delta_T(x)$ the set of all points $\xi \in \mathbf{C}$ such that there are a neighborhood V_ξ of ξ and at least an analytic function $f_x : V_\xi \to X$ such that $(\lambda I - T) f_x(\lambda) \equiv x$, for all $\lambda \in V_\xi$.

We have the sets

$$\rho_T(x) = \delta_T(x) \cap \Omega_T$$
$$\sigma_T(x) = \mathbf{C} \setminus \rho_T(x).$$

We consider the space

$$X_T(F) = \{ x \in X;\ \sigma_T(x) \subset F \}, \text{ where } S_T \subset F \subset \mathbf{C}.$$

Definition 1.4. ([7]) A family of open sets $G_S \cup \{G_i\}_{i=1}^n$ is an $S-$*covering* of the closed set $\sigma \subset \mathbf{C}$ if the following conditions are fulfilled:

1) $G_S \cup \left(\bigcup_{i=1}^{n} G_i \right) \supset \sigma \cup S$
2) $\overline{G}_i \cap S = \varnothing$ $(1 \le i \le n)$ (where $S \subset \mathbf{C}$ is also closed).

Definition 1.5. ([1]) An operator $T \in \mathbf{B}(X)$ is called $S-$*decomposable* (where $S \subset \sigma(T)$ compact) if for any finite open $S-$covering $G_S \cup \{G_i\}_{i=1}^n$ of $\sigma(T)$, there is a system $Y_S \cup \{Y_i\}_{i=1}^n$ of spectral maximal spaces of T such that:

1) $\sigma(T \mid Y_S) \subset G_S$, $\sigma(T \mid Y_i) \subset G_i$ $(1 \le i \le n)$
2) $X = Y_S + \sum_{i=1}^{n} Y_i$.

If $S = \varnothing$, then the operator T is decomposable ([2]).

In Section 2 are summarized several basic properties of $\mathcal{A}_S-$scalar operators.

In the third section, we present some examples of restrictions and quotients of $\mathcal{A}-$scalar operators which become $\mathcal{A}_S-$scalar operators.

2. Basic properties of $\mathcal{A}_S-$ scalar operators

Definition 2.1. ([8]) Let Ω be a set of the complex plane $\mathbf{C}$ and let $S \subset \overline{\Omega}$ be a compact subset.

An algebra $\mathcal{A}_S$ of $\mathbf{C}$-valued functions defined on Ω is said to be $S-$*normal* if for any finite open $S-$covering $G_S \cup \{G_i\}_{i=1}^n$ of $\overline{\Omega}$, there are the functions f_S, $f_i \in \mathcal{A}_S$ having the properties:

1) $f_S(\Omega) \subset [0, 1]$, $f_i(\Omega) \subset [0, 1]$ $(1 \le i \le n)$
2) $\operatorname{supp}(f_S) \subset G_S$, $\operatorname{supp}(f_i) \subset G_i$ $(1 \le i \le n)$

3) $f_S + \sum_{i=1}^{n} f_i = 1$ on Ω

where *the support* of the function $f \in \mathcal{A}_S$ is the set defined as

$$\operatorname{supp}(f) = \overline{\{\xi \in \Omega\,;\, f(\xi) \neq 0\}}.$$

Definition 2.2. ([8]) An algebra $\mathcal{A}_S$ of **C**-valued functions defined on Ω is said to be $S-$*admissible* if the following conditions are established:

1) $\lambda \in \mathcal{A}_S$, $1 \in \mathcal{A}_S$ (where λ, 1 denote the functions $f(\lambda) \equiv \lambda$, $f(\lambda) \equiv 1$)
2) $\mathcal{A}_S$ is $S-$normal
3) for any $f \in \mathcal{A}_S$ and for any $\xi \notin \operatorname{supp}(f)$, the function

$$f_\xi : \Omega \to \mathbf{C},\ f_\xi(\lambda) = \begin{cases} \dfrac{f(\lambda)}{\xi - \lambda}, & \text{for } \lambda \in \Omega \setminus \{\xi\} \\ 0\,, & \text{for } \lambda \in \Omega \cap \{\xi\} \end{cases}$$

belongs to $\mathcal{A}_S$.

Definition 2.3. ([8]) The operator $T \in \mathbf{B}(X)$ is called $\mathcal{A}_S-$*scalar* if there are an $S-$admissible algebra $\mathcal{A}_S$ and an algebraic homomorphism $U : \mathcal{A}_S \to \mathbf{B}(X)$ such that

$$U_1 = I \text{ and } U_\lambda = T$$

(where λ is the identical function $f(\lambda) \equiv \lambda$ on **C**).

The map U is called $\mathcal{A}_S-$*spectral function* for operator T.

If $S = \varnothing$, then we put $\mathcal{A} = \mathcal{A}_\varnothing$ and we obtain an $\mathcal{A}-$spectral function and an $\mathcal{A}-$scalar operator ([2]).

Definition 2.4. ([8]) A subspace Y of X is said to be *invariant* to an $\mathcal{A}_S-$spectral function $U : \mathcal{A}_S \to \mathbf{B}(X)$ if $U_f\, Y \subseteq Y$, for any $f \in \mathcal{A}_S$.

Definition 2.5. ([8]) *The support* of an $\mathcal{A}_S-$spectral function U is denoted by $\operatorname{supp}(U)$ and it is defined as the smallest closed subset of $\overline{\Omega}$ such that $U_f = 0$, for any $f \in \mathcal{A}_S$ with $\operatorname{supp}(f) \cap \operatorname{supp}(U) = \varnothing$.

Theorem 2.1. ([8]) If $U : \mathcal{A}_S \to \mathbf{B}(X)$ is an $\mathcal{A}_S-$spectral function for $T \in \mathbf{B}(X)$, then

$$\operatorname{supp}(U) \subset \sigma(T) \cup S \text{ and } \sigma(T) \subset \operatorname{supp}(U) \cup S.$$

Theorem 2.2. ([8]) Let $T \in \mathbf{B}(X)$ be an $\mathcal{A}_S-$scalar operator and let $U : \mathcal{A}_S \to \mathbf{B}(X)$ be an $\mathcal{A}_S-$spectral function for T. Then

$$S_T \subset S.$$

Theorem 2.3. ([8]) Let $T \in \mathbf{B}(X)$ be an $\mathcal{A}_S-$scalar operator and let $U : \mathcal{A}_S \to \mathbf{B}(X)$ be an $\mathcal{A}_S-$spectral function for T. Then T is an $S-$decomposable operator.

3. Examples of $\mathcal{A}_S -$ scalar operators

Example 3.1. (Restrictions of $\mathcal{A}_S$ – scalar operators)

Let $T \in \mathbf{B}(X)$ be an $\mathcal{A}_S$ – scalar operator, let $U : \mathcal{A}_S \to \mathbf{B}(X)$ be its $\mathcal{A}_S$ – spectral function and let $Y \subset X$ be a spectral maximal space of T such that $Y = X_T(F)$, for any $F \subset \mathbf{C}$ closed, with $S \subset F$.

Then the restriction operator $T \mid Y$ is an $\mathcal{A}_S$ – scalar operator.

It is easily seen that the map $U \mid Y : \mathcal{A}_S \to \mathbf{B}(Y)$ is an $\mathcal{A}_S$ – spectral function for the restriction operator $T \mid Y \in \mathbf{B}(Y)$, hence $T \mid Y$ is $\mathcal{A}_S$ – scalar.

Example 3.2. (Restrictions of $\mathcal{A}$ – **sc**alar operators)

Let $T \in \mathbf{B}(X)$ be an $\mathcal{A}$ – scalar operator, let $U : \mathcal{A} \to \mathbf{B}(X)$ be its $\mathcal{A}$ – spectral function and let $Y \subset X$ be a closed subspace invariant to T, which is not invariant to U.

Then the restriction operator $T \mid Y$ is $\mathcal{A}_S$ – scalar, where $S = \sigma(\dot{T})$ and $\mathcal{A}_S$ is an S – admissible subalgebra of $\mathcal{A}$ composed by all functions f which have one of the following properties:

1) supp $(f) \cap S = \varnothing$
2) $\text{supp}(f) \supset S$.

It is easy to verify that the map $U \mid \mathcal{A}_S : \mathcal{A}_S \to \mathbf{B}(Y)$ defined by $U \mid \mathcal{A}_S(f) = U_f$, $f \in \mathcal{A}_S$, is an $\mathcal{A}_S$ – spectral function for $T \mid Y \in \mathbf{B}(Y)$, therefore $T \mid Y$ is $\mathcal{A}_S$ – scalar.

Example 3.3. (Quotients of $\mathcal{A}$ – scalar operators)

Let $T \in \mathbf{B}(X)$ be an $\mathcal{A}$ – scalar operator, let $U : \mathcal{A} \to \mathbf{B}(X)$ be its $\mathcal{A}$ – spectral function and let $Y \subset X$ be a closed subspace invariant to T, which is not invariant to U.

Then the quotient operator $\dot{T}$ induced by T on the quotient space $\dot{X} = X / Y$ is $\mathcal{A}_S$ – scalar, where $S = \sigma(T \mid Y)$ and $\mathcal{A}_S$ is an S – admissible subalgebra of $\mathcal{A}$ composed by all functions f which have one of the following properties:

1) supp $(f) \cap S = \varnothing$
2) $\text{supp}(f) \supset S$ and $U_f \, Y \subset Y$.

It can be easily to verify that the application $\dot{U} : \mathcal{A}_S \to \mathbf{B}(\dot{X})$ defined by $\dot{U}(f) = \dot{U}_f$, $f \in \mathcal{A}_S$, is an $\mathcal{A}_S$ – spectral function for $\dot{T} \in \mathbf{B}(\dot{X})$, therefore $\dot{T}$ is $\mathcal{A}_S$ – scalar.

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SEQUENCES OF UNCOUNTABLE ITERATED FUNCTION SYSTEMS. PROPERTIES OF CONVERGENCE.

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Abstract In this paper, we consider a sequence of uncountable iterated function systems (U.I.F.S.). Each term of this sequence is built using an uncountable family of contractions and a linear and continuous operator. For each term of the sequence, we have : an associated attractor, a Markov-type operator and a fractal measure. The problem that we solve, is : if the sequence of U.I.F.S. is convergent, in some way, to an U.I.F.S., it is true that the sequences of associated attractors and fractal measures are, also, convergent ?
Key words : iterated function system, attractor, fractal measure, Markov-type operator, vector measure.
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